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# Ramanujan's Summation

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**Abstract**

Ramanujan has brought a number of impressive results to analysis. Some of them have been obtained by a very free use of divergent series, which tends to show that he possessed an intuitive summation process for such divergent series, a process that could even depend of the context. The first step of our analysis is based on some considerations of Ramanujan from Chapter VIII of his Notebooks.

## 1. Introduction

The famous self-taught Indian mathematician Ramanujan (1887–1920) was accustomed to using convergent as well as divergent series freely in his derivation of identities. Most of the results reported on his notebooks were proven to be true, even if the ways he used to find them were not always rigorous. Actually, behind his way of thinking, a few summation schemes have been detected like Borel summation and what is called here the Ramanujan summation.

At the beginning of the 8th chapter of his Notebooks, as it is reported in Berndt's account [2], Ramanujan starts with the Euler–Maclaurin formula

$$a(1) + a(2) + \cdots + a(x-1) = C + \int_1^x a(t)dt + \sum_{k\geq 1} \frac{b_k}{k!} \partial^{k-1} a(x), \tag{1}$$

and remarks that the constant $C$ entertains a mysterious relationship with the series—it is like its "center of gravity"—so that Ramanujan proposes to consider it as the sum of the serie. As an exemple, this process assigns the value $\gamma$ to the sum $\sum_{n=1}^{+\infty} \frac{1}{n}$. The work of Delabaere attempts to make this idea rigorous, in a suitable space of analytic functions.

Let $a(x)$ be a function analytic in the right half-plane $P = \{ x \mid \Re(x) > 0 \}$. First of all, the divergent serie $\sum_{n\geq 1} a(n)$ is considered as a formal expression. Let us introduce the tail of the series $R(x) := \sum_{n\geq 0} a(n+x)$. Then, $R$ is a formal solution of the difference equation $R(x) - R(x+1) = a(x)$ and $\sum_{n\geq 1} a(n) = R(1)$. The problem of summation is then reduced to solving a difference equation.

## 2. Formal Solutions

As a first approach, using the Taylor formula, we write:

$$R(x+1) = \sum_{k\geq 0} \frac{1}{k!} \partial^k f(x) = e^{\partial}(f)(x).$$

If $I$ denotes the identity operator, it follows that $(I - e^{\partial})R = a$. We may now use the formal series expansion:

$$\frac{I}{I - e^{\partial}} = -\partial^{-1} \frac{\partial}{e^{\partial} - I} = -\partial^{-1} \left( I + \sum_{k \geq 1} \frac{b_k}{k!} \partial^k \right),$$

where $b_k$ are the Bernoulli numbers. Finally, we obtain what will be called the formal expansion of $R$:

$$R(x) = -\partial^{-1} a(x) - \sum_{k \geq 1} \frac{b_k}{k!} \partial^{k-1} a(x). \tag{2}$$

Our choice for the bounds of the definite integral in (1) then forces $R$ to satisfy the condition $\int_1^2 R(t)\, dt = 0$. The formal expression in (2) gives us a solution to the difference equation. Observe that in full generality, there can be no uniqueness of solutions since we may add to our solution any periodic non-constant function with mean value 0 over the interval $[1, 2]$. In order to determine a "principal solution," we need to impose suitable conditions on the function $a$.

The second approach uses the Laplace transform. It is classically given by the formula

$$\mathcal{L}(g)(x) = \int_0^{+\infty} e^{-x\xi} g(\xi)\, d\xi.$$

The Laplace transform has the following propertie: if $f(x) = \mathcal{L}(g)(x)$, then

$$f(x+1) = \mathcal{L}\big(\xi \mapsto e^{-\xi} g(\xi)\big)(x).$$

Therefore, if the solution $R$ to the difference equation is a Laplace tranform of a function $f$ and $a$ is a Laplace transform of a function $b$, an expression of $R$ may be obtained, using the inverse transform, by

$$R = \mathcal{L}\left( \xi \mapsto \frac{1}{1 - e^{-\xi}} b(\xi) \right) = \mathcal{L}\left( \xi \mapsto \frac{1}{1 - e^{-\xi}} \mathcal{L}^{-1}(a)(\xi) \right). \tag{3}$$

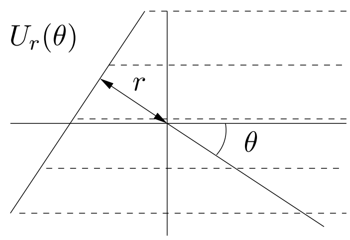
However, here, this formula cannot be applied in general and needs to be adapted, because of the possible singularity induced by $(1 - e^{-\xi})^{-1}$ at $\xi = 0$ in (3). In the following part, another definition of Laplace transform is thus given, together with its inverse transform (called Borel transform) in a suitable space of function. Such transforms are then used to solve the difference equation, yielding a unique principal solution.

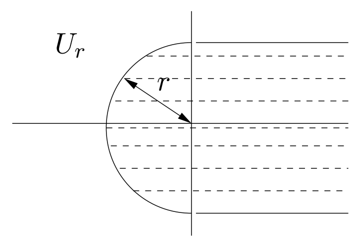
## 3. Ramanujan Summation and Borel–Laplace Transform

Let $a$ be an analytic function over the set $P$ as defined earlier. We will say that $a$ is of exponential type $r$ if for every $\epsilon > 0$, there exists $C > 0$ such that for all $x$ in $P$, there holds $\left|a(x)\right| \leq C e^{(r+\epsilon)|x|}$. The Borel transform of $a$ is then defined by

$$\mathcal{B}_d(a)(\xi) = -\frac{1}{2i\pi} \int_d e^{x\xi} a(x)\, dx$$

where $d$ is a half-line in $P$. It is easy to see that if $\theta$ is the angle of $d$ relatively to the real axis, and if $a$ is exponential of type $r$, then this integral converges for values of $x$ such that $\Re(xe^{i\theta}) < -r$. The Borel transform of $a$ may then be defined in the half-plane $U_r(\theta)$ as in Figure 1. Moreover, if the integral converges for different values of $\theta$, then Cauchy's theorem implies that the integral does not depend on $\theta$. We may then define the Borel transform of $f$, which depends only on the origin $\alpha$ of $d$, in the whole set $U_r = \bigcup_{-\pi/2<\theta<\pi/2} U_r(\theta)$.



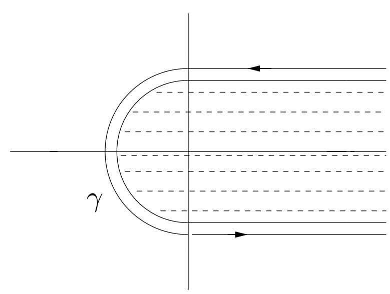




The Borel transform is then of exponential type $k = \Re(\alpha)$. As we can choose $\alpha$ anywhere in the set $P$, we may take $k$ as small as we want. Furthermore, two Borel transforms of the same function differ by an entire function; by Cauchy's theorem, their difference is the integral of the function $\xi \longmapsto -\frac{1}{2i\pi} e^{x\xi} a(x)$ along any closed path joining the two origins of the contours and is thus analytic.

Let us suppose now that $g$ is an analytic function over the set $U_r$ as introduced before and is of exponential type $k$. The Laplace transform of $g$ is given by

$$\mathcal{L}(g)(x) = \int_\gamma e^{-x\xi} g(\xi)\, d\xi$$

where $\gamma$ is the following path:




This formula defines an analytic function over the set $P_k = \{ x \mid \Re(x) > k \}$, which is of exponential type $r$. If $g$ is entire, then by Cauchy's theorem it follows that $\mathcal{L}(g) = 0$. The Laplace transform of two Borel transforms of a function $f$ is thus the same and, as the Borel transform may be choosen to be of any type $k > 0$, is defined over the set $P$. Moreover, we have the identity:

$$\forall x \in P \qquad \mathcal{L}\big(\mathcal{B}(a)\big)(x) = a(x).$$

From this we may now ensure the unicity of the function $R$ that is solution of our difference equation. We have the following theorem:

**Theorem 1.** *Let $a$ be an analytic function over the set $P$ that is of exponential type $\alpha < 2\pi$. The difference equation $R(x) - R(x+1) = a(x)$ admits a unique analytic solution over $P$ that is of exponential type $\alpha$ (denoted as $R_a$), satisfying $\int_1^2 R_a(t)\, dt = 0$.*

Getting back to the difference equation, we apply the Borel transform to the relation between $R$ and $a$ to get $\mathcal{B}_d(R)(\xi) - e^{-\xi}\mathcal{B}_{d'}(R)(\xi) = \mathcal{B}(a)(\xi)$ where $d'$ is the half-line obtained from $d$ by a translation $z \to z + 1$. As mentioned before, we can write

$$\mathcal{B}_d(R)(\xi) - e^{-\xi}\mathcal{B}_{d'}(R)(\xi) = (1-e^{-\xi})\mathcal{B}_d(R)(\xi) + f(x) = a(x)$$

where $f$ is entire. We then apply the Laplace transform to this equality into

$$R(x) = \int_\gamma e^{-x\xi}\frac{1}{1-e^{-\xi}}\mathcal{B}(a)(\xi)\,d\xi - \int_\gamma e^{-x\xi}\frac{1}{1-e^{-\xi}}f(\xi)\,d\xi.$$

As $f$ is entire, the second term of the right member is equal to the residue of $e^{-x\xi}\frac{1}{1-e^{-\xi}}f(\xi)$ in 0 and therefore is a constant equal to $f(1)$. Hence, we have found a solution of the equation that is of the same exponential type as $a$:

$$R(x) = \int_\gamma e^{-x\xi}\frac{1}{1-e^{-\xi}}\mathcal{B}(a)(\xi)\,d\xi - f(1).$$

The only other exponential solutions of order less than $2\pi$ are obtained from this one by adding a constant value, since every entire periodic function of period 1 with an exponential growth less than $2\pi$ is a constant. It is easily checked, using Fourier's formulas that if $a$ is entire of exponential type $\alpha < 2\pi$, each of its Fourier coefficients, except the constant one, are zeros.

From the Borel transform properties, it follows that the function $x \longmapsto -\int_\gamma e^{-x\xi}\frac{1}{\xi}\mathcal{B}(a)(\xi)\,d\xi$ is an antiderivative function of $a$. Hence, the following function is another solution of the difference equation:

$$R(x) = -\int_1^x a(t)\,dt + \int_\gamma e^{-x\xi}\left(\frac{1}{1-e^{-\xi}} - \frac{1}{\xi}\right)\mathcal{B}(a)(\xi)\,d\xi.$$

This last solution does not depend any more on the choice on the Borel transform of $a$. Furthermore, this function satisfies $\int_1^2 R(t)dt = 0$ and thus is the unique solution of our problem. We can then define the Ramanujan summation of a series of general term $a(n)$ as the following:

$$\sum_{n\geq 1}^{[\mathcal{R}]} a(n) = R(1) = \int_\gamma e^{-\xi}\left(\frac{1}{1-e^{-\xi}} - \frac{1}{\xi}\right)\mathcal{B}(a)(\xi)\,d\xi.$$

It is then easy to see that this sum is a linear functional of $a$.

## 4. **Examples and Properties**

For the following functions, we present solutions of the difference equation of exponential order less than $2\pi$, the value of their integral from 1 to 2, and finally the Ramanujan sum of the serie $a(n)$. We will use the Riemann zeta function given for all $x > 0$ and $\Re(z) > 1$ by

$$\zeta(x,z) = \sum_{n=0}^{+\infty}\frac{1}{(n+x)^z}, \qquad \zeta(z) = \zeta(1,z) = \sum_{n\geq 1}\frac{1}{n^z}.$$

| $a(x)$ | $R(x)$ | $\int_1^2 R(t)dt$ | $\sum_{n\geq 1}^{[\mathcal{R}]} a(n)$ |
|---|---|---|---|
| $\frac{1}{x^z}$ | $\zeta(x,z)$ | $\frac{1}{z-1}$ | $\zeta(z) - \frac{1}{z-1}$ |
| $x^k$ | $-\frac{B_k(x)}{k+1}$ | $-\frac{1}{k+1}$ | $\frac{1-B_{k+1}}{k+1}$ |
| $\ln x$ | $-\ln\Gamma(x)$ | $1 - \frac{1}{2}\ln(2\pi)$ | $-1 + \frac{1}{2}\ln(2\pi)$ |
| $e^{\alpha x}$ | $\frac{e^{\alpha x}}{1-e^{\alpha}}$ | $-\frac{e^{\alpha}}{\alpha}$ | $e^{\alpha}\left(\frac{1}{1-e^{\alpha}} + \frac{1}{\alpha}\right)$ |

The first example shows that even if the series is convergent, then we do no have its sum in the usual sense equal to its Ramanujan sum. In fact, we have the following proposition:

**Proposition 1.** *If $R(x)$ tends to a finite limit when $x \longrightarrow +\infty$, then the series $\sum_{n\geq 1} a(n)$ is convergent, and we have:*

$$\sum_{n\geq 1}^{[\mathcal{R}]} a(n) = \sum_{n=1}^{+\infty} a(n) - \lim_{n\to+\infty} \int_1^n a(t)\, dt.$$

It is thus possible to regard the Ramanujan summation scheme as a convenient renormalisation of the usual summation scheme, where enough terms have been subtracted from the usual sum in order to ensure that the result converges at points where it usually diverges (see the example of the function $\zeta$ to the point $z = 1$).

The last example shows another important point. From this last Ramanujan sum, one can compute that

$$\sum_{n\geq 1}^{[\mathcal{R}]} \sin(nt) = \frac{1}{2i}\left(\sum_{n\geq 1}^{[\mathcal{R}]} e^{it} - \sum_{n\geq 1}^{[\mathcal{R}]} e^{-it}\right) = \frac{1}{2}\cot\frac{t}{2} - \frac{\cos t}{t}.$$

Then, if we take $t = \pi$, we get $\sum_{n\geq 1}^{[\mathcal{R}]} \sin(nt) = \frac{1}{\pi}$ whereas $\sum_{n\geq 1}^{[\mathcal{R}]} 0 = 0$. This example shows that the Ramanujan summation depends on the function chosen to represent the sequence we want to sum. In fact, if $a$ and $b$ are two functions that are of exponential type $\alpha < \pi$, if $a(n) = b(n)$ for all $n \geq 1$, then $a = b$, due to a theorem by Carlson [4].

We now have the following properties of Ramanujan summation, considering some classical operations:

**Translation.** The following holds to compute the sum of the series from the $N$th element:

$$\sum_{n\geq 1}^{[\mathcal{R}]} a(n) = a(1) + \cdots + a(N-1) + \sum_{n\geq 0}^{[\mathcal{R}]} a(n+N) - \int_1^N a(t)\, dt.$$

**Derivability.** Considering the solution $R$ as a function of $a$, we get

$$R_{\partial^n a} = \partial^n(R_a) + \partial^{n-1} a(1).$$

As an application of this formula, we have the following (the functions $R$ here are defined up to one constant):

$$\begin{aligned} a(x) = \ln x \quad &\Longrightarrow \quad R(x) = \ln\Gamma(x), \\ a(x) = \tfrac{1}{x} \quad &\Longrightarrow \quad R(x) = \tfrac{\Gamma'(x)}{\Gamma(x)}. \end{aligned}$$

As $\int_1^2 \frac{\Gamma'(t)}{\Gamma(t)}\, dt = \ln\Gamma(2) - \ln\Gamma(1) = \ln(1) - \ln(1) = 0$, we have proved that

$$\sum_{n\geq 1}^{[\mathcal{R}]} \frac{1}{n} = \frac{\Gamma'(1)}{\Gamma(1)} = \gamma.$$

**Summation by parts.** If $a$ and $b$ are both exponential of type less than $\pi$:

$$\sum_{n\geq 1}^{[\mathcal{R}]} a(n)\sum_{k=1}^{n} b(k)+\sum_{n\geq 1}^{[\mathcal{R}]} b(n)\sum_{k=1}^{n} a(k)=\sum_{n\geq 1}^{[\mathcal{R}]} a(n)b(n)+\sum_{n\geq 1}^{[\mathcal{R}]} a(n)\sum_{n\geq 1}^{[\mathcal{R}]} b(n)+\int_1^2 R_a(t)R_b(t)\,dt.$$

This formula is in particular interesting when we take $a(x)=1$ for all $x$. The formula then gives:

$$\sum_{n\geq 1}^{[\mathcal{R}]}\sum_{k=1}^{n} b(k)=\frac{3}{2}\sum_{n\geq 1}^{[\mathcal{R}]} b(n)-\sum_{n\geq 1}^{[\mathcal{R}]} nb(n)-\sum_{n\geq 1}^{[\mathcal{R}]}\partial^{-1}b(n).$$

with $\partial^{-1}b(n)=\int_1^n b(t)dt$. From this last formula, we compute the following harmonic Ramanujan sum where $H_n=1+\cdots+\frac{1}{n}$:

$$\sum_{n\geq 1}^{[\mathcal{R}]} H_n=\frac{3}{2}\gamma+\frac{1}{2}-\frac{1}{2}\ln(2\pi)$$

**Analytic dependence on a variable.**

**Proposition 2.** *Let $D$ be an open set in $\mathbb{C}$ and let the function $a(z,x)$ be analytic in $D\times P$. Suppose for each compact set $K$ in $D$, there exists $C_K$ and $\tau_K$ such that for all $x$ with $|x|>1$, and all $z$ in $K$, we have $\left|a(z,x)\right|\leq C_K e^{\tau|x|}$. Then $z\longmapsto\sum_{n\geq 1}^{[\mathcal{R}]} a(z,n)$ is analytic in $D$ and we have:*

$$\partial_z\left(\sum_{n\geq 1}^{[\mathcal{R}]} a(z,n)\right)=\sum_{n\geq 1}^{[\mathcal{R}]}\partial_z a(z,n).$$

It follows from this theorem that the function $z\longmapsto\sum_{n\geq 1}^{[\mathcal{R}]}\frac{1}{n^z}$ is entire. For all $z$ in $\mathbb{C}$, we have:

$$\sum_{n\geq 1}^{[\mathcal{R}]}\frac{1}{n^z}=\zeta(z)-\frac{1}{z-1}\qquad\sum_{n\geq 1}^{[\mathcal{R}]}\frac{(\ln n)^k}{n^z}=(-1)^k\zeta^{(k)}(z)-\frac{(k-1)!}{(z-1)^k}.$$

These formulas remain true when $z$ assumes the limit value 1.

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