A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract. A proof is given of a conjecture [6] in the theory of certain Hilbert spaces of entire function [1-4] which implies the Riemann hypothesis. Weighted Hardy spaces, whose elements are functions analytic in the upper half–plane, are defined using analytic weight functions which have no zeros in the upper half–plane. The analytic weight function admits an analytic extension without zeros to a larger half–plane when the weighted Hardy space admits a maximal dissipative transformation which is a shift. A Hilbert space, whose elements are entire functions and which is contained isometrically in the weighted Hardy space, is constructed which inherits a maximal dissipative transformation which is a shift. The defining function of the Hilbert space of entire functions admits no pair of distinct zeros which are symmetric about the boundary of the half plane of analyticity determined by the maximal dissipative transformation. The Riemann hypothesis for Hilbert spaces of entire functions denies the existence of paired zeros for the defining functions of Hilbert spaces of entire functions which inherit a maximal dissipative transformation other than a shift. The spaces are constructed from a weighted Hardy space using an Euler product. The factors in an Euler product are determined by entire functions of Pólya class. Although the product converges only in the upper half–plane, the Hilbert spaces of entire functions constructed from the factors converge in the complex plane to a Hilbert space of entire functions [7]. Maximal dissipative transformations are inherited in the approximating spaces of entire functions and in the limit space. Although the maximal dissipative transformations are compressions of shifts, they approximate shifts in a sense determined by the Euler product. The maximal dissipative transformations in the limit spaces deny the existence of paired zeros as conjectured.

The proof of the Riemann hypothesis is an application of the theory [5] of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever an element $F(z)$ of the space has a nonreal zero w, the function

$$F(z)(z - w^-)/(z - w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ for every nonreal number w.

(H3) The function

$$F^*(z) = F(z^-)^-$$

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belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

If an entire function $E(z)$ satisfies the inequality

$$|E^*(z)| < |E(z)|$$

when z is in the upper half–plane, then the entire function

$$K(w, z) = \frac{E(z)E(w)^- - E^*(z)E(w^-)}{2\pi i(w^- - z)}$$

of z has a positive value at w when w is not real. A Hilbert space $\mathcal{H}(E)$ exists whose elements are entire functions $F(z)$ such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

is finite and such that the inequality

$$|F(w)|^2 \leq \|F\|^2 K(w, w)$$

holds for all complex numbers w. The function $K(w, z)$ of z belongs to the space for every complex numbers w and acts as reproducing kernel function for function values at w. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$.

Related Hilbert spaces appear whose elements are functions analytic in the upper half–plane. An analytic weight function is a function $W(z)$ which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ defined by an analytic weight function $W(z)$ is the set of entire functions $F(z)$ such that the least upper bound

$$\|F\|^2 = \sup \int_{-\infty}^{+\infty} |F(x + iy)/W(x + iy)|^2 dx$$

taken over positive numbers y is finite. The function

$$\frac{W(z)W(w)^-}{2\pi i(w^- - z)}$$

belongs to the space as a function of z when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The defining function $E(z)$ of a space $\mathcal{H}(E)$ is an example of an analytic weight function. The space $\mathcal{H}(E)$ is contained isometrically in the space $\mathcal{F}(E)$. The space $\mathcal{H}(E)$ is the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(E)$.

Examples of analytic weight functions are constructed from the gamma function, discovered by Euler, who applied it in the functional identity for the zeta function which he discovered. Although the gamma function was conceived as the solution of a recurrence relation, it is characterized by positivity properties.

A relation T, whose domain and range are contained in a Hilbert space, is said to be maximal dissipative if

$$(T - w)(T + w)^{-1}$$

is an everywhere defined and contractive transformation in the space for some, and hence every, complex number w in the right half–plane. The relation is said to be dissipative if a partially defined contractive transformation is obtained for some, and hence every, complex number w in the right half–plane. A dissipative transformation admits a maximal dissipative extension, which need not be a transformation.

An Euler weight function is an analytic weight function $W(z)$ such that a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined when h is in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ belong to the space.

Maximal dissipative relations in Hilbert spaces of analytic functions permit the construction of derived Hilbert spaces of analytic functions. The derived spaces of a weighted Hardy space $\mathcal{F}(W)$ are spaces of functions analytic in the upper half–plane. A derived space is constructed for some h in the interval $[0, 1]$ from the graph of the adjoint of the maximal dissipative transformation which takes $F(z)$ into $F(z + ih)$. An element of the graph is written as a pair

$$(F(z), G(z + ih))$$

of elements $F(z)$ and $G(z + ih)$ of the space $\mathcal{F}(W)$ such that the adjoint takes $F(z)$ into $G(z + ih)$. The scalar product of elements

$$(F_1(z), G_1(z + ih))$$

and

$$(F_2(z), G_2(z + ih))$$

of the graph is defined as a sum

$$\langle F_1(t), G_2(t + ih)\rangle + \langle G_1(t + ih), F_2(t)\rangle$$

of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative since the adjoint of a maximal dissipative relation is a maximal dissipative relation. Elements of the graph are considered equivalent if the scalar self–product of their difference is zero. The quotient space is a vector space which inherits a nondegenerate scalar product. A fundamental example of an element of the graph is determined by a complex number w such that $w - \frac{1}{2}ih$ belongs to the upper half–plane since the adjoint takes the reproducing kernel function

$$\frac{W(z)W(w - \frac{1}{2}ih)^-}{2\pi i(w^- + \frac{1}{2}ih - z)}$$

for function values at $w - \frac{1}{2}ih$ into the reproducing kernel function

$$\frac{W(z)W(w + \frac{1}{2}ih)^-}{2\pi i(w^- - \frac{1}{2}ih - z)}$$

for function values at $w + \frac{1}{2}ih$. An isometric transformation exists of the quotient space of the graph onto a dense subspace of a derived Hilbert space whose elements are functions analytic in the upper half–plane. The transformation takes an element

$$(F(z), G(z + ih))$$

of the graph into the analytic extension of the function

$$F(z + \tfrac{1}{2}ih) + G(z + \tfrac{1}{2}ih)$$

to the upper half–plane. The transformation is defined on the quotient space since the value of the function at w is a scalar product with the element of the graph determined by w when $w - \frac{1}{2}ih$ is in the upper half–plane. The derived space contains the analytic function

$$\frac{W(z - \frac{1}{2}ih)W(w + \frac{1}{2}ih)^- + W(z + \frac{1}{2}ih)W(w - \frac{1}{2}ih)^-}{2\pi i(w^- - z)}$$

of z in the upper half–plane as reproducing kernel function for function values at w. These conclusions are initially obtained when $w - \frac{1}{2}ih$ is in the upper half–plane, but the extension to the upper half–plane follows from characteristic properties of functions which are analytic and have nonnegative real part in the upper half–plane.

If $F(z)$ and $G(z + ih)$ are elements of the space $\mathcal{F}(W)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$ and if the functions $F(z + \frac{1}{2}ih)$ and $G(z + \frac{1}{2}ih)$ have a common zero in the half–plane $h < iw^- - iw$, then the adjoint of the maximal dissipative transformation in the space takes the element

$$F(z)(z - \tfrac{1}{2}ih - w^-)/(z - \tfrac{1}{2}ih - w)$$

of the space into the element

$$G(z + ih)(z + \tfrac{1}{2}ih - w^-)/(z + \tfrac{1}{2}ih - w)$$

of the space.

If $F(z)$ and $G(z + ih)$ are elements of the space $\mathcal{F}(W)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$ and if the sum

$$\langle F(t), G(t + ih)\rangle + \langle G(t + ih), F(t)\rangle$$

of scalar products in the space $\mathcal{F}(W)$ vanishes, then

$$F(z) + G(z)$$

vanishes identically. When a nonzero element $F(z)$ of the space $\mathcal{F}(W)$ exists such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $-F(z+ih)$, then

$$H(z)(z-w^-)/(z-w)$$

is an element of the derived space which has the same norm as $H(z)$ in the derived space whenever $H(z)$ is an element of the derived space which has a zero w in the upper half–plane. It follows that the functions $W(z)$ and $W(z+ih)$ are linearly dependent.

The derived space is more structured when the functions $W(z)$ and $W(z+ih)$ are linearly independent. If w is in the upper half–plane, continuous linear functionals $H(z)$ into $H_+(w)$ and $H(z)$ into $H_-(w)$ on the derived space exist which take the element

$$H(z) = F(z+\tfrac{1}{2}ih) + G(z+\tfrac{1}{2}ih)$$

of the space into

$$H_+(w) = G(w+\tfrac{1}{2}ih)$$

and

$$H_-(w) = F(w+\tfrac{1}{2}ih)$$

whenever the adjoint of the maximal dissipative transformation in the space $\mathcal{F}(W)$ takes $F(z)$ into $G(z+ih)$. The upper value $H_+(w)$ and the lower value $H_-(w)$ at w of an element $H(z)$ of the derived space are otherwise defined by continuity. Analytic functions $H_+(z)$ and $H_-(z)$ are obtained which decompose an element

$$H(z) = H_+(z) + H_-(z)$$

of the derived space. When $H(z)$ is equal to the reproducing kernel function

$$[W(z-\tfrac{1}{2}ih)W(w+\tfrac{1}{2}ih)^- + W(z+\tfrac{1}{2}ih)W(w-\tfrac{1}{2}ih)^-]/[2\pi i(w^- - z)]$$

for function values at w with w in the upper half–plane, then $H_+(z)$ is equal to

$$W(z-\tfrac{1}{2}ih)W(w+\tfrac{1}{2}ih)^-/[2\pi i(w^- - z)]$$

and $H_-(z)$ is equal to

$$W(z+\tfrac{1}{2}ih)W(w-\tfrac{1}{2}ih)^-/[2\pi i(w^- - z)].$$

When $H_+(z+\frac{1}{2}ih)$ and $H_-(z-\frac{1}{2}ih)$ belong to the space $\mathcal{F}(W)$ for an element $H(z)$ of the derived space, then the adjoint of the maximal dissipative transformation in the space $\mathcal{F}(W)$ takes $H_-(z-\frac{1}{2}ih)$ into $H_+(z+\frac{1}{2}ih)$.

When the functions $W(z)$ and $W(z+ih)$ are linearly dependent, the identity

$$W(z+ih) = \omega W(z)$$

holds for a nonzero number ω whose real part is nonnegative. The adjoint of the maximal dissipative transformation in the space $\mathcal{F}(W)$ takes $F(z)$ into $\omega^- F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space. When the real part of ω vanishes, the maximal dissipative transformation in the space is skew–adjoint and the derived space of the space $\mathcal{F}(W)$ contains no nonzero element. When the real part of ω is positive, every element $H(z)$ of the derived space admits a decomposition

$$H(z) = H_+(z) + H_-(z)$$

with

$$H_+(z) = (1+\omega)^{-1} H(z)$$

and

$$H_-(z) = (1+\omega^-)^{-1} H(z)$$

elements of the space such that

$$H_+(z) = \omega^- H_-(z).$$

A characterization of Euler weight functions results which is an elementary analogue of the Riemann hypothesis [7]. An analytic weight function $W(z)$ is an Euler weight function if, and only if, for every h in the interval $[0,1]$ the function

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

admits an analytic extension to the upper half–plane whose real part is nonnegative in the half–plane.

An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half–plane, if the inequality

$$|E^*(z)| \leq |E(z)|$$

holds when z is in the upper half–plane, and if the modulus of $E(z)$ is a nondecreasing function of distance from the real axis on every vertical line in the upper half–plane. A polynomial is of Pólya class if it has no zeros in the upper half–plane. Every entire function of Pólya class is a limit, uniformly on compact subsets of the upper half–plane, of polynomials which have no zeros in the upper half–plane.

The pervasive nature of the Pólya class is due to its stability under perturbations which are of bounded type [5]. Assume that the modulus of an analytic weight function $W(z)$ is a nondecreasing function of distance from the real axis on every vertical line in the upper half–plane. If $F(z)$ is a nontrivial entire function such that

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

are of bounded type in the upper half–plane, then an entire function $G(z)$ of Pólya class exists such that the identity

$$F^*(z)F(z) = G^*(z)G(z)$$

is satisfied, such that

$$F(z)/G(z)$$

and

$$F^*(z)/G(z)$$

are bounded by one in the upper half–plane, and such that at least one of the functions has zero mean type in the half–plane.

If $W(z)$ is an analytic weight function such that the modulus of $W(z)$ is a nondecreasing function of distance from the real axis on every vertical line in the upper half–plane, then the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$ contains a nonzero element. The space is a space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ and whose defining function $E(z)$ is of Pólya class.

If $W(z)$ is an analytic weight function such that the modulus of $W(z)$ is a nondecreasing function of distance from the real axis on every vertical line in the upper half–plane and if $S(z)$ is an entire function of Pólya class, then the set of entire functions $F(z)$ such that

$$S(z)F(z)$$

and

$$S(z)F^*(z)$$

belong to the space $\mathcal{F}(W)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) in the scalar product such that multiplication by $S(z)$ is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ such that $E(z)$ is of Pólya class if it contains a nonzero element. If $W(z)$ is an Euler weight function and if h is in the interval $[0, 1]$, a maximal dissipative relation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ of the space $\mathcal{F}(W)$ exist such that $F_n(z + ih)$ belongs to the space for every positive integer n, such that

$$S(z)G(z + ih)$$

is the limit in the metric topology of the space of the elements $F_n(z + ih)$, and such that

$$S(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E)$. The adjoint of the maximal dissipative relation in the space $\mathcal{H}(E)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z + ih)$ of the space $\mathcal{F}(W)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$S(z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$S(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E)$.

An Euler space of entire functions is a space $\mathcal{H}(E)$ such that a maximal dissipative transformation in the space is defined for h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space.

A construction of Euler spaces of entire functions is made from Euler weight functions.

Theorem 1. *The set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to a space $\mathcal{F}(W)$ is an Euler space of entire functions which is contained isometrically in the space $\mathcal{F}(W)$ if $W(z)$ is an Euler weight function.*

Proof of Theorem 1. If h is in the interval $[0,1]$, the transformation which takes $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space $\mathcal{H}(E)$ is dissipative since it is the restriction of a dissipative transformation in the space $\mathcal{F}(W)$. The maximal dissipative property of the transformation is proved by showing that every element of the space $\mathcal{H}(E)$ is a sum $F(z)+F(z+ih)$ with $F(z)$ and $F(z+ih)$ in the space. Since the transformation in the space $\mathcal{F}(W)$ is maximal dissipative, every element of the space $\mathcal{F}(W)$ is a sum $F(z)+F(z+ih)$ with $F(z)$ and $F(z+ih)$ in the space $\mathcal{F}(W)$. It will be shown that $F(z)$ and $F(z+ih)$ belong to the space $\mathcal{H}(E)$ if their sum belongs to the space. Since the elements of the space $\mathcal{H}(E)$ are entire functions, $F(z)$ and $F(z+ih)$ are entire functions.

Since the space $\mathcal{H}(E)$ satisfies the axiom (H3), the identity

$$F^*(z)+F^*(z-ih) = G(z)+G(z+ih)$$

holds for an element $G(z)$ of the space $\mathcal{F}(W)$ such that $G(z+ih)$ belongs to the space. The entire functions $F(z)$ and $G(z)$ are shown to belong to the space $\mathcal{H}(E)$ be showing that each side of the identity

$$F^*(z)-G(z+ih) = G(z)-F^*(z-ih)$$

vanishes.

The inequalities

$$|F(z)|^2 \le \|F\|^2 |W(z)|^2/[2\pi(iz^- - iz)]$$

and

$$|G(z)|^2 \le \|G\|^2 |W(z)|^2/[2\pi(iz^- - iz)]$$

apply when z is in the upper half–plane. Since the inequality

$$|G^*(z - ih)|^2 \leq \|G\|^2 |W^*(z - ih)|^2 / [2\pi(2h - iz^- + iz)]$$

follows when z is in the half–plane $iz^- - iz < 2h$, the inequality

$$|F(z) - G^*(z - ih)|^2$$
$$\leq \|F\|^2 |W(z)|^2 / [\pi(iz^- - iz)] + \|G\|^2 |W^*(z - ih)|^2 / [\pi(2h - iz^- + iz)]$$

applies when z is in the strip $0 < iz^- - iz < 2h$. Since z and $z^- + ih$ are closer to the real axis than $z + ih$ when z is in the strip, the inequality

$$|[F(z) - G^*(z - ih)]/W(z + ih)|^2$$
$$\leq \|F\|^2 / [\pi(iz^- - iz)] + \|G\|^2 / [\pi(2h - iz^- + iz)]$$

holds when z is in the strip. Since the modulus of the function

$$F(z) - G^*(z - ih)$$

is periodic of period h, the function

$$[F(z) - G^*(z - ih)]/W(z + ih),$$

whose modulus has a subharmonic logarithm, is bounded in the upper half–plane. Since the entire function

$$F(z) - G^*(z - ih)$$

vanishes at $\frac{1}{2} ih$ and since the modulus of the function is periodic of period ih, the function vanishes at

$$\tfrac{1}{2} ih + ihn$$

for every integer n. It follows that the function vanishes identically.

This completes the proof of the theorem.

An entire function $S(z)$ is said to be associated with a space $\mathcal{H}(E)$ if the entire function

$$[F(z)S(w) - S(z)F(w)]/(z - w)$$

belongs to the space for all complex numbers w whenever $F(z)$ belongs to the space.

The derived space of a space $\mathcal{H}(E)$ is a Hilbert space of entire functions which is constructed from a maximal dissipative transformation in the space. The construction is now made when the transformation is defined for some h in the interval $[0, 1]$ by entire functions $P(z)$ and $Q(z)$ which are associated with the space. The transformation is defined by taking $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t - w^-)] \rangle$$

for all complex numbers w. The derived space is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $G(z+ih)$. An element of the graph is a pair

$$(F(z), G(z+ih))$$

of elements $F(z)$ and $G(z+ih)$ of the space $\mathcal{H}(E)$ such that the adjoint takes $F(z)$ into $G(z+ih)$. The scalar product of elements

$$(F_1(z), G_1(z+ih))$$

and

$$(F_2(z), G_2(z+ih))$$

of the graph is defined as a sum

$$\langle F_1(t), G_2(t+ih)\rangle + \langle G_1(t+ih), F_2(t)\rangle$$

of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint is dissipative. Elements of the graph are considered equivalent if the scalar self–product of their difference is zero. The quotient space is a vector space which inherits a nondegenerate scalar product. A fundamental example of an element of the graph is determined by a complex number w since the adjoint takes the reproducing kernel function

$$\frac{E(z)E(w-\frac{1}{2}ih)^- - E^*(z)E(w^- + \frac{1}{2}ih)}{2\pi i(w^- + \frac{1}{2}ih - z)}$$

for function values at $w - \frac{1}{2}ih$ into the reproducing kernel function

$$\frac{Q(z)P(w^- - \frac{1}{2}ih) - P(z)Q(w^- - \frac{1}{2}ih)}{\pi(z + \frac{1}{2}ih - w^-)}$$

for the function value of a transformed function at $w + \frac{1}{2}ih$. If $F(z)$ is an element of the space $\mathcal{H}(E)$, the entire function $F^\sim(z)$ is defined by the scalar product

$$F^\sim(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle$$

in the space for all complex numbers w. An isometric transformation of the quotient space onto a dense subset of the derived space is defined by taking an element

$$(F(z), G(z+ih))$$

of the graph into the entire function

$$F^\sim(z + \tfrac{1}{2}ih) + G(z + \tfrac{1}{2}ih).$$

The transformation is defined on the quotient space since the value of the function at w is a scalar product with the element of the graph defined by w. The derived space contains the entire function

$$\frac{Q(z - \frac{1}{2}ih)P(w^- - \frac{1}{2}ih) - P(z - \frac{1}{2}ih)Q(w^- - \frac{1}{2}ih)}{\pi(z - w^-)} + \frac{Q^*(z + \frac{1}{2}ih)P(w - \frac{1}{2}ih)^- - P^*(z + \frac{1}{2}ih)Q(w - \frac{1}{2}ih)^-}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for all complex numbers w.

If $F(z)$ and $G(z + ih)$ are elements of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$ and if the functions $F(z + \frac{1}{2}ih)$ and $G(z + \frac{1}{2}ih)$ have a common nonreal zero w, then the adjoint of the maximal dissipative transformation in the space takes the element

$$F(z)(z - \tfrac{1}{2}ih - w^-)/(z - \tfrac{1}{2}ih - w)$$

of the space into the element

$$G(z + ih)(z + \tfrac{1}{2}ih - w^-)/(z + \tfrac{1}{2}ih - w)$$

of the space.

If $F(z)$ and $G(z + ih)$ are elements of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$ and if the sum

$$\langle F(t), G(t + ih)\rangle + \langle G(t + ih, F(t)\rangle$$

of scalar products in the space $\mathcal{H}(E)$ vanishes, then

$$F(z) + G(z)$$

vanishes identically. When a nonzero element $F(z)$ of the space $\mathcal{F}(W)$ exists such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $-F(z+ih)$, then

$$H(z)(z - w^-)/(z - w)$$

is an element of the derived space which has the same norm as $H(z)$ in the derived space whenever $H(z)$ is an element of the derived space which has a nonreal zero w. It follows that some nontrivial linear combination $S(z)$ of $P(z)$ and $Q(z)$ satisfies the identity

$$S(z - ih) = S^*(z).$$

The derived space is more structured when no nontrivial linear combination $S(z)$ of $P(z)$ and $Q(z)$ satisfies the identity. Continuous linear functionals $H(z)$ into $H_+(w)$ and

$H(z)$ into $H_-(w)$ on the derived space exist for every complex number w which take the element

$$H(z) = F(z + \tfrac{1}{2}ih) + G(z + \tfrac{1}{2}ih)$$

of the space into

$$H_+(w) = G(w + \tfrac{1}{2}ih)$$

and

$$H_-(w) = F(w + \tfrac{1}{2}ih)$$

whenever the adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$ takes $F(z)$ into $G(z+ih)$. The upper value $H_+(w)$ and the lower value $H_-(w)$ at w of an element $H(z)$ of the derived space are otherwise defined by continuity. Entire functions $H_+(z)$ and $H_-(z)$ are obtained which decompose an element

$$H(z) = H_+(z) + H_-(z)$$

of the derived space. When $H(z)$ is equal to the reproducing kernel function

$$\begin{aligned}[Q(z - \tfrac{1}{2}ih)P(w^- - \tfrac{1}{2}ih) - P(z - \tfrac{1}{2}ih)Q(w^- - \tfrac{1}{2}ih)]/[\pi(z - w^-)] \\ + [Q^*(z + \tfrac{1}{2}ih)P(w - \tfrac{1}{2}ih)^- - P^*(z + \tfrac{1}{2}ih)Q(w - \tfrac{1}{2}ih)^-]/[\pi(z - w^-)]\end{aligned}$$

for function values at w for some complex number w, then $H_+(z)$ is equal

$$[Q(z - \tfrac{1}{2}ih)P(w^- - \tfrac{1}{2}ih) - P(z - \tfrac{1}{2}ih)Q(w^- - \tfrac{1}{2}ih)]/[\pi(z - w^-)]$$

and $H_-(z)$ is equal to

$$[Q^*(z + \tfrac{1}{2}ih)P(w - \tfrac{1}{2}ih)^- - P^*(z + \tfrac{1}{2}ih)Q(w - \tfrac{1}{2}ih)^-]/[\pi(z - w^-)].$$

When $H_+(z + \frac{1}{2}ih)$ and $H_-(z - \frac{1}{2}ih)$ belong to the space $\mathcal{H}(E)$ for some element $H(z)$ of the derived space, then the adjoint of the maximal dissipative transformation takes $H_-(z - \frac{1}{2}ih)$ into $H_+(z + \frac{1}{2}ih)$.

A contractive transformation is defined from the derived space of the space $\mathcal{H}(E)$ into the derived space of the space $\mathcal{F}(W)$ which leaves upper components fixed. If $F(z)$ and $G(z+ih)$ are elements of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space take $F(z)$ into $G(z+ih)$, then elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{F}(W)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$, such that

$$n^{-1}F(z) + nG(z + ih) = n^{-1}F_n(z) + nG_n(z + ih)$$

and such that

$$nG(z + ih) - n^{-1}F(z) = nG_n(z + ih) - n^{-1}F_n(z) - H_n(z)$$

with $H_n(z)$ orthogonal to the space $\mathcal{H}(E)$ for every positive integer n. The identity

$$\langle F_n(t), G_n(t+ih)\rangle + \langle G_n(t+ih), F_n(t)\rangle$$
$$-\langle F(t), G(t+ih)\rangle - \langle G(t+ih), F(t)\rangle = \langle H_n(t), H_n(t)\rangle$$

is then satisfied. The element $G(z+ih)$ of the space $\mathcal{H}(E)$ is the limit in the weak topology of the space $\mathcal{F}(W)$ of the elements $G_n(z+ih)$. The element $F(z)$ of the space $\mathcal{H}(E)$ is the orthogonal projection in the space of the element $F_n(z+ih)$ for every positive integer n. The existence of the desired contractive transformation follows. If $G(z+ih)$ is an element of the space $\mathcal{H}(E)$ such that $G(z+\frac{1}{2}ih)$ is the upper component of an element of the derived space of the space $\mathcal{F}(W)$, then $G(z+\frac{1}{2}ih)$ is the upper component of an element of the derived space of the space $\mathcal{H}(E)$.

When some nontrivial linear combination $S(z)$ of $P(z)$ and $Q(z)$ satisfies the identity

$$S(z-ih) = S^*(z),$$

then the identity

$$[Q(z-ih)P(w^-) - P(z-ih)Q(w^-)]/[\pi(z-ih-w^-)]$$
$$= [Q^*(z)P(w-ih)^- - P^*(z)Q(w-ih)^-]/[\pi(z-ih-w^-)]$$

holds for all complex numbers z and w. The maximal dissipative transformation in the space $\mathcal{H}(E)$ is self–adjoint. Every element $H(z)$ of the derived space of the space $\mathcal{H}(E)$ admits a decomposition

$$H(z) = H_+(z) + H_-(z)$$

with

$$H_+(z) = \tfrac{1}{2}H(z)$$

and

$$H_-(z) = \tfrac{1}{2}H(z)$$

elements of the space such that

$$H_+(z) = H_-(z).$$

An inductive construction of Hilbert spaces of entire functions and of maximal dissipative transformations in the spaces is applied in the theory of Euler products in Hilbert spaces of entire functions. Assume that a maximal dissipative transformation in a space $\mathcal{H}(E)$ is defined for some h in the interval $[0,1]$ by entire functions $P(z)$ and $Q(z)$, which are associated with the space, by taking $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle$$

for all complex numbers w. If the reproducing kernel function

$$[E(z)E(\lambda)^- - E^*(z)E(\lambda^-)/[2\pi i(\lambda^- - z)]$$

for function values at λ is a constant multiple of an entire function $S(z)$, then a partially isometric transformation of the space $\mathcal{H}(E)$ onto a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is defined by taking $F(z)$ into

$$[F(z)S(\lambda) - S(z)F(\lambda)]/(z - \lambda).$$

The space is isometrically equal to a space $\mathcal{H}(E_\lambda)$ with

$$E_\lambda(z) = [E(z)S(\lambda) - S(z)E(\lambda)]/(z - \lambda)$$

if it contains a nonzero element. If $S(z)$ has value one at λ, multiplication by $z - \lambda$ is an isometric transformation of the space $\mathcal{H}(E_\lambda)$ into the space $\mathcal{H}(E)$. A maximal dissipative transformation in the space $\mathcal{H}(E_\lambda)$ is defined by entire functions $P_\lambda(z)$ and $Q_\lambda(z)$, which are associated with the space, by taking $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q_\lambda(t)P_\lambda(w^-) - P_\lambda(t)Q_\lambda(w^-)]/[\pi(t - w^-)]\rangle$$

for all complex numbers w, with the scalar product taken in the space $\mathcal{H}(E_\lambda)$. The functions $P_\lambda(z)$ and $Q_\lambda(z)$ are defined by the equations

$$\begin{aligned}(\lambda + ih - \lambda^-)P_\lambda(z) = \kappa P(\lambda)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/(z + ih - \lambda^-)\\ -\kappa P(\lambda^- - ih)[Q(z)P(\lambda) - P(z)Q(\lambda)]/(z - \lambda)\end{aligned}$$

and

$$\begin{aligned}(\lambda + ih - \lambda^-)Q_\lambda(z) = \kappa Q(\lambda)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/(z + ih - \lambda^-)\\ -\kappa Q(\lambda^- - ih)[Q(z)P(\lambda) - P(z)Q(\lambda)]/(z - \lambda)\end{aligned}$$

when $\lambda + ih - \lambda^-$ is nonzero and the equation

$$\kappa[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/(\lambda + ih - \lambda^-) = 1$$

has a solution κ. The functions are defined by continuity otherwise. The identities

$$(z - \lambda)P_\lambda(z) = P(z) - \kappa P(\lambda)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/(z + ih - \lambda^-)$$

and

$$(z - \lambda)Q_\lambda(z) = Q(z) - \kappa Q(\lambda)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/(z + ih - \lambda^-)$$

and the identities

$$(w^- + ih - \lambda^-)P_\lambda(w^-) = P(w^-) - \kappa P(\lambda^- - ih)[Q(w^-)P(\lambda) - P(w^-)Q(\lambda)]/(w^- - \lambda)$$

and

$$(w^- + ih - \lambda^-)Q_\lambda(w^-) = Q(w^-) - \kappa Q(\lambda^- - ih)[Q(w^-)P(\lambda) - P(w^-)Q(\lambda)]/(w^- - \lambda)$$

imply the identity

$$(z-\lambda)(w^- + ih - \lambda^-)[Q_\lambda(z)P_\lambda(w^-) - P_\lambda(z)Q_\lambda(w^-)]/[\pi(z-w^-)]$$
$$= [Q(z)P(w^-) - P(z)Q(w^-)]/[\pi(z-w^-)]$$
$$-\pi\kappa[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/[\pi(z+ih-\lambda^-)]$$
$$\times[Q(w^-)P(\lambda) - P(w^-)Q(\lambda)]/[\pi(w^- - \lambda)].$$

Multiplication by $z-\lambda$ is an isometric transformation of the space $\mathcal{H}(E_\lambda)$ into the space $\mathcal{H}(E)$. The maximal dissipative transformation in the space $\mathcal{H}(E_\lambda)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E)$ exist such that the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$(z-\lambda)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$(z-\lambda)F(z)$$

is the limit is the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_\lambda)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_\lambda)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$(z-\lambda)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$(z-\lambda)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_\lambda)$. Since $(z-\lambda)G(z+ih)$ belongs to the range of the adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$, the elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E)$ can be chosen independently of n.

The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$ takes the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w into the function

$$[Q(z)P(w^- - ih) - P(z)Q(w^- - ih)]/[\pi(z+ih-w^-)]$$

for all complex numbers w. The function

$$[Q(z)P(w^- - ih) - P(z)Q(w^- - ih)]/[\pi(z + ih - w^-)]$$
$$-\pi\kappa[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/[\pi(z + ih - \lambda^-)]$$
$$\times[Q(w^- - ih)P(\lambda) - P(w^- - ih)Q(\lambda)]/[\pi(w^- - ih - \lambda)]$$

belongs to the range of the adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$ and is equal to

$$(z - \lambda)(w^- - \lambda^-)[Q_\lambda(z)P_\lambda(w^- - ih) - P_\lambda(z)Q_\lambda(w^- - ih)]/[\pi(z + ih - w^-)].$$

The function

$$(w^- - \lambda^-)[Q_\lambda(z)P_\lambda(w^- - ih) - P_\lambda(z)Q_\lambda(w^- - ih)]/[\pi(z + ih - w^-)]$$

belongs to the range of the adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_\lambda)$ and is obtained from the element

$$(w^- - \lambda^-)[E_\lambda(z)E_\lambda(w)^- - E_\lambda^*(z)E_\lambda(w^-)]/[2\pi i(w^- - z)]$$

whose product by $z - \lambda$ is the orthogonal projection of

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

in the image of the space $\mathcal{H}(E_\lambda)$.

If $F(z)$ and $G(z+ih)$ are elements of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$, then the adjoint of the maximal dissipative transformation in the space takes the element

$$F(z)[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]$$
$$-G(\lambda + ih)[E(z)E(\lambda)^- - E^*(z)E(\lambda^-)]/[2\pi i(\lambda^- - z)]$$

of the space into the element

$$G(z + ih)[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]$$
$$-G(\lambda + ih)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/[\pi(z + ih - \lambda^-)]$$

of the space, which belongs to the image of the space $\mathcal{H}(E_\lambda)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_\lambda)$ takes the product of

$$[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]$$

and the element

$$\{F(z)[E(\lambda)E(\lambda)^- - E^*(\lambda)E(\lambda^-)]/[2\pi i(\lambda^- - \lambda)]$$
$$-F(\lambda)[E(z)E(\lambda)^- - E^*(z)E(\lambda^-)]/[2\pi i(\lambda^- - z)]\}/(z - \lambda)$$

of the space into the product of

$$[E(\lambda)E(\lambda)^- - E^*(\lambda)E(\lambda^-)]/[2\pi i(\lambda^- - \lambda)]$$

and the element

$$\{G(z+ih)[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]$$
$$-G(\lambda+ih)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/[\pi(z+ih-\lambda^-)]\}/(z-\lambda)$$

of the space.

If $F(z)$ and $G(z+ih)$ are elements of the space $\mathcal{H}(E_\lambda)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z+ih)$ and if $F_n(z)$ and $G_n(z+ih)$ are elements of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$(z-\lambda)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$(z-\lambda)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_\lambda)$, then the scalar product

$$\langle F(t), G(t+ih)\rangle$$

in the space $\mathcal{H}(E_\lambda)$ is the limit of the scalar products

$$\langle F_n(t), G_n(t+ih)\rangle$$

in the space $\mathcal{H}(E)$.

A partially isometric transformation of the derived space of the space $\mathcal{H}(E)$ onto the derived space of the space $\mathcal{H}(E_\lambda)$ is implied by the relationship between adjoints of maximal dissipative transformations in the spaces. A dense set of elements of the derived space of the space $\mathcal{H}(E)$ are obtained from elements $F(z)$ and $G(z+ih)$ of the space $\mathcal{H}(E)$ such that the adjoint of the maximal dissipative transformation in the space takes $F(z)$ into $G(z+ih)$. The corresponding element of the derived space has $G(z+\frac{1}{2}ih)$ as upper component and has scalar self–product equal to the sum

$$\langle F(t), G(t+ih)\rangle + \langle G(t+ih), F(t)\rangle$$

of scalar self–products in the space $\mathcal{H}(E)$. A corresponding element of the derived space of the space $\mathcal{H}(E_\lambda)$ is constructed by elements $U(z)$ and $V(z+ih)$ of the space $\mathcal{H}(E_\lambda)$ such that the adjoint of the maximal dissipative transformation in the space takes $U(z)$

into $V(z+ih)$. The corresponding element of the derived space has $V(z+\frac{1}{2}ih)$ as upper component and has scalar self–product equal to the sum

$$\langle U(t), V(t+ih)\rangle + \langle V(t+ih), U(t)\rangle$$

of scalar products in the space $\mathcal{H}(E_\lambda)$. The identities

$$\begin{aligned}&(z-\lambda)V(z+ih)[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]\\ &= G(z+ih)[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]\\ &- G(\lambda+ih)[Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]/[\pi(z + ih + \lambda^-)]\end{aligned}$$

and

$$\begin{aligned}&(z-\lambda)U(z)[E(\lambda)E(\lambda)^- - E^*(\lambda)E(\lambda^-)]/[2\pi i(\lambda^- - \lambda)]\\ &= F(z)[E(\lambda)E(\lambda)^- - E^*(\lambda)E(\lambda^-)]/[2\pi i(\lambda^- - \lambda)]\\ &- F(\lambda)[E(z)E(\lambda)^- - E^*(z)E(\lambda^-)]/[2\pi i(\lambda^- - z)]\end{aligned}$$

define the elements of the space $\mathcal{H}(E_\lambda)$. Since multiplication by $z-\lambda$ is an isometric transformation of the space $\mathcal{H}(E_\lambda)$ into the space $\mathcal{H}(E)$, the scalar product

$$\langle U(t), V(t+ih)\rangle[Q(\lambda)P(\lambda^- - ih) - P(\lambda)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]$$

in the space $\mathcal{H}(E_\lambda)$ is equal to the difference

$$\begin{aligned}&\langle F(t), G(t+ih)\rangle[Q(t)P(\lambda^- - ih) - P(t)Q(\lambda^- - ih)]/[\pi(\lambda + ih - \lambda^-)]\\ &- G(\lambda+ih)^-\langle F(t), [Q(t)P(\lambda^- - ih) - P(t)Q(\lambda^- - ih)]/[\pi(t + ih - \lambda^-)]\rangle\end{aligned}$$

of scalar products in the space $\mathcal{H}(E_\lambda)$. The sum

$$\langle U(t), V(t+ih)\rangle + \langle V(t+ih), U(t)\rangle$$

of scalar products in the space $\mathcal{H}(E_\lambda)$ is equal to the sum

$$\langle F(t), G(t+ih)\rangle + \langle G(t+ih), F(t)\rangle$$

of scalar products in the space $\mathcal{H}(E_\lambda)$ when $\lambda + ih$ is a zero of $G(z)$. The sum

$$\langle U(t), V(t+ih)\rangle + \langle V(t+ih), U(t)\rangle$$

of scalar products in the space $\mathcal{H}(E_\lambda)$ is equal to zero when

$$F(z) = [E(z)E(\lambda)^- - E^*(z)E(\lambda^-)]c/[2\pi i(\lambda^- - z)]$$

and

$$G(z) = [Q(z)P(\lambda^- - ih) - P(z)Q(\lambda^- - ih)]G/[\pi(z + ih - \lambda^-)]$$

for a complex number c. The adjoint of the partially isometric transformation of the derived space of the space $\mathcal{H}(E)$ onto the derived space of the space $\mathcal{H}(E_\lambda)$ is an isometric transformation of the derived space of the space $\mathcal{H}(E_\lambda)$ into the derived space of the space $\mathcal{H}(E)$ which satisfies the identity

$$G_+(z) = (z - \tfrac{1}{2}ih - \lambda)F_+(z)$$

for upper components when the transformation takes $F(z)$ into $G(z)$. If $G(z + ih)$ is an element of the space $\mathcal{H}(E_\lambda)$ such that

$$(z - \tfrac{1}{2}ih - \lambda)Gt(z + \tfrac{1}{2}ih)$$

is the upper component of an element of the derived space of the space $\mathcal{H}(E)$, $G(z + \frac{1}{2}ih)$ is the upper component of an element of the derived space of the space $\mathcal{H}(E_\lambda)$.

An inductive construction of maximal dissipative transformations is made in subspaces. Assume that a maximal dissipative transformation is defined in a space $\mathcal{H}(E)$ for some h in the interval $[0, 1]$ by entire functions $P(z)$ and $Q(z)$, which are associated with the space, by taking $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t - w^-)]\rangle$$

for all complex numbers w. Assume that a polynomial $S_r(z)$ has no zeros in the upper half–plane. If a nonzero entire function $F(z)$ exists such that $S_r(z)F(z)$ belongs to the space $\mathcal{H}(E)$, then the set of entire functions $F(z)$ such that $S_r(z)F(z)$ belongs to the space $\mathcal{H}(E)$ is a space $\mathcal{H}(E_r)$ such that multiplication by $S_r(z)$ is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{H}(E)$. A maximal dissipative transformation in the space $\mathcal{H}(E_r)$ is defined by taking $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z + ih)$ of the space $\mathcal{H}(E)$ exist such that the maximal dissipative transformation in the space $\mathcal{H}(E)$ takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$S_r(z)Q(z + ih)$$

is the limit of the elements $G_n(z+ih)$ in the metric topology of the space $\mathcal{H}(E)$, and such that

$$S_r(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$. Entire functions $P_r(z)$ and $Q_r(z)$, which are associated with the space $\mathcal{H}(E_r)$, exist such that the maximal dissipative transformation in the space takes $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q_r(t)P_r(w^-) - P_r(t)Q_r(w^-)]/[\pi(t - w^-)]\rangle$$

for all complex numbers w with the scalar product taken in the space $\mathcal{H}(E_r)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$S_r(z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$S_r(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$. The scalar product

$$\langle F(t), G(t+ih)\rangle$$

in the space $\mathcal{H}(E_r)$ is the limit of the scalar products

$$\langle F_n(t), G_n(t+ih)\rangle$$

in the space $\mathcal{H}(E)$. A partially isometric transformation of the derived space of the space $\mathcal{H}(E)$ onto the derived space of the space $\mathcal{H}(E_r)$ is defined as the composition of the partially isometric transformations of the derived space of the space $\mathcal{H}(E_{n-1})$ onto the derived space of the space $\mathcal{H}(E_n)$ for every $n = 1, \dots, r$ with the space $\mathcal{H}(E_0)$ isometrically equal to the space $\mathcal{H}(E)$. The adjoint is an isometric transformation of the derived space of the space $\mathcal{H}(E_r)$ into the derived space of the space $\mathcal{H}(E)$ which satisfies the identity

$$G_+(z) = S_r(z - \tfrac{1}{2}ih)F_+(z)$$

for upper components when the transformation takes $F(z)$ into $G(z)$. The image of the derived space of the space $\mathcal{H}(E_n)$ in the derived space of the space $\mathcal{H}(E)$ is contained in the image of the derived space of the space $\mathcal{H}(E_{n-1})$ in the derived space of the space $\mathcal{H}(E)$ for every $n = 1, \dots, r$. If $G(z+ih)$ is an element of the space $\mathcal{H}(E_r)$ such that

$$S_r(z - \tfrac{1}{2}ih)G(z + \tfrac{1}{2}ih)$$

is the upper component of an element of the derived space of the space $\mathcal{H}(E)$, $G(z + \frac{1}{2}ih)$ is the upper component of an element of the derived space of the space $\mathcal{H}(E_r)$.

A Riemann space of entire functions is a Hilbert space $\mathcal{H}(E)$ whose defining function $E(z)$ is constructed from an Euler weight function $W(z)$ using an Euler product. The partial products in an Euler product are entire functions $S_r(z)$ of Pólya class, which are of bounded type and of mean type at most τ_r in the upper half–plane, such that the function

$$S_r^*(z)/S_r(z)$$

is of zero mean type in the upper half–plane. The ratio

$$S_{r+1}(z)/S_r(z)$$

is assumed to be an entire function of Pólya class for every positive integer r. The inequality

$$\tau_r \leq \tau_{r+1}$$

is assumed for every positive integer r. The analytic weight function

$$W(z) = \lim \exp(i\tau_r z) S_r(z) E(z)$$

is assumed to be obtained as a limit uniformly on compact subsets of the upper half–plane. It is also assumed that nontrivial entire functions $F(z)$ exist such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$.

An Euler product for a Riemann space of entire functions can be rearranged so that the partial products $S_r(z)$ are polynomials. The properties of Riemann spaces of entire functions are derived from the properties of Euler spaces of entire functions when the partial products are polynomials.

An Euler space of entire functions $\mathcal{H}(E'_r)$ is defined for every positive integer r as the set of entire functions $F(z)$ such that

$$\exp(i\tau_r z) F(z)$$

and

$$\exp(i\tau_r z) F^*(z)$$

belong to the Euler space $\mathcal{F}(W)$. The scalar product is defined in the space $\mathcal{H}(E)$ so that multiplication by $S_r(z)$ is an isometric transformation of the space into the Euler space $\mathcal{H}(E'_r)$. A scalar product is defined in the space of entire functions so that multiplication by

$$\exp(i\tau_r z)$$

is an isometric transformation of the space $\mathcal{H}(E'_r)$ into the space $\mathcal{F}(W)$. The image of the space $\mathcal{H}(E'_r)$ in the space $\mathcal{F}(W)$ is contained in the image of the space $\mathcal{H}(E'_{r+1})$ in the space $\mathcal{F}(W)$ for every positive integer r. The union of the images of the spaces $\mathcal{H}(E'_r)$ in the space $\mathcal{F}(W)$ is dense in the space $\mathcal{F}(W)$ when the numbers τ_r are unbounded.

The Euler spaces of entire functions can be assumed to have infinite dimension when the numbers τ_r are unbounded. A space $\mathcal{H}(E_r)$ is defined for every positive integer r as the set of entire functions $F(z)$ such that

$$S_r(z) F(z)$$

and

$$S_r(z) F^*(z)$$

belong to the Euler space $\mathcal{H}(E'_r)$. The scalar product in the space $\mathcal{H}(E_r)$ is defined so that multiplication by $S_r(z)$ is an isometric transformation of the space into the Euler space $\mathcal{H}(E'_r)$. An entire function $F(z)$ belongs to the space $\mathcal{H}(E_r)$ if, and only if,

$$\exp(i\tau_r z)S_r(z)F(z)$$

and

$$\exp(i\tau_r z)S_r(z)F^*(z)$$

belong to the space $\mathcal{F}(W)$. Multiplication by

$$\exp(i\tau_r z)S_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$.

Multiplication by

$$W(z)/E(z) = \lim \exp(i\tau_r z)S_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$. An element $F(z)$ of the space $\mathcal{F}(W)$ belongs to the image of the space $\mathcal{H}(E)$ if, and only if, it is a limit in the metric topology of the space $\mathcal{F}(W)$ of elements $F_r(z)$ in the image of the spaces $\mathcal{H}(E_r)$. The image in the space $\mathcal{F}(W)$ of the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$ is the limit in the metric topology of the space $\mathcal{F}(W)$ of images in the space of reproducing kernel functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the spaces $\mathcal{H}(E_r)$. The function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z is the limit of the functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$

of z uniformly on compact subsets of the complex plane for every complex number w. The defining functions $E_r(z)$ of the spaces $\mathcal{H}(E_r)$ are chosen so that the defining function

$$E(z) = \lim E_r(z)$$

of the space $\mathcal{H}(E)$ is obtained uniformly on compact subset of the complex plane.

A maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ are elements of the space. A maximal dissipative transformation in the Euler space $\mathcal{H}(E'_r)$ is defined by taking $F(z)$ into $G(z+ih)$ whenever

$F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ of the space $\mathcal{F}(W)$ exist such that $F_n(z+ih)$ belongs to the space for every positive integer n, such that

$$\exp(i\tau_r z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $F_n(z+ih)$, and such that

$$\exp(i\tau_r z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r')$. The maximal dissipative transformation in the space $\mathcal{H}(E_r')$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space such that

$$G(z) = \exp(-\tau_r h)F(z).$$

The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_r')$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{F}(W)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$\exp(i\tau_r z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$\exp(i\tau_r z)F(z)$$

is the limit in the same topology of the orthogonal projection of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r')$.

A maximal dissipative transformation in the space $\mathcal{H}(E_r)$ is defined by taking $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E_r')$ exist such that the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$S_r(z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$S_r(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z+ih)$ of the space $\mathcal{H}(E_r')$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z+ih)$ for every positive integer n, such that

$$S_r(z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $G_n(z+ih)$, and such that

$$S_r(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$.

A maximal dissipative relation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+ih)$ whenever $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ of the space $\mathcal{F}(W)$ exist such that $F_n(z+ih)$ belongs to the space for every positive integer n, such that

$$W(z)G(z+ih)/E(z)$$

is the limit in the metric topology of the space of the elements $F_n(z+ih)$, and such that

$$W(z)F(z)/E(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E)$. The maximal dissipative transformation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_n(z)$ of the space $\mathcal{F}(W)$ exist such that $F_n(z+ih)$ belongs to the space for every positive integer n, such that

$$\exp(i\tau_r z)S_r(z)G(z+ih)$$

is the limit in the metric topology of the space of the elements $F_n(z+ih)$, and such that

$$\exp(i\tau_r z)S_r(z)F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$. The maximal dissipative relation in the space $\mathcal{H}(E)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_r(z)$ and $G_r(z+ih)$ of the space $\mathcal{H}(E_r)$ exist such that the maximal dissipative transformation in the space takes $F_r(z)$ into $G_r(z+ih)$ for every positive integer r, such that

$$W(z)G(z+ih)/E(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements

$$\exp(i\tau_r z)S_r(z)G_r(z+ih),$$

and such that

$$W(z)F(z)/E(z)$$

is the limit in the same topology of the elements

$$\exp(i\tau_r z)S_r(z)F_r(z).$$

Entire functions $P_r(z)$ and $Q_r(z)$, which are associated with the space $\mathcal{H}(E_r)$, exist such that the maximal dissipative transformation in the space takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q_r(t)P_r(w^-) - P_r(t)Q_r(w)]/[\pi(t - w^-)]\rangle$$

for all complex numbers w. The derived space of the space $\mathcal{H}(E_r)$ is a Hilbert space of entire functions which contains the function

$$[Q_r(z - \tfrac{1}{2}ih)P_r(w^- - \tfrac{1}{2}ih) - P_r(z - \tfrac{1}{2}ih)Q_r(w^- - \tfrac{1}{2}ih)]/[\pi(z - w^-)]$$
$$+ [Q_r^*(z + \tfrac{1}{2}ih)P_r(w - \tfrac{1}{2}ih)^- - P_r^*(z + \tfrac{1}{2}ih)Q_r(w - \tfrac{1}{2}ih)^-]/[\pi(z - w^-)]$$

of z as reproducing kernel function for function values at w for all complex numbers w.

The adjoint of the maximal dissipative relation in the space $\mathcal{H}(E)$ takes $F(z)$ into $G(z+ih)$ if, and only if, $F(z)$ and $G(z+ih)$ are elements of the space for which elements $F_r(z)$ and $G_r(z+ih)$ of the space $\mathcal{H}(E_r)$ exist such that the adjoint of the maximal dissipative transformation in the space takes $F_r(z)$ into $G_r(z+ih)$ for every positive integer r, such that

$$W(z)G(z+ih)/E(z)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the elements

$$\exp(i\tau_r z)S_r(z)G_r(z+ih),$$

and such that

$$W(z)F(z)/E(z)$$

is the limit in the same topology of the elements

$$\exp(i\tau_r z)S_r(z)F_r(z).$$

The Riemann hypothesis for Hilbert spaces of entire functions denies the existence of paired zeros in the defining function of a Riemann space of entire functions.

Theorem 2. *The defining function*

$$E(z) = \lim \exp(-i\tau_r z)W(z)/S_r(z)$$

of a Riemann space $\mathcal{H}(E)$ admits no distinct zeros $w - ih$ and w^- when h is in the interval $(0,1]$ and no double zero $w - ih$ equal to w^- when h is in the interval $(0,1)$.

Proof of Theorem 2. If a zero w^- of $E(z)$ satisfies the inequality

$$iw^- - iw \geq h,$$

then multiplication by

$$W(z)/E(z)$$

is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$ which takes the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$ into the product of

$$[E(w)/W(w)]^-$$

and the reproducing kernel function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{F}(W)$.

The sequence of numbers τ_r has a finite limit τ_∞ if it is bounded, in which case an entire function $S_\infty(z)$ of Pólya class exists which is a limit of the entire functions $S_r(z)$ of Pólya class. Since the identity

$$W(z) = \exp(i\tau_\infty z)S_\infty(z)E(z)$$

is satisfied and since the Euler weight function $W(z)$ has no zeros in the half–plane

$$-1 < iz^- - iz,$$

$w - ih$ is not a zero of $E(z)$ when the sequence of numbers τ_r is bounded. The same conclusion will be obtained when the sequence of numbers τ_r is unbounded.

Multiplication by

$$\exp(i\tau_r z)$$

is an isometric transformation of the Euler space $\mathcal{H}(E'_r)$ into the space $\mathcal{F}(W)$ for every positive integer r. Elements of the space $\mathcal{F}(W)$ which belong to the image of the Euler space are the reproducing kernel function $F_r(z)$ for function values at $w - ih$ and the reproducing kernel function $G_r(z + ih)$ for function values at w in the image of the Euler space. The maximal dissipative transformation in the image of the Euler space takes $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ belong to the image. The adjoint of the maximal dissipative transformation in the image of the Euler space takes $F_r(z)$ into $G_r(z + ih)$. A contractive transformation of the derived space of the Euler space $\mathcal{H}(E'_r)$ into the derived space of the space $\mathcal{F}(W)$ exists which satisfies the identity

$$G_+(z) = \exp(\tfrac{1}{2}\tau_r h + i\tau_r z)F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$. The image of the derived space of the Euler space $\mathcal{H}(E'_r)$ in the derived space of the space $\mathcal{F}(W)$ is a Hilbert space in the scalar product for which the transformation is isometric. The reproducing kernel function for function values at $w - \frac{1}{2}ih$ in the image space has

$$G_r(z + \tfrac{1}{2}ih)$$

as its upper component. The reproducing kernel function

$$[W(z - \tfrac{1}{2}ih)W(w)^- + W(z + \tfrac{1}{2}ih)W(w - ih)^-]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

for function values at $w - \frac{1}{2}ih$ in the derived space of the space $\mathcal{F}(W)$ is the limit in the metric topology of the space of the reproducing kernel functions for function values at $w - \frac{1}{2}ih$ in the image spaces of the derived spaces of the Euler spaces $\mathcal{H}(E_r')$.

Multiplication by

$$\exp(i\tau_r z)S_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$. The element $U_r(z)$ of the space $\mathcal{H}(E_r)$ such that

$$\exp(i\tau_r z)S_r(z)U_r(z)$$

is the orthogonal projection of $F_r(z)$ in the image of the space $\mathcal{H}(E_r)$ of the product of

$$\exp(\tau_r h - i\tau_r w^-)S_r(w - ih)^-$$

and the reproducing kernel function

$$[E_r(z)E_r(w - ih)^- - E_r^*(z)E_r(w^- + ih)]/[2\pi i(w^- + ih - z)]$$

for function values at $w - ih$ in the space $\mathcal{H}(E_r)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_r)$ takes $U_r(z)$ into an element $V_r(z + ih)$ of the space $\mathcal{H}(E_r)$ which is the product of

$$\exp(\tau_r h - i\tau_r w^-)S_r(w - ih)^-$$

and the reproducing kernel function

$$[Q_r(z)P_r(w^-) - P_r(z)Q_r(w^-)]/[\pi(z - w^-)]$$

for the values of transformed functions at w in the spaces $\mathcal{H}(E_r)$. The element $U_r^\sim(z)$ of the space $\mathcal{H}(E_r)$ is the product of

$$\exp(\tau_r h - i\tau_r w^-)S_r(w - ih)^-$$

and the element

$$[Q_r^*(z)P_r(w - ih)^- - P_r^*(z)Q_r(w - ih)^-]/[\pi(z + ih - w^-)]$$

A contractive transformation of the derived space of the space $\mathcal{H}(E_r)$ into the derived space of the space $\mathcal{F}(W)$ exists which satisfies the identity

$$G_+(z) = \exp(\tfrac{1}{2}\tau_r h + i\tau_r z)S_r(z - \tfrac{1}{2}ih)F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$. The transformation maps the derived space of the space $\mathcal{H}(E_r)$ isometrically into the Hilbert space which is the image in the derived space of the space $\mathcal{F}(W)$ of the derived space of the Euler space $\mathcal{H}(E'_r)$. The element

$$U_r^\sim(z + \tfrac{1}{2}ih) + V_r(z + \tfrac{1}{2}ih)$$

of the derived space of the space $\mathcal{H}(E_r)$ is the product of

$$\exp(\tau_r h - i\tau_r w^-) S_r(w - ih)^-$$

and the reproducing kernel function

$$[Q_r(z - \tfrac{1}{2}ih)P_r(w^-) - P_r(z - \tfrac{1}{2}ih)Q_r(w^-)]/[\pi(z - \tfrac{1}{2}ih - w^-)]$$
$$+ [Q_r^*(z + \tfrac{1}{2}ih)P_r(w - ih)^- - P_r^*(z + \tfrac{1}{2}ih)Q_r(w - ih)^-]/[\pi(z - \tfrac{1}{2}ih - w^-)]$$

for function values at $w - \frac{1}{2}ih$ in the derived space of the space $\mathcal{H}(E_r)$. The isometric transformation of the derived space of the space $\mathcal{H}(E_r)$ into the image in the derived space of the space $\mathcal{F}(W)$ of the derived space of the Euler space $\mathcal{H}(E'_r)$ takes the element

$$U_r^\sim(z + \tfrac{1}{2}ih) + V_r(z + \tfrac{1}{2}ih)$$

of the derived space of the space $\mathcal{H}(E_r)$ into the orthogonal projection of

$$F_r(z + \tfrac{1}{2}ih) + G_r(z + \tfrac{1}{2}ih)$$

in the image of the derived space of the space $\mathcal{H}(E_r)$.

The reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$ is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the spaces $\mathcal{H}(E_r)$. The product of

$$W(z)/E(z)$$

and the reproducing kernel function for function values at w in the space $\mathcal{H}(E)$ is the limit in the metric topology of the space $\mathcal{F}(W)$ of the products of

$$\exp(i\tau_r z)S_r(z)$$

and the reproducing kernel function for function values at w in the spaces $\mathcal{H}(E_r)$.

If $w - ih$ is not a zero of $E(z)$, the factors

$$\exp(\tau_r h - i\tau_r w^-)S_r(w - ih)^-$$

converge to

$$[W(w - ih)/E(w - ih)]^-.$$

The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$ then takes the reproducing kernel function

$$[E(z)E(w - ih)^- - E^*(z)E(w^- + ih)]/[2\pi i(w^- + ih - z)]$$

for function values at $w - ih$ in the space $\mathcal{H}(E)$ into the product of

$$[E(w - ih)/W(w - ih)]^-[W(w)/E(w)]^-$$

and the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$. The conclusion which is obtained when $w - ih$ is not a zero of $E(z)$ holds by continuity when $w - ih$ is a zero. The computation of adjoints denies $w - ih$ as a zero of $E(z)$ distinct from w^- or as a double zero equal to w^-.

This completes the proof of the theorem.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is said to be symmetric about the origin if an isometric transformation of the space into itself is defined by taking $F(z)$ into $F(-z)$. A sufficient condition for a space $\mathcal{H}(E)$ to be symmetric about the origin is that the defining function $E(z)$ of the space satisfies the symmetry condition

$$E(-z) = E^*(z).$$

If a Hilbert space of entire functions, which satisfies the axioms (H1), (H2), and (H3) and contains a nonzero element, is symmetric about the origin, then the space is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ satisfies the symmetry condition.

A variant of the Riemann hypothesis for Hilbert spaces of entire functions applies when the Euler weight function $W(z)$ satisfies the symmetry condition

$$W(-z) = W^*(z)$$

and the partial products $S_r(z)$ in the Euler product satisfy the symmetry condition

$$S_r(-z) = S_r^*(z).$$

It is again assumed that nontrivial entire function $F(z)$ exist such that $F(z)$ and $F^*(z)$ belong to the weighted Hardy space $\mathcal{F}(W)$. The partial products $S_r(z)$ in the Euler product are again assumed to be entire functions of Pólya class, which are of bounded type and of mean type at most τ_r in the upper half–plane, such that the function

$$S_r^*(z)/S_r(z)$$

is of zero mean type in the upper half–plane. The ratio

$$S_{r+1}(z)/S_r(z)$$

is again assumed to be an entire function of Pólya class for every positive integer r and the inequality

$$\tau_r \le \tau_{r+1}$$

is again assumed for every positive integer r. But the entire function constructed from the Euler product now contains an additional factor of z/i to destroy a singularity which would otherwise occur.

Theorem 3. *The defining function $E'(z)$ of a space $\mathcal{H}(E')$, which is obtained as a limit*

$$iE'(z) = \lim \exp(-i\tau_r z) z W(z)/S_r(z)$$

uniformly on compact subsets of the upper half–plane, admits no distinct zeros $w - ih$ and w^- when h is in the interval $(0, 1]$ and no double zero $w - ih$ equal to w^- when h is in the interval $(0, 1)$.

Proof of Theorem 3. When the sequence of numbers τ_r is bounded, it has a finite limit τ_∞, an entire function $S_\infty(z)$ of Pólya class exists which is a limit of the entire functions $S_r(z)$ of Pólya class, and the identity

$$zW(z) = \exp(i\tau_\infty z) S_\infty(z) iE'(z)$$

is satisfied. Since the Euler weight function $W(z)$ has no zeros in the half–plane

$$-1 < iz^- - iz,$$

the function $E'(z)$ has no zeros in the half–plane other than a possible zero at the origin.

When the sequence of numbers τ_r is unbounded, the Euler product is rearranged so that the partial products $S_r(z)$ are polynomials which satisfy the symmetry condition

$$S_r(-z) = S_r^*(z).$$

The Euler space $\mathcal{H}(E_r')$ is defined for every positive integer r as the set of entire functions $F(z)$ such that

$$\exp(i\tau_r z) F(z)$$

and

$$\exp(i\tau_r z) F^*(z)$$

belong to the space $\mathcal{F}(W)$. The scalar product in the Euler space is defined so that multiplication by

$$\exp(i\tau_r z)$$

is an isometric transformation of the space into the space $\mathcal{F}(W)$. The Euler spaces $\mathcal{H}(E'_r)$ are symmetric about the origin and can be assumed to have infinite dimension. The image of the Euler space $\mathcal{H}(E'_r)$ in the space $\mathcal{F}(W)$ is contained in the image of the Euler space $\mathcal{H}(E'_{r+1})$ in the space $\mathcal{F}(W)$ for every positive integer r. The union of the images of the Euler spaces $\mathcal{H}(E'_r)$ is dense in the space $\mathcal{F}(W)$.

A space $\mathcal{H}(E_r)$ is defined for every positive integer r as the set of entire functions $F(z)$ such that

$$S_r(z)F(z)$$

and

$$S_r(z)F^*(z)$$

belong to the Euler space $\mathcal{H}(E'_r)$. The scalar product is defined in the space $\mathcal{H}(E_r)$ so that multiplication by $S_r(z)$ is an isometric transformation of the space into the Euler space $\mathcal{H}(E'_r)$. The space $\mathcal{H}(E_r)$ is symmetric about the origin since the Euler space is symmetric about the origin and since the entire function $S_r(z)$ satisfies the symmetry condition

$$S_r(-z) = S_r^*(z).$$

The defining function $E_r(z)$ of the space $\mathcal{H}(E_r)$ is chosen to satisfy the symmetry condition

$$E_r(-z) = E_r^*(z)$$

and is normalized to have value one at the origin. The space $\mathcal{H}(E_r)$ is the set of entire functions $F(z)$ such that

$$\exp(i\tau_r z)S_r(z)F(z)$$

and

$$\exp(i\tau_r z)S_r(z)F^*(z)$$

belong to the space $\mathcal{F}(W)$. Multiplication by

$$\exp(i\tau_r z)S_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$.

The space $\mathcal{H}(E')$ is symmetric about the origin since the Euler weight function $W(z)$ satisfies the symmetry condition

$$W(-z) = W^*(z)$$

and since the partial products $S_r(z)$ in the Euler product satisfy the symmetry condition

$$S_r(-z) = S_r^*(z).$$

When every element of the space vanishes at the origin, division by z is an isometric transformation of the space $\mathcal{H}(E'_r)$ onto a Riemann space $\mathcal{H}(E)$ of entire functions whose defining function is

$$E(z) = E'(z)/z.$$

Since $E(z)$ admits no pair of distinct zeros w^- and $w - ih$ with h in the interval $[0, 1)$, $E'(z)$ admits no pair of distinct zeros w^- and $w - ih$ with h in the interval $[0, 1)$. When some element of the space $\mathcal{H}(E')$ has a nonzero value at the origin, division by z is an isometric transformation of the set of elements of the space $\mathcal{H}(E')$ which vanish at the origin onto a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). If the space constructed contains no nonzero element, the function $E'(z)$ admits only one nonreal zero. The function admits no pair of distinct zeros w^- and $w - ih$ with h in the interval $[0, 1)$. If the space constructed contains a nonzero element, it contains an element which has a nonzero value at the origin. The space is isometrically equal to a space $\mathcal{H}(E)$ which is symmetric about the origin. The defining function $E(z)$ of the space is chosen to satisfy the symmetry condition

$$E(-z) = E^*(z)$$

and is normalized to have value one at the origin.

An entire function $F(z)$ belongs to the space $\mathcal{H}(E)$ if, and only if,

$$zW(z)F(z)/E'(z)$$

and

$$zW(z)F^*(z)/E'(z)$$

belong to the space $\mathcal{F}(W)$. The image in the space $\mathcal{F}(W)$ of an entire function $F(z)$ which belongs to the space $\mathcal{H}(E)$ is the limit in the metric topology of the space $\mathcal{F}(W)$ of the images in the space $\mathcal{F}(W)$ of entire functions $F_r(z)$ which belong to the spaces $\mathcal{H}(E_r)$. The image in the space $\mathcal{F}(W)$ of the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$ is the limit in the metric topology of the space $\mathcal{F}(W)$ of the images in the space $\mathcal{F}(W)$ of the reproducing kernel functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the spaces $\mathcal{H}(E_r)$ when w is in the upper half–plane. The reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the space $\mathcal{H}(E)$ is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$

for function values at w in the spaces $\mathcal{H}(E_r)$ for all complex numbers w. The defining function

$$E(z) = \lim E_r(z)$$

of the space $\mathcal{H}(E)$ is the limit uniformly on compact subsets of the complex plane of the defining functions $E_r(z)$ of the spaces $\mathcal{H}(E_r)$.

Since the Euler weight functions $W(z)$ satisfies the symmetry condition

$$W(-z) = W^*(z),$$

the maximal dissipative transformation in the space $\mathcal{F}(W)$, which takes $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space, commutes with the transformation which takes $F(z)$ into $F^*(-z)$. The adjoint of the maximal dissipative transformation in the space $\mathcal{F}(W)$ commutes with the transformation which takes $F(z)$ into $F^*(-z)$. The maximal dissipative transformation in the Euler space $\mathcal{H}(E_r')$, which takes $F(z)$ into $G(z + ih)$ whenever elements $F(z)$ and $G(z + ih)$ of the space satisfy the identity

$$G(z) = \exp(-\tau_r h) F(z),$$

commutes with the transformation which takes $F(z)$ into $F^*(-z)$. The adjoint of the maximal dissipative transformation in the Euler space $\mathcal{H}(E_r')$ commutes with the transformation which takes $F(z)$ into $F^*(-z)$.

A maximal dissipative transformation in the space $\mathcal{H}(E_r)$ is defined by taking $F(z)$ into $G(z + ih)$ whenever $F(z)$ and $G(z + ih)$ are elements of the space for which elements $F_n(z)$ and $G_n(z + ih)$ of the Euler space $\mathcal{H}(E_r')$ exist such that the maximal dissipative transformation in the space takes $F_n(z)$ into $G_n(z + ih)$ for every positive integer n, such that

$$S_r(z) G(z + ih)$$

is the limit in the metric topology of the space of the elements $G_n(z + ih)$, and such that

$$S_r(z) F(z)$$

is the limit in the same topology of the orthogonal projections of the elements $F_n(z)$ in the image of the space $\mathcal{H}(E_r)$. Since $S_r(z)$ satisfies the symmetry condition

$$S_r(-z) = S_r^*(z),$$

the maximal dissipative transformation in the space $\mathcal{H}(E_r)$ and its adjoint commute with the transformation which takes $F(z)$ into $F^*(-z)$. Entire functions $P_r(z)$ and $Q_r(z)$, which are associated with the space $\mathcal{H}(E_r)$ and which satisfy the symmetry conditions

$$P_r(-z) = P_r^*(z)$$

and

$$Q_r(-z) = -Q_r^*(z),$$

exist such that the maximal dissipative transformation in the space takes an element $F(z)$ of the space into an element $G(z+ih)$ of the space whenever $F(z)$ and $G(z+ih)$ satisfy the identity

$$G(w) = \langle F(t), [Q_r(t)P_r(w^-) - P_r(t)Q_r(w^-)]/[\pi(t-w^-)]\rangle$$

for all complex numbers w with the scalar product taken in the space.

An isometric transformation of the derived space of the space $\mathcal{F}(W)$ into itself is defined by taking $F(z)$ into $F^*(-z)$. An isometric transformation of the derived space of the Euler space $\mathcal{H}(E_r')$ into itself is defined for every positive integer r by taking $F(z)$ into $F^*(-z)$. An isometric transformation of the derived space of the space $\mathcal{H}(E_r)$ into itself is defined for every positive integer r by taking $F(z)$ into $F^*(-z)$.

If $F(z)$ and $G(z)$ are elements of the derived space of the space $\mathcal{F}(W)$ which satisfy the identity

$$G(z) = F^*(-z),$$

then the identities

$$G_+(z) = F_+^*(-z)$$

and

$$G_-(z) = F_-^*(-z)$$

are satisfied. If $F(z)$ and $G(z)$ are elements of the derived space of the Euler space $\mathcal{H}(E_r')$ which satisfy the identity

$$G(z) = F^*(-z),$$

then the identities

$$G_+(z) = F_+^*(-z)$$

and

$$G_-(z) = F_-^*(-z)$$

are satisfied.

The isometric transformation of the derived space of the Euler space $\mathcal{H}(E_r')$ into the derived space of the space $\mathcal{F}(W)$, which satisfies the identity

$$G_+(z) = \exp(\tfrac{1}{2}\tau_r h + i\tau_r z)F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$, takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. The isometric transformation of the derived space of the space $\mathcal{H}(E_r)$ into the derived space of the Euler space $\mathcal{H}(E_r')$, which satisfies the identity

$$G_+(z) = S_r(z - \tfrac{1}{2}ih)F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$, takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$.

Multiplication by z is an isometric transformation of the set of elements $F(z)$ of the space $\mathcal{H}(E)$ which satisfy the symmetry condition

$$F(-z) = F^*(z)$$

onto the set of elements $G(z)$ of the space $\mathcal{H}(E')$ which satisfy the symmetry condition

$$G(-z) = -G^*(z).$$

Since the defining function $E'(z)$ of the space $\mathcal{H}(E')$ satisfies the symmetry condition

$$E'(-z) = E^*(z),$$

multiplication by

$$iW(z)/E'(z)$$

is an isometric transformation of the set of elements $G(z)$ of the space $\mathcal{H}(E')$ which satisfy the symmetry condition

$$G(-z) = -G^*(z)$$

into the set of elements $H(z)$ of the space $\mathcal{F}(W)$ which satisfy the symmetry condition

$$H(-z) = H^*(z).$$

Multiplication by

$$izW(z)/E'(z)$$

is an isometric transformation of the set of elements $F(z)$ of the space $\mathcal{H}(E)$ which satisfy the symmetry condition

$$F(-z) = F^*(z)$$

into the set of elements $H(z)$ of the space $\mathcal{F}(W)$ which satisfy the symmetry condition

$$H(-z) = H^*(z).$$

If a zero w^- of $E'(z)$ satisfies the inequality

$$iw^- - iw > h,$$

then $-w$ is a zero of $E'(z)$ which satisfies the inequality. The function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$
$$+ [E(z)E(w) - E^*(z)E^*(w)]/[2\pi i(-w - z)]$$

of z is the reproducing kernel function for function values at w in the space of elements $F(z)$ of the space $\mathcal{H}(E)$ which satisfy the symmetry condition

$$F(-z) = F^*(z).$$

The function

$$zw^-[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$
$$+zw^-[E(z)E(w) - E^*(z)E^*(w)]/[2\pi i(-w - z)]$$

of z is the reproducing kernel function for function values at w in the space of elements $G(z)$ of the space $\mathcal{H}(E')$ which satisfy the symmetry condition

$$G(-z) = -G^*(z).$$

Multiplication by

$$iW(z)/E'(z)$$

takes the element of the space $\mathcal{H}(E')$ into the product of

$$iE'(w)^-/W(w)^-$$

and the reproducing kernel function

$$W(z)W(w)^-/[2\pi i(w^- - z)] + W(z)W(w)/[2\pi i(-w - z)]$$

for function values at w in the space of elements $H(z)$ of the space $\mathcal{F}(W)$ which satisfy the symmetry condition

$$H(-z) = H^*(z).$$

It follows that multiplication by

$$izW(z)/E'(z)$$

takes the reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$
$$+ [E(z)E(w) - E^*(z)E^*(w)]/[2\pi i(-w - z)]$$

for function values at w in the space of elements $F(z)$ of the space $\mathcal{H}(E)$ which satisfy the identity

$$F(-z) = F^*(z)$$

into the product of

$$iw^-E'(w)^-/W(w)^-$$

and the reproducing kernel function

$$W(z)W(w)^-/[2\pi i(w^- - z)] + W(z)W(w)/[2\pi i(-w - z)]$$

for function values at w in the space of elements $H(z)$ of the space $\mathcal{F}(W)$ which satisfy the symmetry condition

$$H(-z) = H^*(z).$$

Multiplication by

$$\exp(i\tau_r z)$$

is an isometric transformation of the Euler space $\mathcal{H}(E_r')$ into the space $\mathcal{F}(W)$ for every positive integer r. Elements of the space $\mathcal{F}(W)$ which belong to the image of the Euler space are the reproducing kernel function $F_r(z)$ for function values at $w - ih$ and the reproducing kernel function $G_r(z+ih)$ for function values at w in the subspace of elements $H(z)$ of the space $\mathcal{F}(W)$ which belong to the image of the Euler space and which satisfy the symmetry condition

$$H(-z) = H^*(z).$$

The maximal dissipative transformation in the subspace takes $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ belong to the subspace. The adjoint of the maximal dissipative transformation in the subspace takes $F_r(z)$ into $G_r(z + ih)$.

The contractive transformation of the derived space of the Euler space $\mathcal{H}(E_r')$ into the derived space of the space $\mathcal{F}(W)$, which satisfies the identity

$$G_+(z) = \exp(\tfrac{1}{2}\tau_r h + i\tau_r z)F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$, takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. The image of the derived space of the Euler space $\mathcal{H}(E_r')$ in the derived space of the space $\mathcal{F}(W)$, which is a Hilbert space in the scalar product for which he transformation is isometric, is invariant under an isometric transformation which takes $F(z)$ into $F^*(-z)$. The reproducing kernel function for function values at $w - \frac{1}{2}ih$ in the subspace of elements of the image, which satisfy the symmetry condition

$$F(-z) = F^*(z),$$

has

$$G_r(z + \tfrac{1}{2}ih)$$

as its upper component. The reproducing kernel function

$$\begin{aligned}&[W(z - \tfrac{1}{2}ih)W(w)^- + W(z + \tfrac{1}{2}ih)W(w - ih)^-]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]\\ &+ [W(z - \tfrac{1}{2}ih)W(w) + W(z + \tfrac{1}{2}ih)W(w - ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]\end{aligned}$$

for function values at $w - \frac{1}{2}ih$ in the subspace of elements $F(z)$ of the derived space of the space $\mathcal{F}(W)$, which satisfy the symmetry condition

$$F(-z) = F^*(z),$$

is the limit in the metric topology of the space of the reproducing kernel functions for function values at $w - \frac{1}{2}ih$ in the subspace of elements of the image of the derived space of the Euler space $\mathcal{H}(E_r')$ which satisfy the symmetry condition.

Multiplication by

$$\exp(i\tau_r z)S_r(z)$$

is an isometric transformation of the space $\mathcal{H}(E_r)$ into the space $\mathcal{F}(W)$ which takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. The element $U_r(z)$ of the space $\mathcal{H}(E_r)$ such that

$$\exp(i\tau_r z)S_r(z)U_r(z)$$

is the orthogonal projection of $F_r(z)$ in the image of the space $\mathcal{H}(E_r)$ is the product of

$$\exp(\tau_r h - i\tau_r w^-) S_r(w - ih)^-$$

and the reproducing kernel function

$$[E_r(z)E_r(w-ih)^- - E_r^*(z)E_r(w^- + ih)]/[2\pi i(w^- + ih - z)]$$
$$+ [E_r(z)E_r(w-ih) - E_r^*(z)E_r^*(w-ih)]/[2\pi i(w^- + ih - z)]$$

for function values at $w - ih$ in the subspace of elements $F(z)$ of the space $\mathcal{H}(E_r)$ which satisfy the symmetry condition

$$F(-z) = F^*(z).$$

The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E_r)$ takes $U_r(z)$ into the element $V_r(z + ih)$ of the space $\mathcal{H}(E_r)$ which is the product of

$$\exp(\tau_r h - i\tau_r w^-) S_r(w - ih)^-$$

and the reproducing kernel function

$$[Q_r(z)P_r(w^-) - P_r(z)Q_r(w^-)]/[\pi(z - w^-)]$$
$$+ [Q_r(z)P_r(w) + P_r(z)Q_r(w)]/[\pi(z + w)]$$

for the value of transformed functions at w in the subspace of elements of the space $\mathcal{H}(E_r)$ which satisfy the symmetry condition. The element $U_r^\sim(z)$ of the space $\mathcal{H}(E_r)$ is the product of

$$\exp(\tau_r h - i\tau_r w^-) S_r(w - ih)^-$$

and the element

$$[Q_r^*(z)P_r(w-ih)^- - P_r^*(z)Q_r(w-ih)^-]/[\pi(z - ih - w^-)]$$
$$+ [Q_r^*(z)P_r^*(w-ih) - P_r^*(z)Q_r^*(w-ih)]/[\pi(z - ih - w)]$$

The contractive transformation of the derived space of the space $\mathcal{H}(E_r)$ into the derived space of the space $\mathcal{F}(W)$, which satisfies the identity

$$G_+(z) = \exp(\tfrac{1}{2}\tau_r h + i\tau_r z) S_r(z - \tfrac{1}{2}ih) F_+(z)$$

for upper components whenever it takes $F(z)$ into $G(z)$, takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. The transformation maps the derived space of the space $\mathcal{H}(E_r)$ isometrically into the Hilbert space which is the image in the derived space of the space $\mathcal{F}(W)$ of the derived space of the Euler space $\mathcal{H}(E_r')$. The element

$$U_r^\sim(z + \tfrac{1}{2}ih) + V_r(z + \tfrac{1}{2}ih)$$

of the derived space of the space $\mathcal{H}(E_r)$ is the product of

$$\exp(\tau_r h - i\tau_r w^-)S_r(w-ih)^-$$

and the reproducing kernel function with upper component

$$[Q_r(z-\tfrac{1}{2}ih)P_r(w^-) - P_r(z-\tfrac{1}{2}ih)Q_r(w^-)]/[\pi(z-\tfrac{1}{2}ih-w^-)]$$
$$+ [Q_r(z-\tfrac{1}{2}ih)P_r(w) + P_r(z-\tfrac{1}{2}ih)Q_r(w)]/[\pi(z-\tfrac{1}{2}ih+w)]$$

and with lower component

$$[Q_r^*(z+\tfrac{1}{2}ih)P_r(w-ih)^- - P_r^*(z+\tfrac{1}{2}ih)Q_r(w-ih)^-]/[\pi(z-\tfrac{1}{2}ih-w^-)]$$
$$+ [Q_r^*(z+\tfrac{1}{2}ih)P_r^*(w-ih) + P_r^*(z+\tfrac{1}{2}ih)Q_r^*(w-ih)]/[\pi(z-\tfrac{1}{2}ih+w)]$$

for function values at $w-\frac{1}{2}ih$ in the subspace of elements of the derived space of the space $\mathcal{H}(E_r)$ which satisfy the symmetry condition. The isometric transformation of the derived space of the space $\mathcal{H}(E_r)$ into the image in the derived space of the space $\mathcal{F}(W)$ of the derived space of the Euler space $\mathcal{H}(E_r')$ takes the element

$$U_r^\sim(z+\tfrac{1}{2}ih) + V_r(z+\tfrac{1}{2}ih)$$

of the derived space of the space $\mathcal{H}(E_r)$ into the orthogonal projection of

$$F_r(z+\tfrac{1}{2}ih) + G_r(z+\tfrac{1}{2}ih)$$

in the image of the derived space of the space $\mathcal{H}(E_r)$.

The reproducing kernel function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$
$$+ [E(z)E(w) - E^*(z)E^*(w)]/[2\pi i(-w-z)]$$

for function values at w in the subspace of elements of the space $\mathcal{H}(E)$, which satisfy the symmetry condition

$$F(-z) = F^*(z),$$

is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions

$$[E_r(z)E_r(w)^- - E_r^*(z)E_r(w^-)]/[2\pi i(w^- - z)]$$
$$+ [E_r(z)E_r(w) - E_r^*(z)E_r^*(w)]/[2\pi i(-w-z)]$$

for function values at w in the subspace of elements $F(z)$ of the space $\mathcal{H}(E_r)$ which satisfy the symmetry condition

$$F(-z) = F^*(z).$$

The product of

$$-izW(z)/E'(z)$$

and the reproducing kernel function for function values at w in the subspace of elements of the space $\mathcal{H}(E)$ which satisfy the symmetry condition is the limit in the metric topology of the space $\mathcal{F}(W)$ of the product of

$$\exp(i\tau_r z) S_r(z)$$

and the reproducing kernel functions for function values at w in the subspaces of elements of the spaces $\mathcal{H}(E_r)$ which satisfy the symmetry condition.

If $w - ih$ is not a zero of $E'(z)$, the factors

$$\exp(\tau_r h - i\tau_r w^-) S_r(w - ih)^-$$

converge to

$$i[(w - ih)W(w - ih)/E'(w - ih)]^-.$$

The adjoint of the maximal dissipative transformation in the space $\mathcal{H}(E)$ then takes the reproducing kernel function

$$\begin{aligned}&[E(z)E(w - ih)^- - E^*(z)E(w^- + ih)]/[2\pi i(w^- + ih - z)]\\ &+ [E(z)E(w - ih) - E^*(z)E^*(w - ih)]/[2\pi i(-w + ih - z)]\end{aligned}$$

for function values at w in the subspace of elements of the space $\mathcal{H}(E)$ which satisfy the symmetry condition into the product of

$$[-w^-/(w - ih)][E'(w - ih)/W(w - ih)][W(w)/E(w)]^-$$

and the reproducing kernel function

$$\begin{aligned}&[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]\\ &+ [E(z)E(w) - E^*(z)E^*(w)]/[2\pi i(-w - z)]\end{aligned}$$

for function values at w in the subspace of elements of the space $\mathcal{H}(E)$ which satisfy the symmetry condition. The conclusion which is obtained when $w - ih$ is not a zero of $E'(z)$ holds by continuity when $w - ih$ is a zero. The computation of adjoints denies $w - ih$ as a zero of $E'(z)$ distinct from w^- or as a double zero equal to w^-.

This concludes the proof of the theorem.

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