# The Dimension of the Symmetric $k$-tensors

Cesar O. Aguilar

1.1 Proposition: *Let $V$ be a $n$-dimensional vector space. The dimension of the symmetric $k$-tensors on $V$ is*

$$\dim S_k(V) = \binom{n+k-1}{k}.$$

Proof: Let $\sigma \in S_k(V)$ and write $\sigma = \sigma_{i_1\cdots i_k}\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$ for some basis $\{\varepsilon^1, \ldots, \varepsilon^n\}$ of $V^*$. By definition of a symmetric tensor, the coefficients $\sigma_{i_1\cdots i_k}$ are unchanged by any permutation of the indices $i_1, \ldots, i_k$. Thus we need to determine the number of $k$ combinations, with possible repetitions, of the integers $\{1, 2, \ldots, n\}$.

1.2 Lemma: *Let $A$ be an alphabet containing $k$ distinct letters. The first letter will be denoted by 1, the second letter will be denoted by 2, etc. The number of all possible words of length $n$ containing $n_1$ times the first letter, $n_2$ times the second letter, ..., $n_k$ times the $k$th letter is*

$$P(n; n_1, n_2, \ldots, n_k) := \frac{n!}{n_1!n_2!\cdots n_k!},$$

*where $n = n_1+n_2+\cdots+n_k$ (the length of the word). In other words, the number of permutations of $n$ elements with repetitions as described above is $P(n; n_1, n_2, \ldots, n_k)$.*

Proof: Let $w$ be a word and let us distinguish the different copies of a letter in $w$ with subscripts, that is, write $w = 1_1 1_2 \cdots 1_{n_1} 2_1 2_2 \cdots 2_{n_2} \cdots k_1 k_2 \cdots k_{n_k}$. Let us generate the number of permutations of this $n$ letter word as follows: 1) choose the position of each kind of letter, then 2) choose an ordering of the subscripts of the first letter, then 3) choose an ordering of the subscripts of the second letter, ..., and finally $k+1$) choose an ordering of the subscripts of the $k$th letter. The first step can be done in $P(n; n_1, \ldots, n_k)$ ways, the second step in $n_1$ ways, the third in $n_2$ ways, ..., the last step in $n_k!$ ways. Thus,

$$n! = P(n; n_1, \ldots, n_k)n_1!n_2!\cdots n_k!.$$

■

1.3 Lemma: *The number of combinations of $n$ objects taken $k$ at a time with repetition is*

$$P(n+k-1; k, n-1) = \binom{n+k-1}{k}.$$

Proof: The number of combinations of $n$ objects taken $k$ at a time with repetition is equal to the number of ways $k$ identical objects can be distributed among $n$ distinct containers. The latter is equal to the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$. Write a solution to $x_1 + x_2 + \cdots + x_n = k$ as

$$\underbrace{|||\cdots||}_{x_1 \text{ times}} + \underbrace{|||\cdots||}_{x_2 \text{ times}} + \;\cdots\; + \underbrace{|||\cdots||}_{x_n \text{ times}}.$$

The number of such solutions is an arrangements of $k$ "bars" $|$ and $(n-1)$ "+" signs, which is equal to $P(n+k-1; k, n-1) = \frac{n+k-1}{k!(n-1)!} = \binom{n+k-1}{k}$. ■

Using the above lemmas, the proof of Proposition 1.1 is now easy. The number of distinct coefficients $\sigma_{i_1\cdots i_k}$ is exactly the number of combinations from $n$ objects taken $k$ at a time with repetition, which is the number we claim to be. ■