Technical Note No. 2*
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Properties of Lognormal Distribution

A variable Q has a lognormal distribution if $V = \ln(Q)$ has a normal distribution. Suppose that V is $\phi(m, s)$; that is, it has a normal distribution with mean m and standard deviation, s. The probability density function for V is

$$\frac{1}{\sqrt{2\pi}s}\exp\left(-\frac{(V-m)^2}{2s^2}\right)$$

The probability density function for Q is therefore

$$h(Q) = \frac{1}{\sqrt{2\pi}sQ}\exp\left(-\frac{[\ln(Q)-m]^2}{2s^2}\right)$$

Consider the nth moment of Q

$$\int_0^{+\infty} Q^n h(Q)dQ$$

Substituting $Q = \exp V$ this is

$$\int_{-\infty}^{+\infty} \frac{\exp(nV)}{\sqrt{2\pi}s}\exp\left(-\frac{(V-m)^2}{2s^2}\right)dV$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}s}\exp\left(-\frac{(V-m-ns^2)^2}{2s^2}\right)\exp\left(\frac{2mns^2+n^2s^4}{2s^2}\right)dV$$

$$= \exp(nm + n^2s^2/2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}s}\exp\left(-\frac{(V-m-ns^2)^2}{2s^2}\right)dV$$

The integral in this expression is the integral of a normal density function with mean $m + ns^2$ and standard deviation s and is therefore 1.0. It follows that

$$\int_0^{+\infty} Q^n h(Q)dQ = \exp(nm + n^2s^2/2) \tag{1}$$

The expected value of Q is given when $n = 1$. It is

$$\exp(m + s^2/2)$$

The result in equation (13.4) follows by setting $m = \ln(S_0) + (\mu - \sigma^2/2)T$ and $s = \sigma\sqrt{T}$

The variance of Q is $E(Q^2) - [E(Q)]^2$. Setting $n = 2$ in equation (1) we get

$$E(Q^2) = \exp(2m + 2s^2)$$

The variance of Q is therefore

$$\exp(2m + 2s^2) - \exp(2m + s^2) = \exp(2m + s^2)[\exp(s^2) - 1]$$

The result in equation (13.5) follows by setting $m = \ln(S_0) + (\mu - \sigma^2/2)T$ and $s = \sigma\sqrt{T}$.