Laser Beams and Resonators: Beyond the 1960s

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Abstract—**This paper describes the continuing advances in laser resonators and optical beam propagation that emerged in the decades following the 1960s, growing out of the fundamental concepts from that era reviewed in an earlier paper. It also presents a brief look forward at some of the continued innovations now emerging in these areas.**

Keywords— **Optical resonators, laser beams, history.**

I. Introduction

All or nearly all the basic concepts associated with real stable and unstable optical resonators and with the propagation of real gaussian beams were in place by the mid to late 1960s, as described in the first of this pair of reviews [1].

The development of new concepts related to resonators and beam propagation nonetheless continued at a rapid pace through the following three decades, and continues even today, motivated by new kinds of lasers, by requirements for more complex and multielement laser resonators, by considerations of laser beam propagation in practical applications, and by a continuing desire for basic understanding of optical beams. This second paper attempts to provide an overview of the most important of these further advances from the late 1960s onward.

II. Beyond the 1960s: "Complex and Twisted"

A significant fraction of the developments in optical resonators and beam propagation from the late 1960s onward can be described as "complex," "twisted," and even "nonnormal" in character, to use only slightly hyperbolic terms whose meanings will emerge in the following sections.

A. Complex and Twisted Paraxial Optics

During the late 1960s laser researchers became increasingly interested in gain-guided lasers, along with lasers having transversely tapered internal apertures and laser mirrors having transversely tapered reflectivity. It was soon realized that all these systems could be described using existing analytical tools simply by converting most of the real-valued parameters in gaussian beam theory into complex-valued quantities.

At about the same time some optics researchers also became interested in optical structures characterized by general astigmatism or "twist" about the axis of propagation. This was motivated in part by the development of nonplanar ring resonators as described below, along with optical isolators or optical diodes employing Faraday rotation.

The intellectual history of these combined developments beginning in the late 1960s is nearly as "complex and twisted" as the topics themselves. The basic ideas emerged in overlapping papers from many different authors, making it difficult to trace primary origins in the extensive literature that has developed. The following summary therefore cannot always identify just which individuals contributed which concepts.

B. Complex Ducts and Complex Gaussian Beams

The concept of an elementary gaussian beam as a spherical wave diverging from a complex-valued source point emerged early in the laser era, as did the recognition that one could combine the wavefront radius of curvature R and gaussian spot size w into a single complex-valued gaussian $\tilde{q}$ parameter defined by $1/\tilde{q} \equiv 1/R - j\lambda/\pi w^2$. The conceptual relationship between narrow optical beams and complex-valued eikonal rays was also recognized in more general approaches to optical propagation [2-6].

Early discussions of propagation through parabolic gain-guided or "complex parabolic" lensguides were given in 1965 by Kogelnik [7] and a few years later by Casperson and Yariv [8]. A slightly later discussion of laser resonators having transversely varying mirror amplitude reflectivity ("VRM resonators") was published by Zucker [9]. The author also made an early suggestion for using Hermite-gaussian modes with complex argument as free-space eigenfunctions [10].

Further advances came as laser workers realized that an aperture with a parabolic tranmission profile was formally equivalent to a thin lens with an imaginary focal length, and a mirror with a parabolic reflectivity profile was formally equivalent to spherical mirror with an imaginary-valued radius of curvature. More generally, systems with parabolic transverse gain or loss profiles were recognized as formally equivalent to optical fibers or ducts having imaginary-valued transverse index variations.

Essentially all the formal mathematical results for real-valued optical systems could then be converted to handle these complex-valued systems simply by adopting complex values for the ABCD matrix elements [11-13], the gaussian spot size w, and the arguments of the Hermite-gaussian and Laguerre-gaussian functions. One can refer to these complex-parabolic systems in general as complex gaussian ducts and their eigenmodes as complex gaussian beams, or if necessary one can speak of generalized complex-valued ducts for those more general cases where the transverse variations are not simply parabolic [14-18].

Figure 1 is a schematic illustration of a complex-gaussian beam with a fixed amplitude profile propagating in a complex duct. A particularly important aspect of complex paraxial optics is that systems with positive gain or loss guiding can still support transversely confined and guided gaussian modes even in the presence of divergent or unsta-

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ble mirrors, as demonstrated by Casperson and Yariv [8]. (Note that the essential point is that the system have higher transmission on axis and decreasing trasnmission off axis, whether this is accomplished by radially decreasing gain or radially increasing attenuation.)

Casperson also appears to have been the first to note that there was an important distinction between tranverse confinement and perturbation stability for complex gaussian systems [19-23]. Complex gaussian concepts were also applied to one-dimensional modes in gain-guided semiconductor diode lasers [24], to free-space propagation and to other forms of complex gaussian resonators [25-30], and to the propagation of partially coherent light [31-33]. Additional work along these lines has also continued into the past decade [34-42]

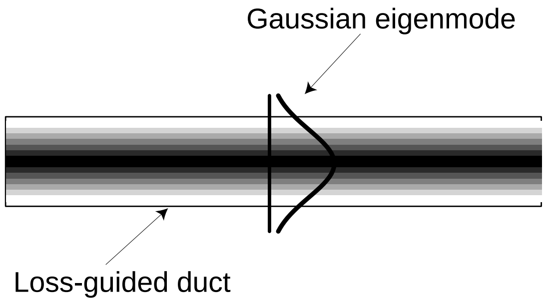


Fig. 1. Schematic drawing of the lowest-order complex-gaussian beam in a loss-guided or gain-guided duct. The lowest and higher-order modes have diverging phase fronts (not shown), but maintain fixed intensity profiles.

The variable reflectivity mirror (VRM) concept which was pioneered by Zucker [9], used in combination with a geometrically unstable resonator design [43], provides perhaps the best current design approach for obtaining the desired combination of large mode volume, moderate output coupling, and good mode discrimination and beam quality in higher-power and higher-energy lasers [37, 38, 44-51]. This approach does, however, remain somewhat hampered by difficulties in obtaining the necessary variable-reflectivity mirrors.

C. *Twisted Beams and General Astigmatism*

For the elementary case of an optical system having ordinary astigmatism, one can simply make a separation of variables along the transverse principal axes of the system, and apply ordinary gaussian beam arguments (complex-valued if appropriate) separately along the resulting orthogonal x and y axes.

Suppose, however, that one assembles an optical resonator or lensguide containing astigmatic or anisotropic paraxial elements, for example cylindrical lenses and Brewster plates, with the principal axes of individual simple astigmatic elements rotated about the system axis by various angles other than multiples of 90 degrees. The result is a general astigmatic system, or an optical system with an inherent "twist". Twisted optical systems of this sort have nonorthogonal principal axes and their transverse eigenmodes are characterized by image rotation and polarization rotation properties.

Optical beams propagating in free space can also possess an inherent "twist" independent of any optical system (although some sort of twisted optical system may be necessary to create a twisted free-space optical beam). Even in a lowest-order gaussian beam, for example, the principal axes of the gaussian intensity profile may be rotated by an arbitrary angle from the principal axes of the toroidal rather than spherical phase profile. The principal axes for the intensity profile of such a beam may then rotate by up to 180 degrees as the beam propagates from $z = -\infty$ to $+\infty$. Such beams will then no longer have a unique waist, or even two unique orthogonal waists where the transverse spot sizes are minimum and the wavefront curvatures simultaneously flat. One can also create twisted optical beams in which the intensity profile does not visibly twist but in which there is a twisted or helical character to the phase profile within the beam [52-57], leading in general to an increased diffraction spreading for the beam.

Recent decades have brought forth extensive discussions of twisted optical systems and twisted optical beams. Much of the early literature in this area is associated with non-planar ring resonators as described elsewhere in this paper. Of note are publications by Arnaud [5, 52, 58-63]; and much fundamental work on the classification and analysis of stigmatic, simple astigmatic, and general astigmatic beams and systems has been done by Nemes [64-70].

In the most general case, where the optical system and the propagating beam may have both complex-valued and twisted (general astigmatic) character, the ray matrices [71] must be expanded from four potentially complex ABCD coefficients (actually three independent values) to four complex-valued two-by-two tensor coefficients **ABCD** [52, 55, 57] with up to ten independent elements [65, 67-69, 72]. The complex-valued gaussian $\tilde{q}$ value similarly becomes a complex-valued tensor quantity $\tilde{\mathbf{Q}}$ [12, 58, 73, 74]. Methods for transforming the properties of general astigmatic beams in various ways using general astigmatic systems are of interest [66, 75] as are the angular momentum properties of twisted beams. Additional papers in this area include [54, 76-85]. A recent overview of optical twist has been given by Friberg [86].

III. Additional Advances beyond the 1960s

A. *Numerical Beam Propagation: Fast Transforms*

Fox and Li's original calculations employed straightforward numerical integration of the Huygens' integral for N sample points across each transverse dimension. The same propagation calculations can also be carried out using gaussian mode expansions [87], but both of these methods are computationally intensive and sensitive to round-off errors. From another viewpoint, Huygens's integral in scalar form is a convolution integral which can be evaluated using Fourier transform methods [88, 89]. The required Fourier transforms for a uniformly sampled beam profile can then be converted to discrete Fourier transforms, which can be evaluated using the Fast Fourier Transform (FFT) algorithm. The merits of this FFT approach include not only reducing the N^2 computational problem for one transverse dimension to an $N \log N$ problem, where N is the number of sample points needed to accurately describe the beam profile, but also substantial reductions in required computer storage and especially in round-off errors.

The first applications of the FFT algorithm to optical resonator and beam calculations were made by Sziklas and Siegman [90, 91] and by Johnson [92], and this has now become the standard approach for commercial propagation codes, for example those of Lawrence [34, 93-95]. A quasi "Fast Hankel Transform" (FHT) algorithm which uses the FFT algorithm internally and captures many of its advantages, was also developed a few years later for propagation calculations in cylindrical coordinates [34, 96-98].

B. Finding Higher-Order Modes: The Prony Method

The Fox and Li method in its simplest form eventually converges to the lowest-order or lowest-loss eigenmode of a resonator, but convergence can be slow if there are multiple modes with very similar losses. In some cases one can also extract individual higher-order modes, for example with the convergence method illustrated in Fig. 2 of [1], or with the resonance excitation approach used in later calculations by Fox and Li [99]. Powerful numerical algorithms for extracting all the eigenvalues of large complex matrices can also be applied to the resonator problem [100].

The general problem in any case is to fit the results from repeated round trips inside the resonator with a superposition of exponentially decaying complex-valued eigenmodes. This problem arises in other situations as well, and is known to be mathematically difficult. One relatively simple procedure for fitting N round trips exactly with N decaying exponentials was invented by the Baron de Prony in 1795 [101] and introduced serendipitously to the author by Tuttle [102] in the late 1960s. This Prony method has since been applied to the calculation of optical resonator eigenmodes by a number of workers [103-108].

It is also possible to apply the Fox and Li approach to nonlinear and multimode problems, for example competition between transverse modes in a laser oscillator. Klein and coworkers [109] accomplished this by carrying forward two simultaneous but independent Fox and Li calculations representing two possible modes inside the same resonator, together with a gain sheet that saturated based on the total intensity profile of both modes. They observed that, in accurate mimicry of a real laser, under some circumstances one array built up to a dominant mode while the other decayed to noise level as a result of mode competition, whereas in other cases both arrays built up to steady-state levels in two different transverse modes. The same technique has subsequently been used by others [110].

C. Operator Methods and Quantum Analogies

A gaussian beam propagating off axis in a parabolic index profile is formally equivalent to a gaussian quantum wave packet in a parabolic potential well. A number of workers have exploited the close formal analogies between paraxial beam propagation and quantum theory by applying quantum operator techniques and algebraic methods as alternative approaches to paraxial beam and resonator problems [26, 27, 111-122].

D. Nonnormal Modes

As first noted by Fox and Li, the transverse eigenmodes of an open-sided or gain-guided lensguide or resonator are not power-orthogonal in the usual sense, and thus do not form the usual set of "normal modes" characteristic of most other linear systems. Rather, the resonator eigenmodes are biorthogonal to a set of adjoint modes which correspond physically to eigenmodes propagating in the reverse direction along the same waveguide or around the same resonator [63, 123-128]. The purely real Hermite-gaussian or Laguerre-gaussian functions that are widely used to analyse elementary stable cavities do form complete and orthonormal sets; but these functions are only approximations (though often extremely good approximations) to the slightly nonorthogonal exact eigenmodes in these cavities. The complex-gaussian transverse modes in complex-gaussian laser cavities and also the eigenmodes of unstable resonators are distinctly nonorthogonal in character. As a general principle, purely index-guided systems such as conventional optical fibers have normal modes, but gain-guided, loss-guided, or finite open-sided systems have nonnormal modes.

The nonnormal character of optical resonator eigenmodes leads to a number of nonstandard and quite nonintuitive mathematical and physical properties for these systems. For example, the total power or energy in a nonnormal lensguide or resonator is not simply the sum of the powers or energies in the individual modes, even if the eigenmode functions are individually normalized. In addition, maximum excitation of a given eigenmode by an externally injected signal is obtained not with the usual matched coupling but with so-called "adjoint coupling" [42, 129-133]. Injecting a reversed or adjoint wave into a nonnormal system, as shown for a gain-guided duct in the upper part of Fig. 2, causes the lowest-order eigenmode to be excited with more power or energy than the input wave itself, as shown by the extrapolated portion of the $n = 0$ line in the lower part of Fig./ 2. This adjoint coupling to the lowest mode is necessarily accompanied by simultaneous excitation of other higher-order eigenmodes which then die out relative to the lowest-order mode

The second quantization procedure that is widely used for quantizing the electromagnetic fields of a laser oscillator in most fully quantum laser analyses [134] is also called into serious question in nonnormal optical resonators [132, 133]; and the procedure for expanding the fields in an optical resonator or lensguide in terms of its eigenmodes must also be substantially modified from the usual least-squares normal-mode expansion [29, 42]. The Rayleigh Ritz procedure used for estimating eigenvalues in many other systems also cannot be applied to optical resonator modes [135].

Finally, laser oscillators having distinctly nonorthogonal modes will experience a substantial excess spontaneous emission per mode or excess quantum noise factor, lead-

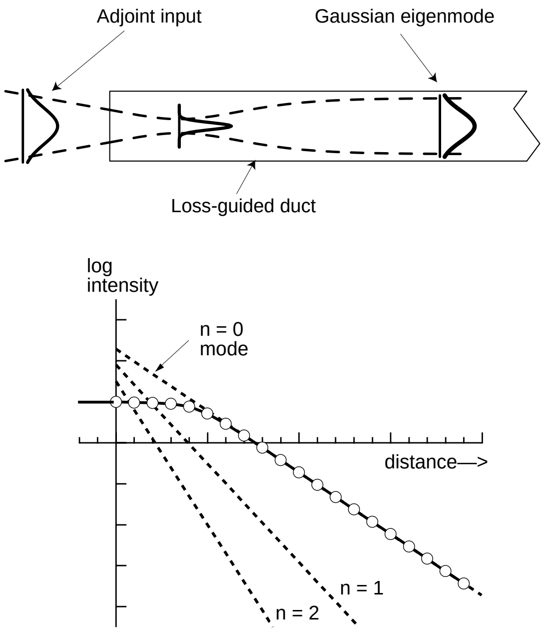


Fig. 2. Maximum excitation of the lowest-order eigenmode in a complex-gaussian duct is obtained not by the usual mode-matched coupling, but by so-called adjoint coupling.

ing to a corresponding increase in their Schawlow-Townes linewidth. This excess noise effect was first proposed by Petermann for gain-guided diode lasers [136], and has since been shown by the author to be entirely determined by the nonnormal or nonorthogonal character of the eigenmodes [137-139]. Large increases in the Schawlow-Townes linewidth in otherwise conventional lasers with nonorthogonal transverse modes have been confirmed by groups at Stanford [140-143], Leiden [144-146], and Rennes [147]. Recent work has shown that the polarization eigenmodes in certain twisted-mode cavities are also nonnormal or nonorthogonal in polarization space, leading to similarly increased quantum noise properties when used as laser oscillators [145, 148, 149].

E. Nonplanar Ring Resonators

A ring resonator having four or more mirrors can be converted into a nonplanar ring resonator by warping the ring, for example by lifting any one of the mirrors out of the plane formed by any three of the other mirrrors. At some time in the early 1960s it was realized that as an optical beam travels around such a nonplanar ring the beam profile will rotate or twist about the direction of propagation by an amount that depends on the nonplanarity of the ring. The polarization axes for the beam will also rotate about the propagation axis by a related amount. Both effects are manifestations of Berry's geometrical phase [150].

Early papers on nonplanar ring resonators for lasers include those of Danileiko and Lobachev, Popov, Arnaud, and subsequent authors [59-61, 151-157]. The image rotation aspect of these resonators has led to a useful class of unstable ring resonators, sometimes called "UR90" resonators [158-168], in which the image rotation around the ring path is 90 degrees and the output beam emerges as a single completely filled rectangular beam, in contrast to the annular output beam from a conventional unstable resonator.

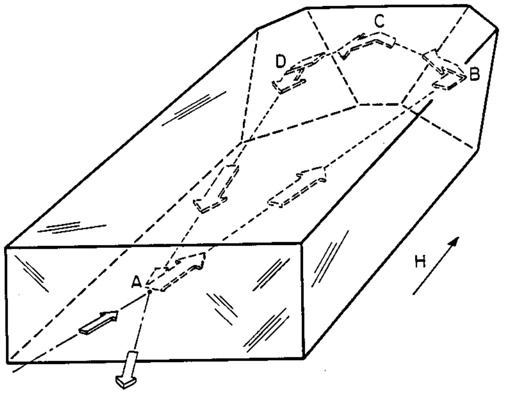


Fig. 3. Monolithic nonplanar ring oscillator (NPRO) (from Kane and Byer).

The polarization rotation properties of nonplanar rings have been employed to create more complex types of ring laser gyroscopes [169-173]. They have also been used in conjunction with a small amount of Faraday rotation to create unidirectional ring laser oscillators [174, 175], notably the very compact and stable monolithic nonplanar ring oscillator (NPRO) of Kane and Byer [176, 177] shown in Fig. 2. Nonplanar rings also display modified misalignment or axis stability properties as analyzed by several authors [59, 178-181].

F. Lasers and the Sagnac Interferometer

The Sagnac interferometer, aka the "antiresonant ring," is another classic optical element that has found widespread applications in lasers. In addition to prism end reflectors [182, 183], it has been employed as a laser end mirror for laser cavity dumping and switching and for active and passive (CPM) mode locking [184-194]. With nonlinear elements inserted it can also be used for saturation spectroscopy and nonlinear optics [195-199]. The availability of fiber couplers and ring resonators has extended these concepts to fiber optic implementations [200, 201]. Most recently the Sagnac interferometer has emerged as a promising concept for laser-interferometer gravity wave observations or LIGO [202-204].

G. Chaotic Resonators

Stable resonators and lensguides are designated as such in part because their ray paths, including even the off-axis ray paths, are stable against small perturbations in initial position or direction. There are elementary closed geometrical shapes or closed resonators, however, in which all the internal ray paths that might be taken by a bouncing light beam or a classical billiard ball are completely chaotic or perturbation-unstable in character. That is, no matter how close together two rays may start out in such a structure, their trajectories will eventually diverge in a chaotic fashion. The statistical properties of the rays and resonant modes in these chaotic structures are of substantial interest to workers in chaos and nonlinear dynamics.

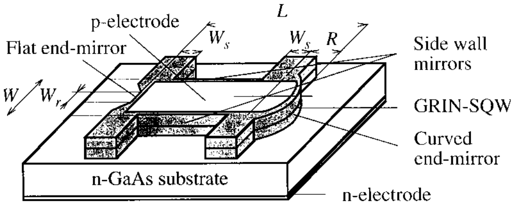


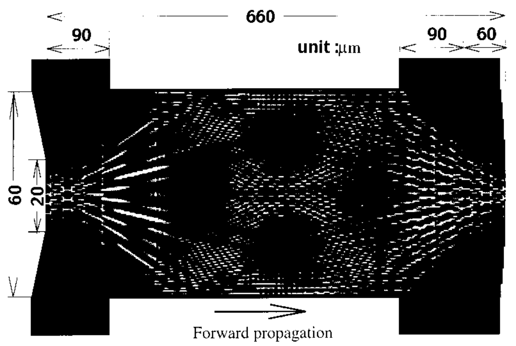
Fig. 4. Chaotic stadium resonator with internal gain created by reactive-ion etching on the surface of a GaAs diode laser.

It can be of interest, therefore, to see if there are any connections between this chaotic ray behavior and the oscillation modes that will be estabished in such a structure if the structure can be somehow filled with a laser gain medium. One widely discussed chaotic structure is the "stadium billiard" or stadion [205-208] which consists essentially of two cylindrical or spherical end caps forming a one-dimensional unstable optical resonator with side walls added. Preliminary experiments to study the oscillation behavior in such a stadium structure have recently been carried out by using reactive ion etching techniques to fabricate a "stadium laser" on the surface of a semiconductor laser as shown in Fig. 4 [40, 209]. The resulting oscillation properties of this chaotic stadium-resonator laser as shown in Fig. 5 can then be observed and explained in some detail using an extended Fox and Li analysis [40, 210]. There appear to be possibilities for further exploration of this subject [211].

H. Laser Arrays, Higher-Order Modes, and Talbot Mirrors

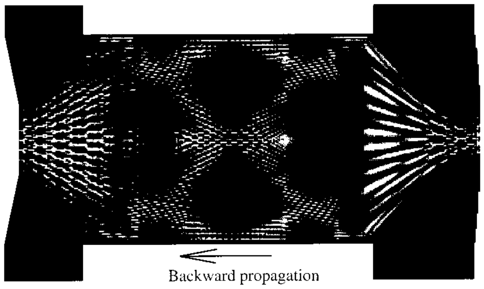
The lowest-order gaussian mode in any conventional stable optical resonator has a mode diameter $d \sim (L\lambda)^{1/2}$ which for a length L on the order of a meter and a wavelength λ on the order of one micron means a mode diameter on the order of a millimeter or less. Laser gain media are readily available, however, in the form of laser rods or gas laser tubes with diameters of centimeters or more. Extracting all the available laser power or energy from these large-volume gain media while maintaining single-mode operation and thus good beam quality can, therefore, be a difficult problem, not to mention the damage problems associated with handling very large powers or energies in small diameter beams.

The unstable optical resonator offers one useful solution to this dilemma, although not without creating other difficulties; and much effort has gone into the search for other approaches to high power or high energy, large mode volume lasers. One approach is to not only allow but encourage laser oscillation in a single higher-order (and thus larger-diameter) cavity mode, and then to use some form of phase correction to convert the highly divergent but nonetheless fully coherent output beam from this laser into a more desirable beam profile. Laser oscillation in a single controlled higher-order mode has in fact been achieved in several types of lasers, using either various types of modal filters within the laser resonator or variations on Talbot mirrors [212] to select a single oscillating mode.



Forward propagation

(a)



Backward propagation

(b)

Fig. 5. Oscillation profile within the chaotic resonator of Fig. 4. The oscillating "scars" are different in the two directions along the structure.

Converting the resulting output beam into a truly high-quality beam is, however, a more uncertain proposition. Even if the near-field phase profile can be converted to an ideal uniphase wavefront so as to obtain a narrow central lobe in the far field, the transverse intensity variations inherently associated with a higher-order mode mean that a large amount of the total beam energy may still reside in weak but numerous side lobes surrounding this main central lobe.

Another generic approach that comes naturally to those familiar with phased array radars at microwave frequencies is to construct a large number of lower-power, small-diameter, single-mode lasers, and then attempt to combine their beams into a single phased array output. Much effort has gone into variations on this approach, using for example coupled laser oscillators with conventional cavities, or arrays of coupled waveguided lasers such as for example closely adjacent laser stripes on semiconductor diode laser chips.

One generic problem associated with this approach is that success requires that the individual oscillators operate not only all at exactly the same frequency but all with precisely controlled phases relative to each other—and not only are the dimensional tolerances involved in precise phase control much more severe at optical frequencies than at microwave frequencies, but the problems involved in obtaining adequately single-frequency operation are also much more difficult in lasers than in typical microwave devices. Locking together some significant number N of quasi independent lasers requires either potentially herculean control system efforts, or some form of optical coupling between the individual lasers in order to lock together oscillators that must already be highly stabilized in frequency. Even if the latter can be achieved the generic situation is that N coupled oscillators can potentially operate in any of N “super-modes” of the coupled system, only one of which is wanted, and the rest of which must be adequately suppressed or discriminated against—and even if this can be achieved the beam-quality or side-lobe problem mentioned above still remains.

A review of some of the ideas discussed in this section has been given by Pashinin [213].. I have not attempted to survey the very large literature devoted to the above topics, and to laser arrays in particular, in part because many of the publications are devoted to the complexities of array theory rather than modal concepts, in part because it is difficult to identify the seminal or key papers in the field—and in part because I have some concern that this general approach, despite the effort devoted to it, may not in the end provide the kind of results that are sought after.

IV. Conclusions and a Brief Look Forward

My objective in these two reviews has been to provide a historical look back at at least some of the most important earlier and continuing advances in laser resonators and laser beam propagation during the past four decades. In doing this I have slighted certain topics, including roof resonators and distributed-feedback and Bragg-mirror resonators as mentioned in the first review. Among developments in more recent decades, I have not attempted to review phase-conjugate resonators or resonators employing stimulated back-scattering, despite their interesting properties and extensive literature, in part because they fall more in the domain of nonlinear optics; and I have only given general comments on the broad topic of coupled laser resonators and arrays. I have also not attempted to cover recent developments in the obviously very important and promising subject of fiber lasers.

It is traditional to conclude any such look backwards in a given area with at least a brief look forward in the same area, and my own look forward will be brief. In addition to fiber lasers, particularly significant areas for continued progress in laser resonators and beams would seem to include at least the following topics.

A. *Custom Resonator Designs*

In the early years of the laser era research efforts focused on understanding the modal properties of simple resonator structures. In recent years the advent of more powerful computers, new ways of fabricating complex mirror shapes, and new concepts in tomography and adaptive optics have led to increased interest in the synthesis of customized resonator structures and especially of custom mirror profiles. The objective in these efforts is to achieve resonator modes having predetermined mode profiles, such as for example uniform or flat-topped mode profiles, that may be better suited to particular applications or to extracting more energy from a given laser medium.

Practical examples of this approach include mirrors with graded or nonspherical phase profiles to achieve individual passive “mode conjugation”, graded amplitude reflectivity profiles, diffractive optical elements, and various kinds of adaptive resonator mirrors. Major contributions to research in this area have come from groups led by Belanger [38, 214-217] and Leger [218-223]. Continued advances in this area of customized resonators and of passive mode conjugation can be expected.

B. *Diode Laser Resonators*

Semiconductor diode lasers, including both conventional horizontal stripe lasers and the newer vertical-cavity surface-emitting (VCSEL) lasers, represent a dominant portion of current laser research and also of the future market potential for lasers. Many of the resonator concepts described in this paper are also relevant to diode lasers (although this may not always be recognized by diode laser workers having only semiconductor fabrication backgrounds).

Resonator designs for diode lasers are constrained by the almost invisibly small size of diode lasers and by the need in most cases to fabricate the resonator structure directly on or within the diode laser itself. Great progress has been made in developing diode laser resonators under these constraints but many interesting challenges and possibilities remain.

C. *Laser Beam Quality*

The skeptical observer might note that over the past four decades the modal properties of practical laser devices have been much more frequently calculated than measured. The meaningful measurement of laser beam properties has, however, made substantial progress during the past decade with the emergence of the rigorously defined, second-moment-based “M-squared” parameter [224], together with many new instruments to measure laser beam properties.

The M^2 value in particular, although a useful parameter for characterizing a laser beam, is best viewed as providing only a “propagation factor” for a laser beam, rather than a unique or universal measure of laser beam quality, since the “quality” of a laser beam depends on the application for which it is intended, and this is not in every case directly related to the M^2 value. The accurate measurement of M^2 and other beam spatial properties can nonetheless provide much useful information on the merits of a given laser beam for practical applications, and also on the modal characteristics and the basic laser physics of laser devices. Continued attention to methods for characterizing real laser beams, along with increased emphasis on really measuring the modal characteristics of real laser devices, should be encouraged.

D. Ultra Low Loss Resonators

Finally, advances in mirror coating technology in recent years have made possible near perfect optical mirrors with power reflections in the "five nines" range, leading to optical resonators with parts per million losses per round trip. This in turn has made possible new advances in ring laser gyroscopes, laser gravity wave (LIGO) interferometers, laser electron accelerators, and sophisticated experiments that explore the basic quantum properties of atoms in extraordinarily high-Q optical cavities. Advances in this area will obviously become more difficult as the technologies near perfection, but the possibilities of new fundamental results are also similarly enhanced.

Acknowledgments

The author appreciates the many opportunities he has had over the years to interact with colleagues and to learn from them about laser beams and resonators. Much of the material for these reviews was first assembled for an R. V. Pole Memorial Lecture at the 1990 CLEO meeting, and I particularly appreciate many suggestions made by J. A. Arnaud during preparation of that talk, as well as very helpful comments offered by numerous colleagues on draft versions of these reviews. I am particularly grateful for the sustained support provided by the U.S. Air Force Office of Scientific Research for my own research on lasers and laser resonators over much of the period surveyed.

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