High order statistics

Tim Horbury
Imperial College, London

- Distributions and moments
- Intermittency
- Structure functions
- Intermittency models
- Fractal and multifractals

Probability density functions

- For a continuous distribution, probability density function is defined as

$$p(x) = \lim_{\Delta x \to 0}\left[\frac{\text{Prob}\,[x < x(t) \leqslant x + \Delta x]}{\Delta x}\right]$$

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1$$

$$p(x) \geqslant 0$$

- If stationary and ergodic, then the estimator from a single time sample will be unbiased:

$$P\left[x, W\right] = \text{Prob}\left[\left(x - \frac{W}{2}\right) \leqslant x(t) \leqslant \left(x + \frac{W}{2}\right)\right]$$

- So, as $T \to \infty$ and $W \to 0$, this estimator approaches the true value

- Can calculate for discrete time series,

$$p(x) = \frac{N_x}{NW}$$

- Where N_x is number of points in interval centred on x, of width W. Recall, N is total number of data points.

- Definitions: **histogram** is just N_x, **probability density estimate** (or probability density function) is $p(x)$. Can also plot **probability per bin**

- Note: bins need not be of equal width

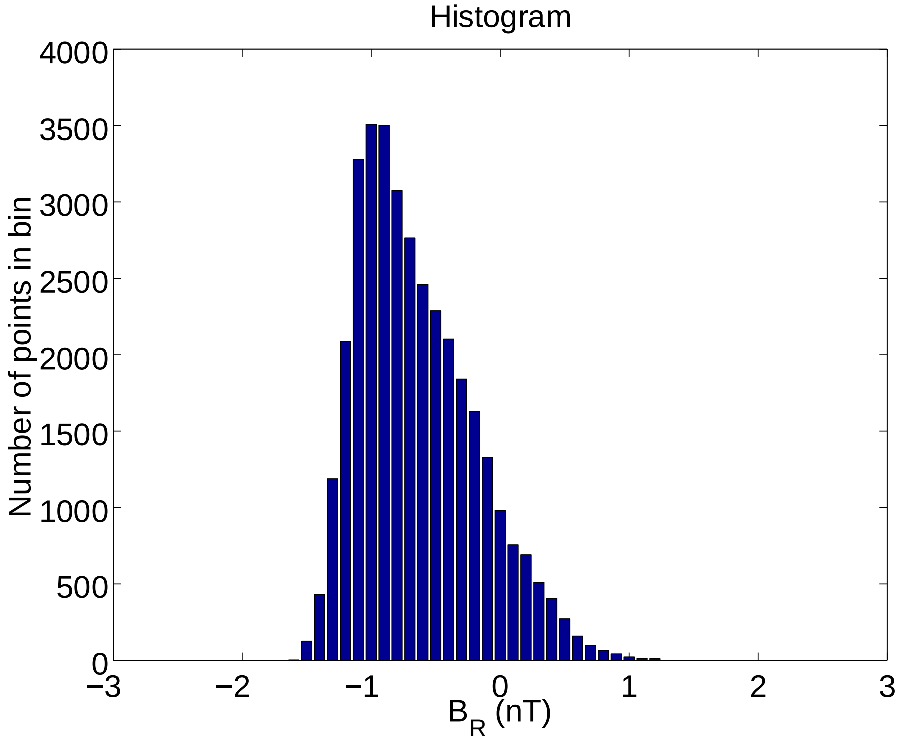
Probability density functions

- Gotcha 1: don't make the bins too wide. Often compare with model distributions: remember that the probability will not be flat across a bin
- Gotcha 2: probability densities can be above 1.
- Recall that $\int_{-\infty}^{+\infty} p(x)\,dx = 1$, so if total width of distribution is <1, $p(x)$ can be >1.
- Example: Gaussian:

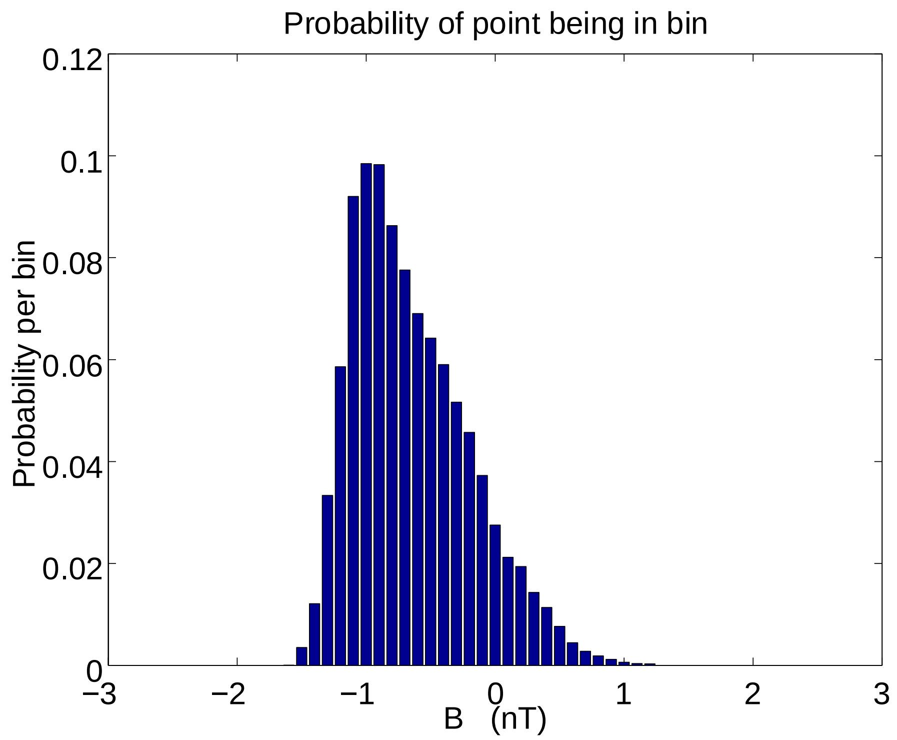
$$p(x) = \frac{e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}}{\left(\sigma_x\sqrt{2\pi}\right)}$$

- Peak of probability density function, at $x = \mu_x$, has value $1/(\sigma_x\sqrt{2\pi})$. This can be greater than 1!

- Examples, with Ulysses data:



- N_x / N:



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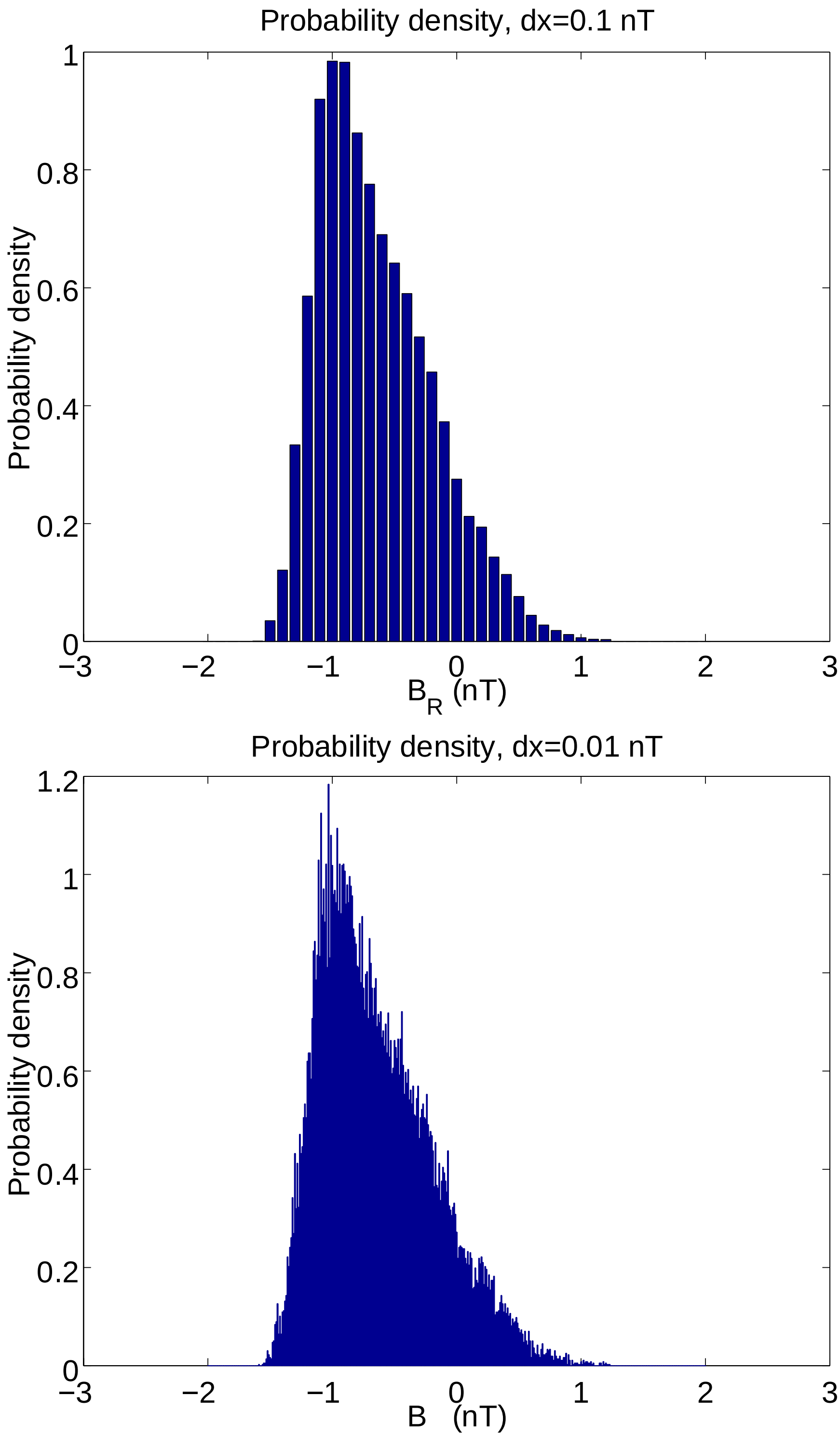


Probability density, dx=0.1 nT
Probability density
1
0.8
0.6
0.4
0.2
0
−3
−2
−1
0
1
2
3
B_R (nT)
Probability density, dx=0.01 nT
Probability density
1.2
1
0.8
0.6
0.4
0.2
0
−3
−2
−1
0
1
2
3
B (nT)

Moments

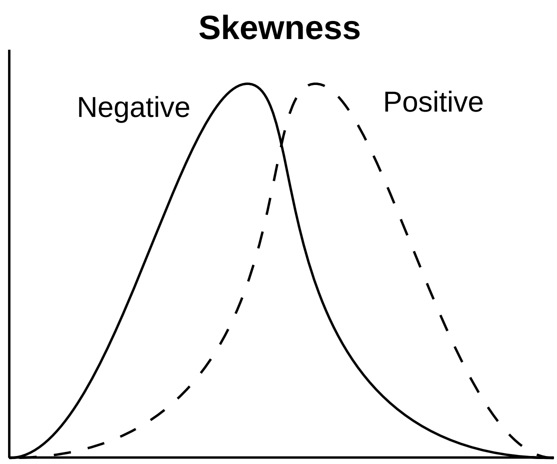
- **Can calculate an infinite number of moments of a time series:**

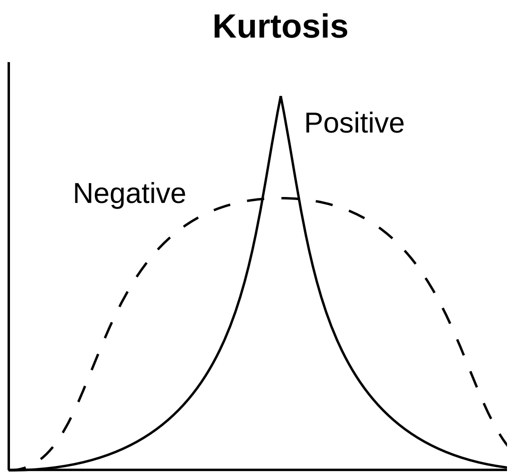
$$m_n = \int_{-\infty}^{+\infty} x^n p(x)\, dx \text{ or, } m_n = \sum x^n$$

- **Can calculate from a calculated distribution, or from each data point individually**

- **Assuming zero mean,**

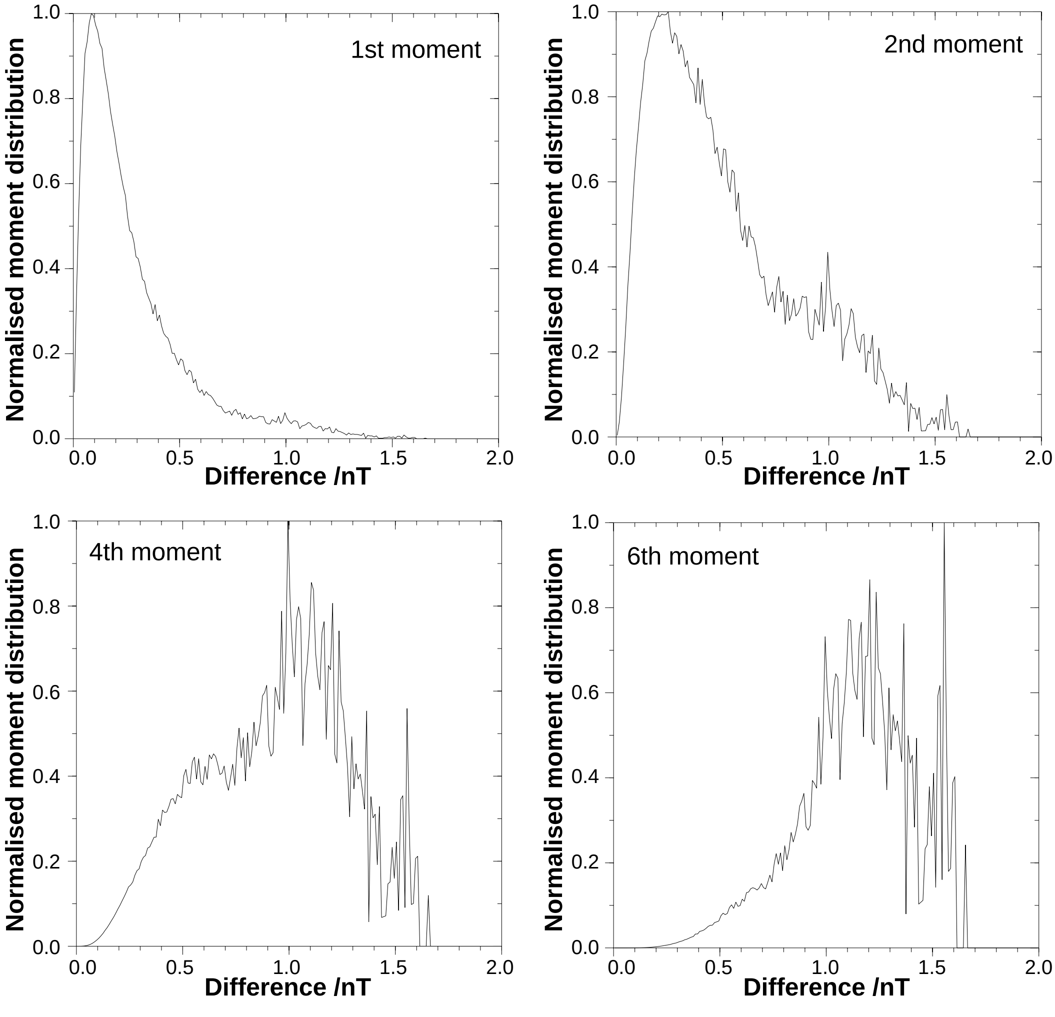
Moment	**Name**	**Gaussian value**
1	mean	0
2	variance	σ^2
3	skewness	0
4	flatness (kurtosis+3)	$3\sigma^2$
5	-	0
6	-	$15\sigma^2$

- 



Importance of outliers

- Higher moments emphasise outliers of the distribution
- Eventually, moment is dominated by a small number of points
- If we want to calculate high moments, need long, stationary data sets



Errors on high moments

- **When calculating errors** in e.g. means, we usually assume Gaussian statistics
- Similarly, when calculating errors in higher order moments, can assume that even higher orders are distributed as a Gaussian
- However, this is usually not the case (that's why we're looking at higher moments)
- Therefore, error estimation is hard

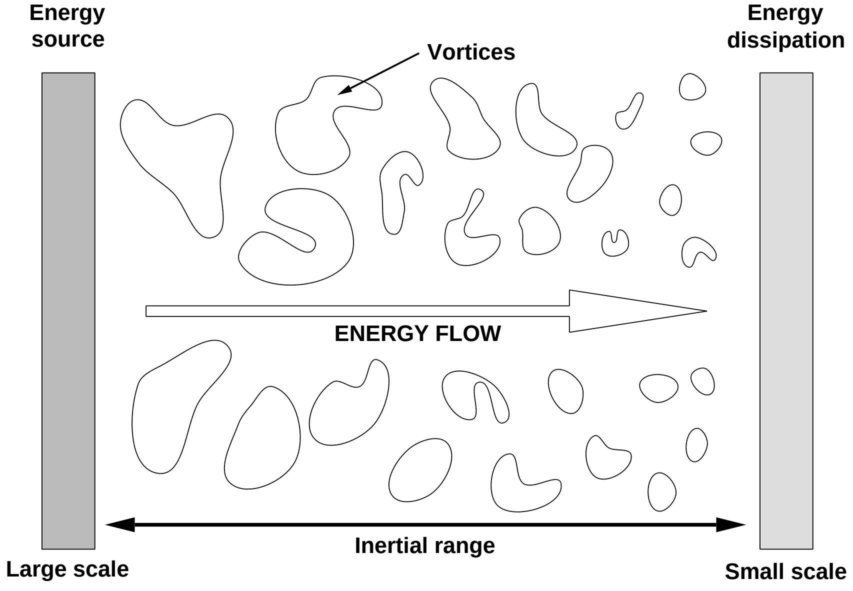
Distributions in practice

- Often of interest in turbulence to study increments (more later)
- See progressively more non-Gaussian distributions, with extended tails higher than for Gaussians, at smaller scales
- Typically, see non-Gaussian distributions when turbulent - **intermittency**

What is intermittency?

- Spatial inhomogeneity of fluctuations in turbulent fluid
- "Burstiness", e.g. gusts on a windy night
- Extensively studied in hydrodynamics; also in solar wind
- Related to generation of **structures** in the fluid

What is turbulence?

- Random, chaotic fluid motion
- Energy transfer between scales (typically, large to small)
- Inertial range: large separation between input and output scales, $\rightarrow$ fully developed turbulence
- 

Kolmogorov (1941) theory

- Foundation of our understanding of turbulence
- Scaling argument - almost no physics
- Scale l, velocity $u(l)$, energy transfer rate $\varepsilon(l)$
- Steady state: $\varepsilon(l) = \varepsilon(l') = \varepsilon$.
- Energy transfer time,

$$\tau_T(l) \propto \frac{E(l)}{\varepsilon}; \; E(l) \propto u^2(l)$$

- Eddy decay, so

$$\tau_T(l) \propto \tau_E(l) \propto \frac{l}{u(l)}$$

- So,

$$u^2(l) \propto E(l) \propto \varepsilon\tau_T(l) \propto \varepsilon\frac{l}{u(l)}$$

- Therefore, velocity dependence on scale,

$$u(l) \propto l^{m/3}$$

- So, energy, $u^2(l) \propto l^{2/3}$.
- In wavenumbers, $u^2(k) \propto k^{2/3}$ - famous K41 power law!

Experimental verification of K41

- Observe $k^{-5/3}$ power spectra in turbulent fluids:
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- Example from Frisch - already photocopied (have 2)
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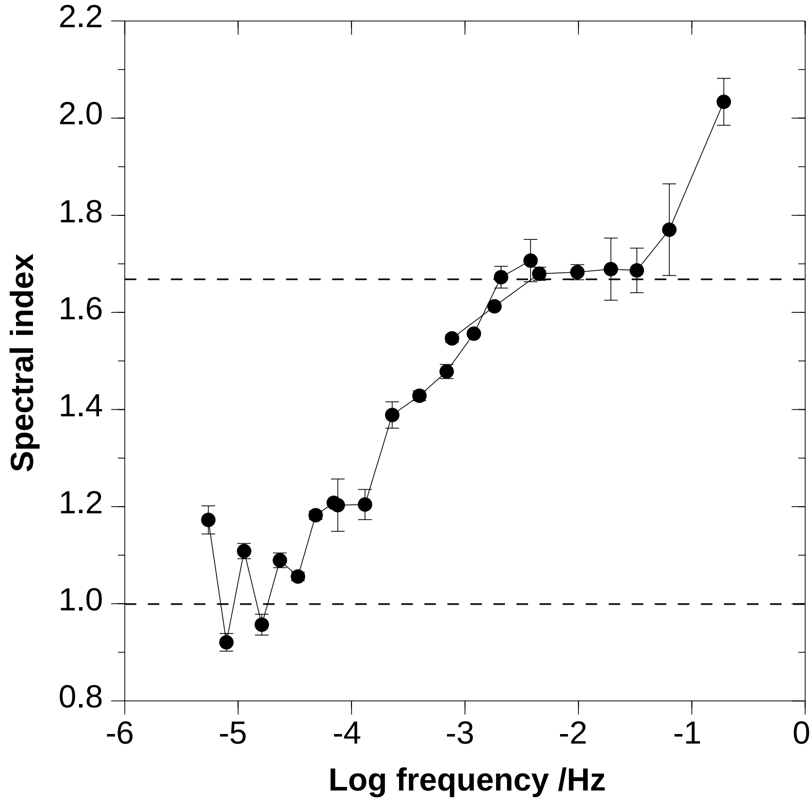
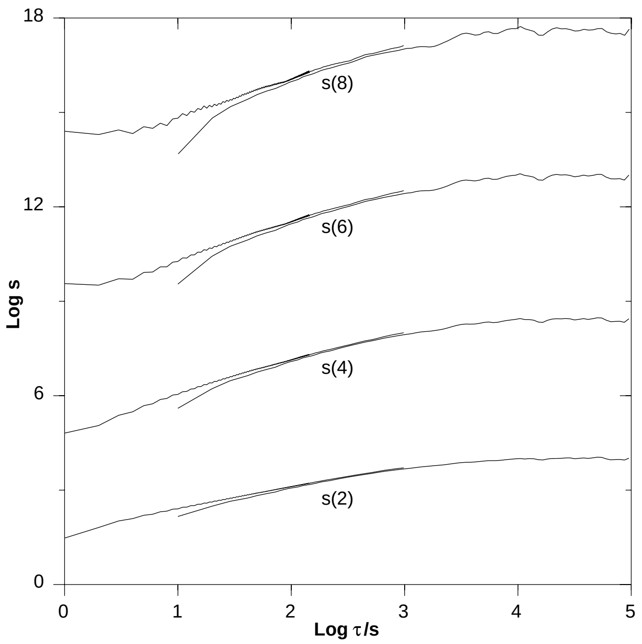
- Also in solar wind (Ulysses):



Structure functions

- Use to analyse turbulence

$$S(\tau, m) = \langle |x(t + \tau) - x(t)|^m \rangle$$

- Other definitions exist (not modulus, etc.)
- This is a high order analysis method
- Essentially, take moments of increments (of velocity, magnetic field, etc.)
- In this way, measure levels of fluctuations
- We are interested in how these **scale**
- Example, Ulysses:
- 

- Note scaling over range of τ:

$$S(\tau, m) \propto \tau^{\zeta(m)}$$

- We want to measure the $\zeta(m)$...

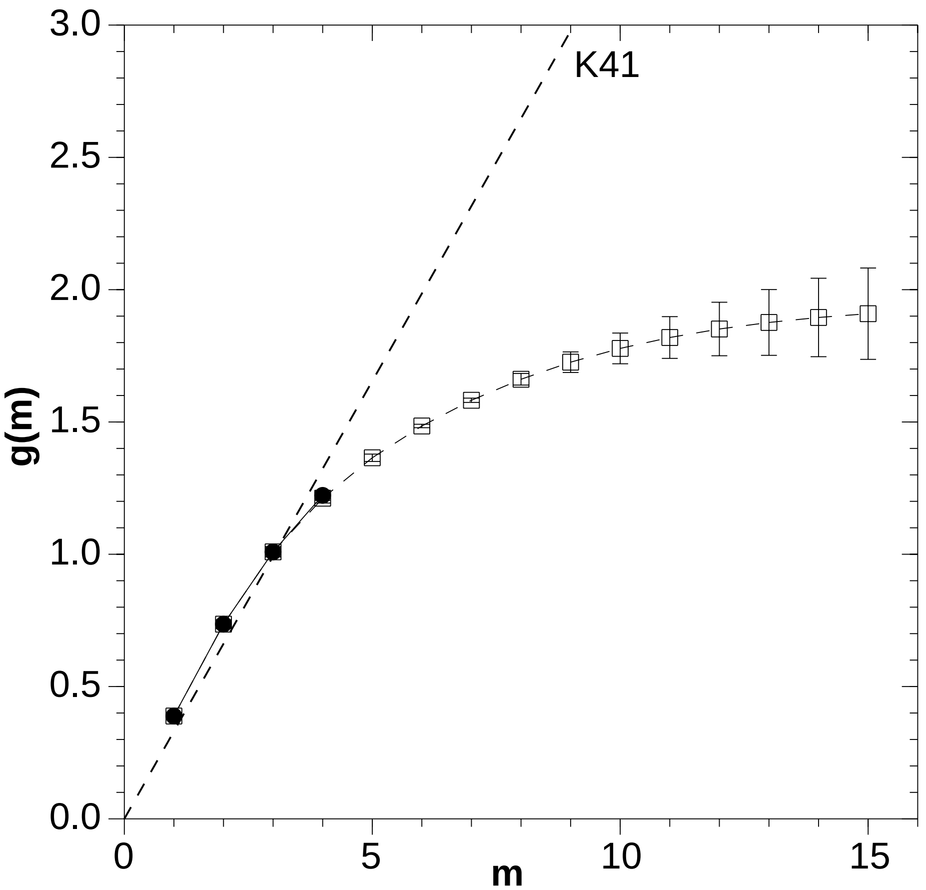
Comparison with K41

- Recall, $u(l) \propto l^{1/3}$
- Therefore, expect

$$u^m(l) \propto l^{m/3}$$

- And so $S(\tau, m) \propto l^{m/3}$ and,

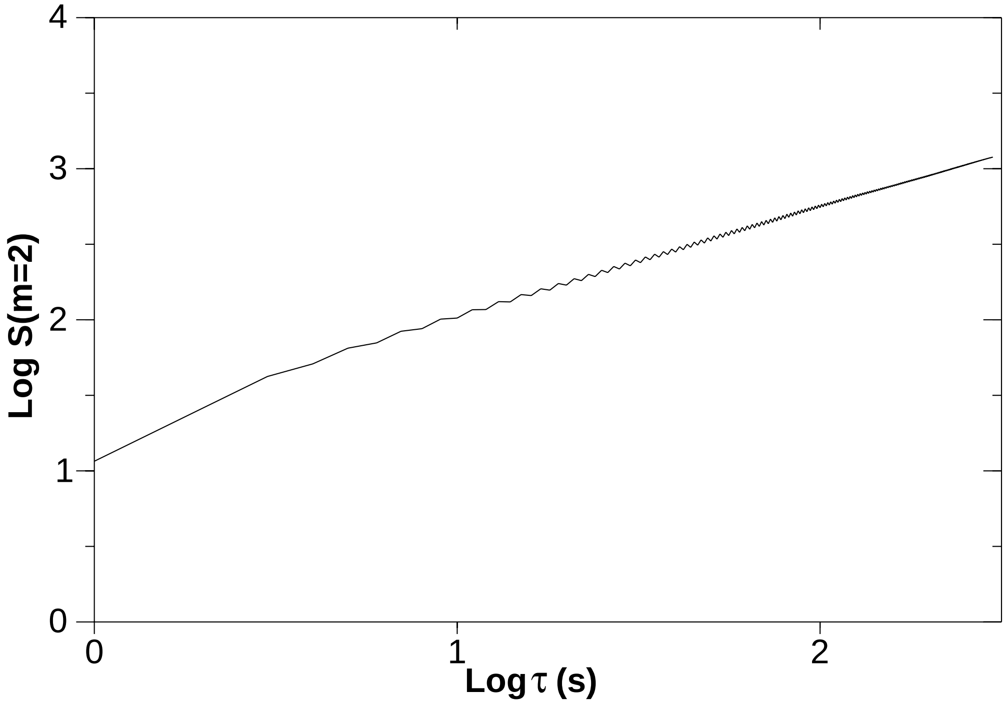
$$\zeta(m) = \frac{m}{3}$$

- This is not what is observed - we see a curve!
- 

- This is due to intermittency!
- Note: exact result: $\zeta(3) = 1$
- This is satisfied experimentally

Structure functions - problems

- Can't use very high moments - they are unreliable
- In previous example, only $m \leqslant 4$ were reliable
- Structure functions have a wide spectral response - so always look nice and smooth and well behaved:



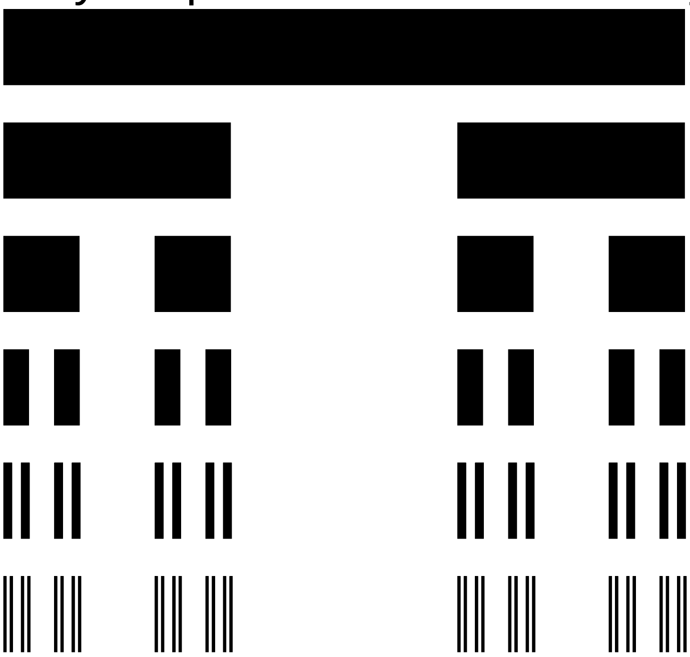
- Note wiggle - this is a data rate change issue!
- Data are **oversampled** - points are not independent
- Calculating error on gradient from line fitting is therefore dangerous...
- Error bars can be small, but values can still be wrong!

Fractals

- Self-similarity
- Examples: clouds, coastlines, turbulence...
- Intermittency is related to **scaling**, and therefore to fractals
- Can better understand intermittency (and non-Gaussian behaviour in many situations) through fractals - and later, multifractals...

The Cantor set

- Very simple fractal: remove every centre third



- Embedding dimension, $D_E = 1$
- Topological dimension, $D_T = 0$ (it's a dust)
- Similarity (fractal) dimension, $D_S = 0.6309...$

Similarity dimension

- How many copies do we need to replicate at a different scale:

$$N = r^{D_S}$$

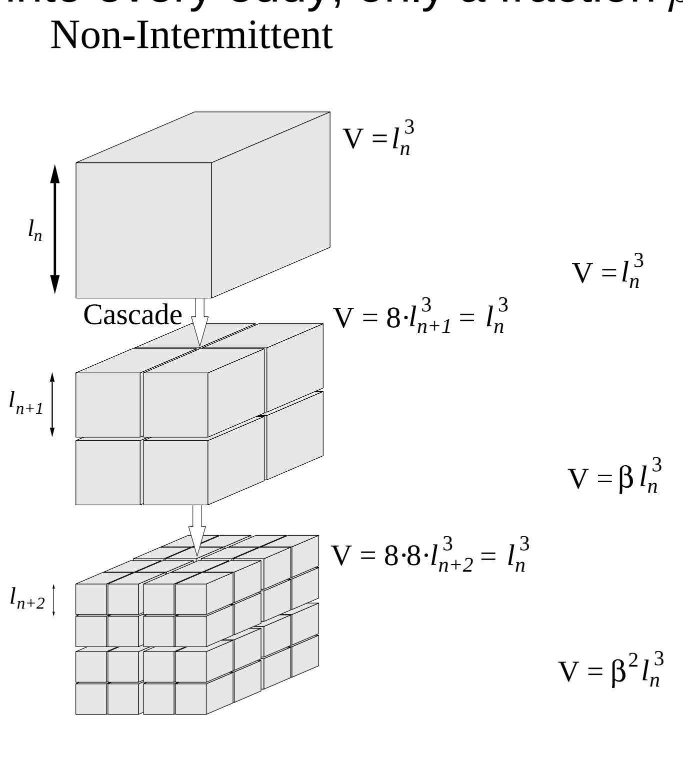
- Need N copies to reproduce from a fraction r
- $D_S = 1$ for a line (need 2 half metre rulers to cover 1 metre: $N = 2, \ r = 1/2$)
- $D_S = 3$ for a cube (need 8 half metre cubes to fill a 1 metre cube: $= 8, \ r = 1/2$)
- So,

$$D_S = \frac{\log N}{\log(1/r)}$$

- Cantor set: $N = 2, r = 1/3$
- So, $D_S = \log 2 / \log 3 = 0.6309...$
- $D_T < D_S < D_E$ for all fractals
- Note: can have different N, r choices for Cantor set

The β model

- Frisch et al (1978)
- Fractal model of intermittency
- Only a fractal subset of all space filled with active turbulent eddies
- Concept of support: a measure (e.g. turbulence) can be supported on a fractal set
- e.g. Cantor set as a bar with mass: hammering rather than cutting...
- β model: take parent eddy
- Split it up into 2 daughters - but do not put energy into every eddy, only a fraction β:



The β model, contd.

- At scale $l = l_0 2^{-n}$, only a fraction β^n of space is filled with *active* eddies
- This is a fractal subset!
- Its fractal dimension is $D = \dfrac{\log N}{\log(1/r)} = \dfrac{\log(2^3 \cdot \beta)}{\log(1/2)}$
- The fraction of the volume filled with active eddies at scale l is $\beta(l) = l^{3-D}$
- Only a fraction of space is filled with active eddies: energy at scale l:

$$E(l) \propto \beta(l) \cdot u^2(l)$$

- Energy flux,

$$\varepsilon \propto \frac{E(l)}{\tau(l)} \propto \frac{u^2(l)\beta(l)}{l/u(l)} \propto \frac{u^3(l)\beta(l)}{l} \propto u^3(l)\, l^{3-D} l^{-1}$$

- Since $\varepsilon \neq \varepsilon(l)$,

$$u(l) \propto l^{\frac{1}{3} - \frac{3-D}{3}}$$

- If D=3, we recover K41
- So, turbulence is **supported** on a fractal subset of the whole space at each scale - only parts of the fluid are "active"
- The fluctuations are intermittent

How can we tell if this is happening?

- High order moments allow us to probe this behaviour. Structure functions:

$$S(\tau, m) = \langle |v(t+\tau) - v(t)|^m \rangle \propto \beta(\tau) u^m(\tau)$$

- Since the velocity in *active eddies* is

$$u(l) \propto \varepsilon^{1/3} l^{1/3} l^{-(3-D)/3}$$

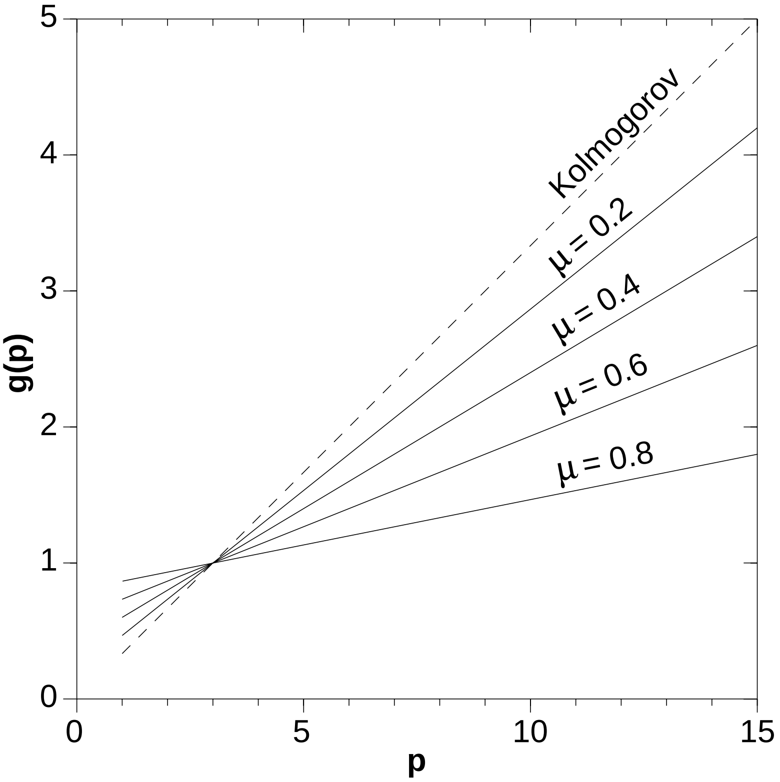
- And they exist in only a fraction l^{3-D} of the space,

$$S(\tau, m) \propto \varepsilon^{m/3} \tau^{m/3} \tau^{-(3-D)m/3} \tau^{(3-D)}$$

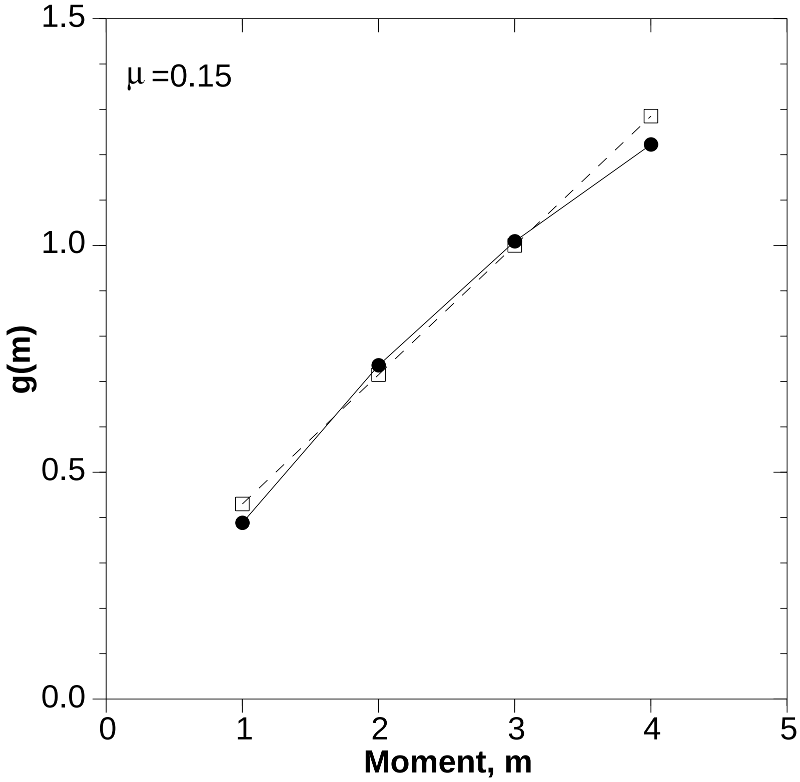
- Using $\mu = 3 - D$ as a measure of the intermittency ($\beta = 2^{-\mu}$),

$$\zeta(m) = m/3 + \mu(1 - m/3)$$

- If $\mu = 0$, we have K41, and $\zeta = m/3$



Testing the β model

- Can't detect using power spectrum - alters the spectral index, but still gives power law behaviour
- Compare with experimental structure functions:
- 

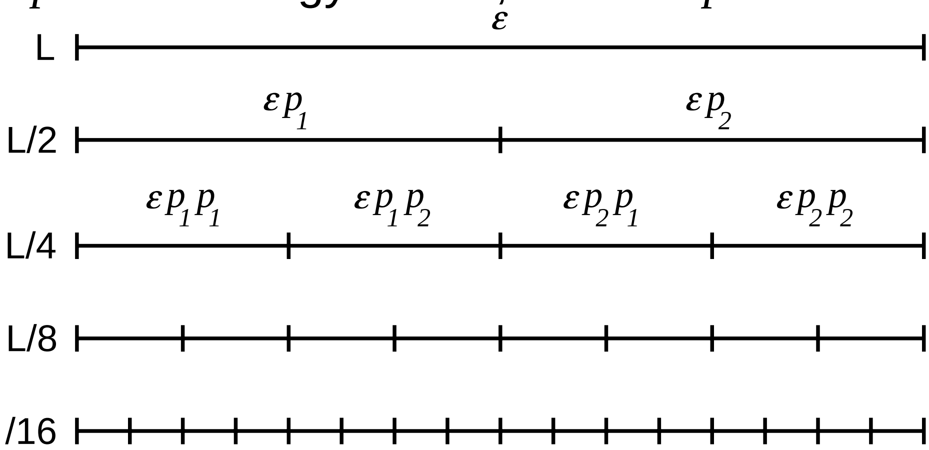
- Not bad, but observations show a *curved* $\zeta(m)$ behaviour, while the β model is straight
- Can't reproduce the observations - need something more complicated
- Multifractals... (deep breath...)

What is a multifractal?

- A fractal supporting a measure (e.g. turbulent energy)
- There are (many) different values of the measure at the same scale (simple fractal has only one), distributed with a probability distribution
- Each set with a particular measure is supported on a fractal set
- Each fractal has a *different dimension*, and...
- The infinite number of fractal subsets exactly fill all of the space!
- We can detect different value of the measure by taking moments: higher moments select high value measures
- Examine the scalings to detect the "spectrum" of fractal subsets
- In practice, this gives us curved $\xi(m)$ behaviour, which is what we want...

The p model

- Meneveau and Sreenivasan, 1987
- A multifractal intermittency model, which is in good agreement with structure function observations
- Here, the turbulence fills all of space at every scale, but the energy in different eddies is different
- Like K41, split each eddy into 2 pieces, but give a fraction p of the energy to one, and $1 - p$ to the other:



- Then, at each scale, we have a distribution of energies
- The fraction of energy in each box in given by the binomial expansion, $\binom{n}{m}$
- By changing p we can change the intermittency of the energy: $p = 0.5$ corresponds to K41
- $p > 0.5$ is more "bursty" - $p = 1$ is the maximally intermittent case

Which eddies are most important?

- At any given scale n, there is one eddy with the maximum energy, p^n and one with the minimum, $(1 - p)^n$
- For large n, these eddies have negligible probability. There is some intermediate energy which is most probable
- By taking different moments, however, we can become sensitive to different energies
- Taking an infinitely large moment, $m \to +\infty$, we are only sensitive to the single eddy with the most energy
- If we take $m \to -\infty$, we are sensitive only to the large energetic eddy
- Since different parts of the energy distribution have a different number of eddies at each scale (e.g. least energy has only one at each scale), their fractal dimensions are different
- All these sets together constitute a multifractal - this is what we measure...

Deriving $\zeta(m)$ for the p model

- At scale l_n, we have 2^n eddies
- Each eddy i has an energy transfer rate

$$\varepsilon_i = \varepsilon_n \cdot f_i$$

- where f_i is the fraction of energy in eddy i
- ε_n is the average energy, $\varepsilon_n = \sum_{i=1}^{N} \varepsilon_i = \varepsilon_L L / l_n$
- Clearly,

$$\sum_{i=1}^{i=2^n} f_i = 1 = (p + (1 - p))^n$$

- Taking moments,

$$\sum_{i=1}^{i=2^n} f^m{}_i = \left(p^m + (1 - p)^m\right)^n$$

- So,

$$\sum_{i=1}^{i=2^n} \varepsilon^m{}_i = \sum_{i=1}^{N} f_i^m \varepsilon_n^m = \sum_{i=1}^{N} \varepsilon_n^m \cdot \left(p^m + (1 - p)^m\right)^n$$

p model - structure functions

- Energy transfer rate is related to velocity fluctuations, as before:

$$\varepsilon_i \propto \frac{u_i^3}{l_n}$$

- Therefore structure functions,

$$S(\tau, m) = \frac{1}{N_n} \sum_{i=1}^{i=2^n} |u_i|^m = \sum_{i=1}^{i=2^n} \varepsilon^{m/3} l_n^{m/3} l_n / L$$

- Since

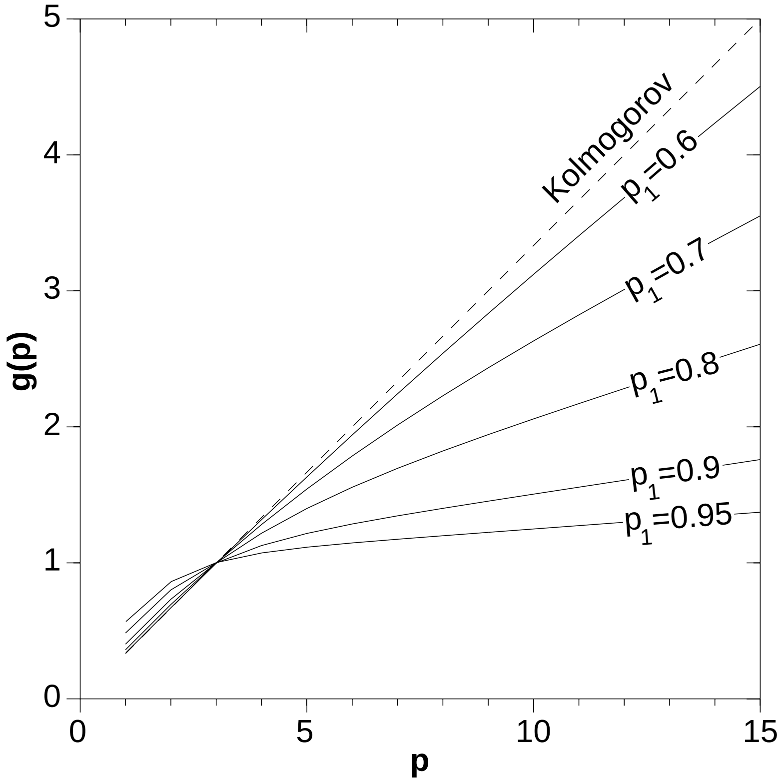
$$\left(p^m + (1-p)^m\right)^n = \left(\frac{l_n}{L}\right)^{-\log_2\left(p^m + (1-p)^m\right)}$$

- Therefore,

$$S(\tau, m) = \frac{l_n}{L} \cdot \left(\frac{l_n}{L}\right)^{-\log_2\left(p^m + (1-p)^m\right)}$$

- We have $\zeta(m) = 1 - \log_2\left(p^{m/3} + (1-p)^{m/3}\right)$
- As with β model, the zero intermittency case (here, $p = 0.5$) gives us the K41 case

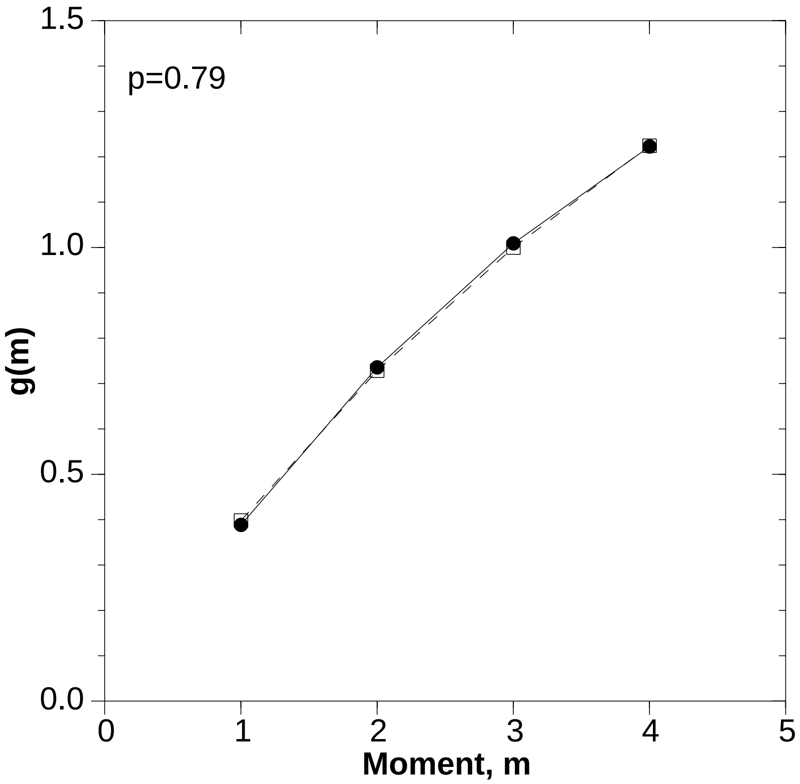
The p model - ζ(m)

- p model structure function scaling parameters are curves - this is what we want
- 

- $p = 0.5$ is K41 case
- $p = 1$ is maximally intermittent case
- As with all intermittency models, $\zeta(2) > 2/3$ - power spectrum is steeper than simple K41 case

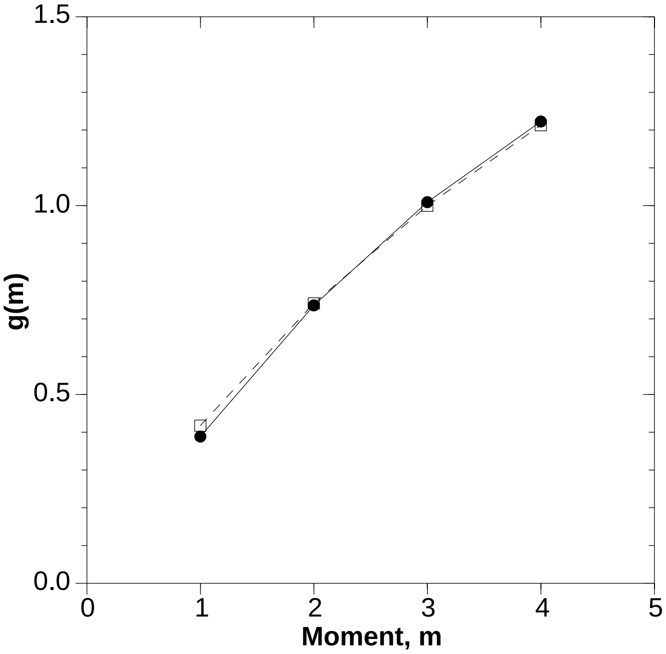
Testing the p model

- Agrees well with hydrodynamic turbulence - also with solar wind:

- 

- Value of intermittency parameter, p, is typically 0.7-0.8
- This is similar to values found in neutral fluids
- Does this mean that the intermittency is "the same" in neutral fluids and plasmas?
- Almost definitely not - more likely, it means that structure functions aren't good at distinguishing between different structures, and models like the p model are just generic multifractals

Other models of intermittency

- β and p models contain almost no physics, but describe how eddies decay in a phenomenological way
- Other ways of looking at this problem:
- She and Leveque (1994): dimension of dissipation structures. Using 2D ("current sheet") structures, good agreement with observations, and no free parameters:
- 

- Are structure functions really good enough to distinguish different fractal structures?
- Other approaches: distributions, e.g. Castaing
- In solar wind, attempt to identify individual structures which are intermittent (e.g. Bruno et al.) - they seem to be discontinuities

Summary

- High order moments allow us to probe non-Gaussian distributions
- Care must be taken in calculating these moments, however
- In practice, want to study scaling of these moments
- Measurement of fractal and multifractal dimensions
- Particularly of interest in turbulent fluids
- Difficult to distinguish between different models - we need to do better
- Other analysis methods...
- Identification of intermittent structures
- Other turbulence properties, e.g. anisotropy