



Figure 8.1 The domain D and contour C and the singular points z_1, z_2, \dots, z_n in the statement of Cauchy's residue theorem.

If f has a removable singularity at z_0 , then $a_{-n} = 0$, for $n = 1, 2, \dots$. Therefore, $\text{Res}[f, z_0] = 0$. Theorem 8.2 gives methods for evaluating residues at poles.

► Theorem 8.2 (Residues at poles)

i. If f has a simple pole at z_0 , then

$$\text{Res}[f, z_0] = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

ii. If f has a pole of order 2 at z_0 , then

$$\text{Res}[f, z_0] = \lim_{z \rightarrow z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

iii. If f has a pole of order k at z_0 , then

$$\text{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z).$$

Proof If f has a simple pole at z_0 , then the Laurent series is

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

If we multiply both sides of this equation by $(z - z_0)$ and take the limit as $z \rightarrow z_0$, we obtain

$$\begin{aligned} \lim_{z \rightarrow z_0} (z - z_0) f(z) &= \lim_{z \rightarrow z_0} \left[a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \dots \right] \\ &= a_{-1} = \text{Res}[f, z_0], \end{aligned}$$

which establishes part (i). We proceed to part (iii), as part (ii) is a special case of it. Suppose that f has a pole of order k at z_0 . Then f can be written as

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots.$$

Multiplying both sides of this equation by $(z - z_0)^k$ gives

$$(z - z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z - z_0)^{k-1} + a_0(z - z_0)^k + \cdots.$$

If we differentiate both sides $k - 1$ times, we get

$$\begin{aligned} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right] &= (k-1)!a_{-1} + k!a_0(z - z_0) \\ &\quad + (k+1)!a_1(z - z_0)^2 + \cdots, \end{aligned}$$

and when we let $z \rightarrow z_0$, the result is

$$\lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right] = (k-1)!a_{-1} = (k-1)!\text{Res}[f, z_0],$$

which establishes part (iii).