Trinity Mathematical Society

Gödel's Theorem

How much does it matter for mathematicians?

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- The (First) Incompleteness Theorem
- Some philosophical implications and non-implications
- How the Theorem is proved
- Are Gödel sentences arithmetically interesting?
- An unprovable arithmetically interesting truth?
- The speed-up theorem

First Order Peano Arithmetic

- The language L_A of basic arithmetic contains '0', a symbol for the successor function, symbols for addition and multiplication, identity, and logical symbols (particularly the quantifiers for quantifying over numbers).

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- The benchmark theory of basic arithmetic is PA, First Order Peano Arithmetic. PA knows that different natural numbers have different successors, that 0 isn't a successor; it knows the recursive definitions of addition and multiplication; it knows about instances of induction.
- PA is strong enough to capture all facts about the decidable properties of particular numbers.
- (S) Suppose P is a decidable numerical property. Then there will be an expression $\varphi(x)$ of L_A such that
  1. If n is P, then PA $\vdash \varphi(n)$
  2. If n is not P, then PA $\vdash \neg\varphi(n)$

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  2. properly formalized (so that it is a decidable matter whether a putative T-proof really is a proof according to the rules of the game)
  3. includes 'First Order Peano Arithmetic'
- Kurt Gödel (1931) shows how to take any nice theory T and construct an arithmetic sentence G_T, such that,
  1. If T *is* consistent, $T \nvdash G_T$ (i.e. T doesn't prove G_T).
  2. If T *is* consistent, G_T is true.

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  4. There will be another true arithmetical sentence G_{T^+} such that $T^+ \nvdash G_{T^+}$ (and so $T \nvdash G_{T^+}$ too).
- T's incompleteness is incurable (except at the price of inconsistency or no longer being a properly axiomatized theory).

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- His theorem sabotages the project of *Principia Mathematica* which aims to make good Bertrand Russell's programmatic claim:

  "All mathematics deals exclusively with concepts definable in terms of a very small number of logical concepts, and . . . all its propositions are deducible from a very small number of fundamental logical principles."

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- (2) "We are smarter than any arithmetically competent machine. For the output of such a machine corresponds to the output of some nice theory T, and we can always see to be true something it can't prove, namely its Gödel sentence G_T."
- Not so. Gödel shows that G_T is true if T is consistent. To see G_T *is* true we have to be able to see that T *is* consistent. In general we won't be able to do that if T is complex.

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- Gödel also proves that if T is nice, it can express the numerical property *codes-for-a-provable-sentence-of-T*.
- (E) In other words, there's a predicate $prov_T(x)$ such that $prov_T(\ulcorner S \urcorner)$ is true just if S is a T-theorem.

The undecidability of nice theories

- So take the predicate $\neg prov_T(x)$ (which says the sentence with code number x is *not* provable in T). By the fixed point lemma (D) there is a sentence G_T such that

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- By result (S)
  1. If G_T is provable in T, then $T \vdash prov_T(\ulcorner G_T \urcorner)$
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  2. If G_T is not provable in T, then $T \vdash \neg prov_T(\ulcorner G_T \urcorner)$
- Those three are contradictory. Hence ...
  Theorem 1: there can't be a way of deciding theoremhood for a nice theory T.

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- To repeat: take the predicate $\neg prov_T(x)$ (which says x is *not* provable in T). By (D) there is a sentence G_T such that

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- Now ask: can T prove G_T? If so it also proves $\neg prov_T(\ulcorner G_T \urcorner)$. So being a theorem, that will be true. But it says that the sentence G_T is not provable. Contradiction!

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- So Theorem 2: G_T is unprovable. So it is true that $\neg prov_T(\ulcorner G_T \urcorner)$, and also true that $G_T \leftrightarrow \neg prov_T(\ulcorner G_T \urcorner)$. So G_T is true.

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  3. At the cost of either slightly strengthening the assumption that T is consistent, or slightly complicating the construction of G_T, we can show that neither G_T or $\neg G_T$ is provable. There is a 'formally undecidable' sentence of T.

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- Looked at in the light of our choice of "Gödel numbering", this G_T decodes as ‘I am unprovable in G_T’. So are we tangling with self-referential paradox here?
- No! When definitional abbreviations are unpacked G_T is just a long, complicated arithmetical sentence involving the successor, addition, multiplication function symbols plus logical notation. The semantics for G_T is entirely normal: G_T is a sentence about numbers (not about sentences).

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- Then *Con* will also be true-but-unprovable in nice T. Which is essentially Gödel's Second Incompleteness Theorem.
- Its significance is that, if T can't even prove that T is consistent, it can't be used to prove a *stronger* theory is consistent. (For example, we can't use 'safe' PA-level reasoning to prove e.g. that ZFC is consistent.)

Are there 'arithmetically interesting' undecidable sentences?

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- Natural question arising. If we take a standard formal theory of arithmetic like Peano Arithmetic, Gödel tells that there are there are arithmetical truths that can't be proved in PA. But are there arithmetically interesting claims – not constructed e.g. by coding logical facts about provability – which can't be decided in PA?

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- In 1977, Jeff Paris and Leo Harrington found a new combinatorial statement (a not particularly natural version of the finite Ramsey Theorem) which is true, statable in the language of basic arithmetic, but not provable in PA.
- But a few years later it was shown that an already-known theorem about arithmetic was independent of PA: *every Goodstein sequence terminates* (which is provable in ZF) isn't provable in PA. To explain . . .

Hereditary base representation

- Define the hereditary base k representation of n as follows: write n as a sum of powers of k, then write the *exponents* as sums of powers of k, then write *those* exponents as sums of powers of k, and keep going . . . .

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- Example:

$$268 = 2^8 + 2^3 + 2^2$$

So the pure base 2 representation of 268 is

$$266 = 2^{2^{2^{2^0}} + 2^0} + 2^{2^{2^0} + 2^0} + 2^{2^{2^0}}$$

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- Similarly:

$$266 = 3^5 + 3^2 + 3^2 + 3^1 + 1 + 1$$

  So the pure base 3 representation is

$$266 = 3^{3^{3^0}+3^0+3^0} + 3^{3^0+3^0} + 3^{3^0+3^0} + 3^{3^0} + 3^0 + 3^0$$

The Goodstein bump function

- We define the Goodstein bump function $G(n, k)$ as the result of
  i. taking the hereditary base k representation of n;
  ii. bumping up every k to $k+1$,
  iii. subtracting 1 from the resulting number.

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  1. taking the hereditary base k representation of n;
  2. bumping up every k to $k+1$,
  3. subtracting 1 from the resulting number.
- Example: we'll calculate $G(19, 2)$.
  1. $19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0$
  2. bump up the base: $3^{3^{3^{3^0}}} + 3^{3^0} + 3^0$
  3. subtract 1 to get

     $$G(19, 2) = 3^{3^{3^{3^0}}} + 3^{3^0} = 7625597484990$$

The Goodstein sequence

The bump function G: bump up the base by one, then subtract one.

The Goodstein sequence starting at n is got by repeatedly applying the bump function:

$$\begin{aligned} g_1 &= n \\ g_2 &= G(g_1, 2) \\ g_3 &= G(g_2, 3) \\ g_4 &= G(g_3, 4) \\ g_5 &= G(g_4, 5) \\ &\vdots \end{aligned}$$

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g_1 &= 3 = 2^{2^0} + 2^0 \\
g_2 &= 3^{3^0} + 3^0 - 1 = 3^{3^0} \\
g_3 &= 4^{4^0} - 1 = 4^0 + 4^0 + 4^0 \\
g_4 &= 5^0 + 5^0 \\
g_5 &= 6^0 \\
g_6 &= 0
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$$\begin{aligned} g_1 &= 19 = 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0 \\ g_2 &= 3^{3^{3^{3^0}}} + 3^{3^0} \approx 7 \cdot 10^{13} \\ g_3 &= 4^{4^{4^{4^0}}} + 4^{4^0} - 1 \\ &= 4^{4^{4^{4^0}}} + 4^0 + 4^0 + 4^0 \approx 7 \cdot 10^{154} \\ g_4 &= 5^{5^{5^{5^0}}} + 5^0 + 5^0 \quad \text{(which is enormous!)} \\ &\vdots \end{aligned}$$

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We'll write '1' for 'x^0', and 'x' for 'x^{x^0}' for brevity. Here's the Goodstein sequence for 19 again:

$$\begin{aligned} g_1 &= 2^{2^{2^{2^0}}} + 2^{2^0} + 2^0 \\ g_2 &= 3^{3^{3^{3^0}}} + 3^{3^0} \\ g_3 &= 4^{4^{4^{4^0}}} + 4^0 + 4^0 + 4^0 \\ g_4 &= 5^{5^{5^{5^0}}} + 5^0 + 5^0 \\ &\vdots \end{aligned}$$

$$\begin{aligned} g_1 &= 2^{2^2} + 2 + 1 \\ g_2 &= 3^{3^3} + 3 \\ g_3 &= 4^{4^4} + 1 + 1 + 1 \\ g_4 &= 5^{5^5} + 1 + 1 \\ &\vdots \end{aligned}$$

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For <u>every</u> n, the Goodstein sequence starting with n terminates at zero!!!

Proved by mapping the Goodstein sequence for n to a sequence of ordinals: at each step, replace k with ω.

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On r.h.s. we get strictly decreasing sequence of ordinals. By ZF, must bottom out at zero. So l.h.s. must bottom out too.

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- (An aside about Fermat's Last Theorem.)

- The (First) Incompleteness Theorem
- Some philosophical implications and non-implications
- How the Theorem is proved
- Are Gödel sentences arithmetically interesting?
- An unprovable arithmetically interesting truth?
- The speed-up theorem

Speeding up proofs

- Let's turn from the question of whether there are arithmetically interesting sentences which are not provable in PA to question about sentences that are provable in PA.

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Speeding up proofs

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- The same fixed point lemma that quickly yields the Incompleteness Theorem also gets us to the following speed-up theorem (quick and dirty version):
- For any T which extends PA, there will be sentences φ which are provable in PA but whose shortest PA-proof is vastly longer than their shortest T-proofs.

Speed-up more carefully

- Let's say that a theory T_1 exhibits ultra speed-up over T_2 if for any computable function f, there is some corresponding wff φ such that
  1. both $T_1 \vdash \varphi$ and $T_2 \vdash \varphi$
  2. while there is a T_1-proof of φ with g.n. p, there is no T_2-proof with g.n. less than or equal to $f(p)$.
- In other words, there are indefinitely many wffs for which T_1 gives 'much shorter' proofs than T_2.
- Theorem 3: If T is nice theory, and γ is some sentence such that neither $T \vdash \gamma$ nor $T \vdash \neg\gamma$. Then the theory $T + \gamma$ got by adding γ as a new axiom exhibits ultra speed-up over T.

The moral

- Number theorists have long been familiar with cases where arithmetical theorems provable in e.g. complex analysis seem only to have very long and messy proofs in ‘pure’ arithmetic. The speed-up theorem shows is that there is an inevitability about this kind of situation.

The moral

- Number theorists have long been familiar with cases where arithmetical theorems provable in e.g. complex analysis seem only to have very long and messy proofs in 'pure' arithmetic. The speed-up theorem shows is that there is an inevitability about this kind of situation.
- The moral: even if PA in principle implies all the 'arithmetically interesting' claims expressible in the language of basic arithmetic, there will never be a shortage of work for mathematicians to make new truths accessible by developing richer theories which extend PA.

Proving ultra speed-up – 1

- Suppose, for reductio, that there is a sentence γ which is undecided by T, and there is also a computer function f such that for every wff φ, if φ has a proof in $T + \gamma$ with g.n. p, then it has a proof in the original T with g.n. number no greater than $f(p)$.

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- For any wff φ, $(\gamma \vee \varphi)$ is trivially provable in $T + \gamma$. And there will be a very simple computation, with no open-ended searching, that takes us from the g.n. of φ to the g.n. of the trivial proof of $(\gamma \vee \varphi)$. In other words, the g.n. of the proof will $h(\ulcorner\varphi\urcorner)$, for some computable function h.

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- For any wff φ, $(\gamma \vee \varphi)$ is trivially provable in $T + \gamma$. And there will be a very simple computation, with no open-ended searching, that takes us from the g.n. of φ to the g.n. of the trivial proof of $(\gamma \vee \varphi)$. In other words, the g.n. of the proof will $h(\ulcorner\varphi\urcorner)$, for some computable function h.
- So, by our supposition, $(\gamma \vee \varphi)$ must have a proof in T with g.n. no greater than $f(h(\ulcorner\varphi\urcorner))$.

Proving ultra speed-up – 2

- Next consider the theory $T + \neg\gamma$. Trivially again, for any φ, $T + \neg\gamma \vdash \varphi$ iff $T \vdash (\gamma \vee \varphi)$.

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- So we have a decision procedure for telling whether an arbitrary φ is a theorem of $T + \neg\gamma$. Just run a 'for' loop examining in turn all the T-proofs with g.n. up to $f(h(\ulcorner\varphi\urcorner))$ and see if a proof of $(\gamma \vee \varphi)$ turns up.

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- So we have a decision procedure for telling whether an arbitrary φ is a theorem of $T + \neg\gamma$. Just run a 'for' loop examining in turn all the T-proofs with g.n. up to $f(h(\ulcorner\varphi\urcorner))$ and see if a proof of $(\gamma \vee \varphi)$ turns up.
- But $T + \neg\gamma$ is still a nice theory: it is consistent (else we'd have $T \vdash \gamma$, contrary to hypothesis), it is properly axiomatized, and it contains PA since T does. So our earlier theorem applies, and there can't be a computational procedure for testing theoremhood in $T + \neg\gamma$. Contradiction.