# Jacobi's Ideas on Eigenvalue Computation in a modern context

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#### **General remarks**

 $Ax = \lambda x$ 

Nonlinear problem: for n > 4 no explicit solution Essentially iterative methods

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Masterthesis of Anjet de Boer, 1991, Utrecht

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## Early paper by Leverrier (1811-1877)

- Sur les Variations sèculaires des Eléments elliptiques
- des sept Planètes principales: Mercure, Vénus, la Terre, Mars,
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Perturbations to the orbits of planets caused by the

presence of other planets

linear eigensystem from system of 7 diff. equations

coefficients of characteristic polynomial

He neglected some small elements: factors of degree 3 and 4

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## Papers by Jacobi (1804-1851)

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New method for solution of sym. linear systems;

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He announces the application for eigenproblems

## Second paper by Jacobi

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Über ein leichtes Verfahren, die in der Theory der Säculärstörungen

vorkommenden Gleichungen numerisch aufzulösen, 1846

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He applied his 1845-method to the system studied by Leverrier
Claim: easier and more accurate method (unsupported)
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Bodewig (1951): Jacobi knew his methods before 1840 (inconclusive) evidence: letter of Schumacher to Gauss (1842)



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The Jacobi (rotation) method was forgotten, but J. described the two methods as one single algorithm



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joint work with Murray and Von Neumann



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**Bodewig** (1950, 1951) described the full J-method He claimed the rediscovery •

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Jacobi in Matrix Notation

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(1) First plane rotations to make A diag. dom. suppose that  $a_{1,1}$  is largest element then  $\lambda \approx a_{1,1}$  and  $x \approx e_1$  ( $Ax = \lambda x$ ) (2) Consider orthogonal complement of  $c_1$ :  $A\begin{pmatrix} 1\\ w \end{pmatrix} = \begin{pmatrix} a_{1,1} & c^T\\ c & F \end{pmatrix} \begin{pmatrix} 1\\ w \end{pmatrix} = \lambda \begin{pmatrix} 1\\ w \end{pmatrix}$ 

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leads to

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 $\lambda = a_{1,1} + c^T w$  $(F - \lambda I)w = -c$ 

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start with w = 0,  $\theta = a_{1,1}$ Solve w from  $(F - \theta I)w = -c$  with G-J iterations

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Goldstine suggested J's rotations only for proving real eigenvalues

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## **Krylov subspaces (1)**

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Krylov suggested in 1931 the subspace:

 $K_m(A;x) = \operatorname{span}\{x, Ax, \dots, A^{m-1}x\}$ 

for some convenient starting vector  $\boldsymbol{x}$ 

for construction of characteristic polynomial

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illconditioned basis, but in his case: m = 6

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How to make things work for large m?

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Start with  $v_1 = x/||x||$ Form  $Av_1$  and orthogonalize w.r.t.  $v_1$ Normalize:  $v_2$  (so far nothing new!)

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Orthogonalize w.r.t  $v_1$ ,  $v_2$  and normalize:  $v_3$ 

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**Krylov subspaces (3)** 

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The general step is:

Form  $Av_i$ , Orth. w.r.t  $v_1, \ldots, v_i$ , normalize:  $v_{i+1}$ 

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Results in well-conditioned basis (Stewart, SIAM books)

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A symmetric: LANCZOS METHOD (1952)

A unsymmetric: ARNOLDI METHOD (1952)

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# **Davidson's subspace**

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Davidson opens ways for other subspaces

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# **Davidson - num. analysis**

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Idea: apply preconditioner instead of Jacobi rotations and

use Jacobi's idea for new update of z

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In Jacobi's case:  $e_1$  is the approximation for x

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Sleijpen en VDV (1996): compute update in  $z^{\pm}$ 

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 $(A - \theta I)$  restricted to  $z^{\perp}$  is given by  $B = (I - zz^*)(A - \theta I)(I - zz^*)$ 

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Expand subspace with (approx.) solution of Bt = rJacobi-Davidson method, SIMAX 1996

Newton method for RQ

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n = 100, A = tridiag(1, 2.4, 1) $x = (1, 1, ..., 1)^T$ 

$$n=100$$
,  $A={
m tridiag}(1,2.4,1)$   
 $x=(1,1,\ldots,1)^T$ 

Davidson:  $M_k = A - \theta_k I$ : stagnation

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Davidson, prec. with GMRES(5) for  $(A - \theta_k I)\tilde{t} = r$ : slow convergence (since  $\theta_k \approx \lambda$ )

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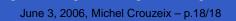
Davidson, prec. with GMRES(5) for  $(A - \theta_k I)\tilde{t} = r$ : slow convergence (since  $\theta_k \approx \lambda$ )

Jac.Dav., GMRES(5) for  $F\tilde{t} = r$  with  $F = (I - zz^T)(A - \theta_k I)(I - zz^T):13$ 

Note that F has no small eigenvalues

Acoustics, attachment line:

 $Ax + \lambda Bx + \lambda^2 Cx = 0$ 



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### For problem coming from acoustics:

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#### **Results for interior isolated eigenvalue (resonance)**

### on a CRAY T3D

Processors	Elapsed time (sec)
16	206.4
32	101.3
64	52.1

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For n = 274625, on 64 processors: 93.3 seconds

1 invert step  $\approx 3$  hours