Trinity Mathematical Society

Gödel's Theorem How much does it matter for mathematicians?

Peter Smith, Philosophy Faculty

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Peter Smith, Philosophy Faculty: Gödel's Theorem, How much does it matter for mathematicians?

- The (First) Incompleteness Theorem
- Some philosophical implications and non-implications
- How the Theorem is proved
- Are Gödel sentences arithmetically interesting?
- An unprovable arithmetically interesting truth?
- The speed-up theorem

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- The benchmark theory of basic arithmetic is PA, First Order Peano Arithmetic. PA knows that different natural numbers have different successors, that 0 isn't a successor; it knows the recursive definitions of addition and multiplication; it knows about instances of induction.
- PA is strong enough to capture all facts about the decidable properties of particular numbers.
- ► (S) Suppose *P* is a decidable numerical property. Then there will be an expression $\varphi(x)$ of L_A such that
 - 1. If *n* is *P*, then $\mathsf{PA} \vdash \varphi(n)$
 - 2. If *n* is not *P*, then $\mathsf{PA} \vdash \neg \varphi(n)$

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 - properly formalized (so that it is a decidable matter whether a putative *T*-proof really is a proof according to the rules of the game)
 - 3. includes 'First Order Peano Arithmetic'
- ► Kurt Gödel (1931) shows how to take any nice theory T and construct an arithmetic sentence G_T, such that,
 - 1. If T is consistent, $T \nvDash G_T$ (i.e. T doesn't prove G_T).
 - 2. If T is consistent, G_T is true.

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 - 4. There will be another true arithmetical sentence G_{T^+} such that $T^+ \nvDash G_{T^+}$ (and so $T \nvDash G_{T^+}$ too).
- T's incompleteness is incurable (except at the price of inconsistency or no longer being a properly axiomatized theory).

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- Gödel's original paper was called 'On formally undecidable propositions of *Principia Mathematica*.'
- His theorem sabotages the project of *Principia Mathematica* which aims to make good Bertrand Russell's programmatic claim:

"All mathematics deals exclusively with concepts definable in terms of a very small number of logical concepts, and ... all its propositions are deducible from a very small number of fundamental logical principles."

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- ► (2) "We are smarter than any arithmetically competent machine. For the output of such a machine corresponds to the output of some nice theory *T*, and we can always see to be true something it can't prove, namely its Gödel sentence *G_T*."
- Not so. Gödel shows that G_T is true if T is consistent. To see G_T is true we have to be able to see that T is consistent. In general we won't be able to do that if T is complex.

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- ► Gödel also proves that if T is nice, it can express the numerical property codes-for-a-provable-sentence-of-T.
- ► (E) In other words, there's a predicate prov_T(x) such that prov_T(¬S¬) is true just if S is a T-theorem.

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Now suppose the property of being a theorem of the nice theory T is decidable. That is to say, given a number n we can mechanically decide whether n is code number of a provable sentence of T.

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- By result (S)
 - 1. If G_T is provable in T, then $T \vdash prov_T(\ulcorner G_T \urcorner)$
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 - 2. If G_T is not provable in T, then $T \vdash \neg prov_T(\ulcorner G_T \urcorner)$
- Those three are contradictory. Hence ... Theorem 1: there can't be a way of deciding theoremhood for a nice theory T.

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- Now ask: can *T* prove *G_T*? If so it also proves ¬*prov_T*([¬]*G_T*[¬]). So being a theorem, that will be true. But it says that the sentence *G_T* is not provable. Contradiction!

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- ▶ So Theorem 2: G_T is unprovable. So it is true that $\neg prov_T(\ulcorner G_T \urcorner)$, and also true that $G_T \leftrightarrow \neg prov_T(\ulcorner G_T \urcorner)$. So G_T is true.

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- But we can improve this result in three ways.
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 - 2. We can show that G_T is a Π_1 sentence of basic arithmetic (is of 'Goldbach type'), i.e. is just a universal generalization whose instances are all mechanically decidable arithmetical statements.
 - 3. At the cost of either slightly strengthening the assumption that T is consistent, or slightly complicating the construction of G_T , we can show that neither G_T or $\neg G_T$ is provable. There is a 'formally undecidable' sentence of T.

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Are Gödel sentences 'paradoxical'?

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- ► No! When definitional abbreviations are unpacked G_T is just a long, complicated arithmetical sentence involving the successor, addition, multiplication function symbols plus logical notation. The semantics for G_T is entirely normal: G_T is a sentence about numbers (not about sentences).

Are Gödel sentences arithmetically interesting?

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▶ Worth noting that there are other fixed point sentences *C* s.t.

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- ► Then *Con* will also be true-but-unprovable in nice *T*. Which is essentially Gödel's Second Incompleteness Theorem.
- Its significance is that, if T can't even prove that T is consistent, it can't be used to prove a *stronger* theory is consistent. (For example, we can't use 'safe' PA-level reasoning to prove e.g. that ZFC is consistent.)

Are there 'arithmetically interesting' undecidable sentences?

► G_T is an *immensely* long, complicated arithmetical sentence. Its fine details are dependent on entirely arbitrary choices about our Gödel numbering scheme. G_T is not a proposition of intrinsic arithmetical interest: we wouldn't antecedently have wondered about its truth/provability.

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- Natural question arising. If we take a standard formal theory of arithmetic like Peano Arithmetic, Gödel tells that there are there are arithmetical truths that can't be proved in PA. But are there arithmetically interesting claims – not constructed e.g. by coding logical facts about provability – which can't be decided in PA?

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Introducing Goodstein

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- It look forty-six years after the First Theorem for anyone to find a truth expressible in the language of basic arithmetic which is independent of PA.
- In 1977, Jeff Paris and Leo Harrington found a new combinatorial statement (a not particularly natural version of the finite Ramsey Theorem) which is true, statable in the language of basic arithmetic, but not provable in PA.
- But a few years later it was shown that an already-known theorem about arithmetic was independent of PA: every Goodstein sequence terminates (which is provable in ZF) isn't provable in PA. To explain ...

Hereditary base representation

Define the hereditary base k representation of n as follows: write n as a sum of powers of k, then write the exponents as sums of powers of k, then write those exponents as sums of powers of k, and keep going

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- Example:

$$268 = 2^8 + 2^3 + 2^2$$

So the pure base 2 representation of 268 is

$$266 = 2^{2^{2^{2^{0}}+2^{0}}} + 2^{2^{2^{0}}+2^{0}} + 2^{2^{2^{0}}}$$

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Similarly:

$$266 = 3^5 + 3^2 + 3^2 + 3^1 + 1 + 1$$

So the pure base 3 representation is

$$266 = 3^{3^{3^0}+3^0+3^0} + 3^{3^0+3^0} + 3^{3^0+3^0} + 3^{3^0} + 3^0 + 3^0$$

The Goodstein bump function

- ► We define the Goodstein bump function G(n, k) as the result of
 - i. taking the hereditary base k representation of n;
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 - iii. subtracting 1 from the resulting number.

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The Goodstein sequence

The bump function G: bump up the base by one, then subtract one.

The Goodstein sequence starting at n is got by repeatedly applying the bump function:

$$g_{1} = n$$

$$g_{2} = G(g_{1}, 2)$$

$$g_{3} = G(g_{2}, 3)$$

$$g_{4} = G(g_{3}, 4)$$

$$g_{5} = G(g_{4}, 5)$$

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$$\begin{array}{rcl} g_1 &=& n & g_1 &=& 3 = 2^{2^0} + 2^0 \\ g_2 &=& G(g_1, 2) & g_2 &=& 3^{3^0} + 3^0 - 1 = 3^{3^0} \\ g_3 &=& G(g_2, 3) & g_3 &=& 4^{4^0} - 1 = 4^0 + 4^0 + 4^0 \\ g_4 &=& G(g_3, 4) & g_4 &=& 5^0 + 5^0 \\ g_5 &=& G(g_4, 5) & g_5 &=& 6^0 \\ \vdots & g_6 &=& 0 \end{array}$$

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$$g_{1} = n \qquad g_{1} = 19 = 2^{2^{2^{2^{0}}}} + 2^{2^{0}} + 2^{0}$$

$$g_{2} = G(g_{1}, 2) \qquad g_{2} = 3^{3^{3^{0}}} + 3^{3^{0}} \approx 7 \cdot 10^{13}$$

$$g_{3} = G(g_{2}, 3) \qquad g_{3} = 4^{4^{4^{0}}} + 4^{4^{0}} - 1$$

$$g_{4} = G(g_{3}, 4) \qquad = 4^{4^{4^{0}}} + 4^{0} + 4^{0} + 4^{0} \approx 7 \cdot 10^{154}$$

$$g_{5} = G(g_{4}, 5) \qquad g_{4} = 5^{5^{5^{0}}} + 5^{0} + 5^{0} \text{ (which is enormous!)}$$

$$\vdots \qquad \vdots$$

For every n, the Goodstein sequence starting with n terminates at zero $\boxed{!!}$

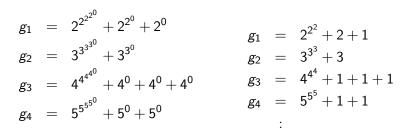
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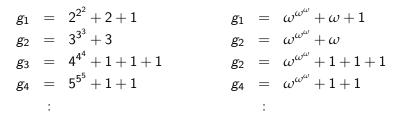
We'll write '1' for ' x^{0} ', and 'x' for ' $x^{x^{0}}$ ' for brevity. Here's the Goodstein sequence for 19 again:



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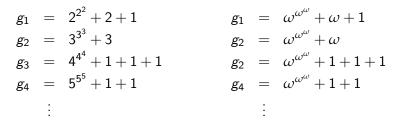
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On r.h.s. we get strictly decreasing sequence of ordinals. By ZF, must bottom out at zero. So l.h.s. must bottom out too.

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- However, to prove it seems essentially to involve 'higher order' ideas about infinite ordinals, rather than adding more purely arithmetical ideas to PA. (Goodstein was exploring induction over ordinals up to \varepsilon_0).
- ??? Perhaps the unprovability of Goodstein's Theorem in PA is too much like the unprovability of G_{PA} in PA both concern the unprovability of sentences which are arithmetically expressible but whose interest is that they are related, by some coding device, to non-arithmetical facts (about proofs, about ordinals).

But how 'arithmetic' is Goodstein's Theorem?

- Goodstein's Theorem can be expressed in language of arithmetic but can't be proved in PA.
- However, to prove it seems essentially to involve 'higher order' ideas about infinite ordinals, rather than adding more purely arithmetical ideas to PA. (Goodstein was exploring induction over ordinals up to \varepsilon_0).
- ??? Perhaps the unprovability of Goodstein's Theorem in PA is too much like the unprovability of G_{PA} in PA both concern the unprovability of sentences which are arithmetically expressible but whose interest is that they are related, by some coding device, to non-arithmetical facts (about proofs, about ordinals).
- (An aside about Fermat's Last Theorem.)

- The (First) Incompleteness Theorem
- Some philosophical implications and non-implications
- How the Theorem is proved
- Are Gödel sentences arithmetically interesting?
- An unprovable arithmetically interesting truth?
- The speed-up theorem

Speeding up proofs

Let's turn from the question of whether there are arithmetically interesting sentences which are not provable in PA to question about sentences that are provable in PA. Speeding up proofs

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Speeding up proofs

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- The same fixed point lemma that quickly yields the Incompleteness Theorem also gets us to the following speed-up theorem (quick and dirty version):
- For any T which extends PA, there will be sentences φ which are provable in PA but whose shortest PA-proof is vastly longer than their shortest T-proofs.

Speed-up more carefully

- Let's say that a theory T₁ exhibits ultra speed-up over T₂ if for any computable function f, there is some corresponding wff φ such that
 - 1. both $T_1 \vdash \varphi$ and $T_2 \vdash \varphi$
 - 2. while there is a T_1 -proof of φ with g.n. p, there is no T_2 -proof with g.n. less than or equal to f(p).
- In other words, there are indefinitely many wffs for which T₁ gives 'much shorter' proofs than T₂.
- Theorem 3: If *T* is nice theory, and *γ* is some sentence such that neither *T* ⊢ *γ* nor *T* ⊢ ¬*γ*. Then the theory *T* + *γ* got by adding *γ* as a new axiom exhibits ultra speed-up over *T*.

The moral

Number theorists have long been familiar with cases where arithmetical theorems provable in e.g. complex analysis seem only to have very long and messy proofs in 'pure' arithmetic. The speed-up theorem shows is that there is an inevitability about this kind of situation.

The moral

- Number theorists have long been familiar with cases where arithmetical theorems provable in e.g. complex analysis seem only to have very long and messy proofs in 'pure' arithmetic. The speed-up theorem shows is that there is an inevitability about this kind of situation.
- The moral: even if PA in principle implies all the 'arithmetically interesting' claims expressible in the language of basic arithmetic, there will never be a shortage of work for mathematicians to make new truths accessible by developing richer theories which extend PA.

Proving ultra speed-up -1

Suppose, for reductio, that there is a sentence γ which is undecided by *T*, and there is also a computer function *f* such that for every wff φ, if φ has a proof in *T* + γ with g.n. p, then it has a proof in the original *T* with g.n. number no greater than *f*(*p*). Proving ultra speed-up -1

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- For any wff φ, (γ ∨ φ) is trivially provable in T + γ. And there will be a very simple computation, with no open-ended searching, that takes us from the g.n. of φ to the g.n. of the trivial proof of (γ ∨ φ). In other words, the g.n. of the proof will h([¬]φ[¬]), for some computable function h.

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- ▶ So, by our supposition, $(\gamma \lor \varphi)$ must have a proof in T with g.n. no greater than $f(h(\ulcorner \varphi \urcorner))$.

Proving ultra speed-up – 2

▶ Next consider the theory $T + \neg \gamma$. Trivially again, for any φ , $T + \neg \gamma \vdash \varphi$ iff $T \vdash (\gamma \lor \varphi)$. Proving ultra speed-up – 2

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- So we have a decision procedure for telling whether an arbitrary φ is a theorem of T + ¬γ. Just run a 'for' loop examining in turn all the T-proofs with g.n. up to f(h(¬φ¬)) and see if a proof of (γ ∨ φ) turns up.

Proving ultra speed-up – 2

- ▶ Next consider the theory $T + \neg \gamma$. Trivially again, for any φ , $T + \neg \gamma \vdash \varphi$ iff $T \vdash (\gamma \lor \varphi)$.
- So we have a decision procedure for telling whether an arbitrary φ is a theorem of T + ¬γ. Just run a 'for' loop examining in turn all the T-proofs with g.n. up to f(h(¬φ¬)) and see if a proof of (γ ∨ φ) turns up.
- But T + ¬γ is still a nice theory: it is consistent (else we'd have T ⊢ γ, contrary to hypothesis), it is properly axiomatized, and it contains PA since T does. So our earlier theorem applies, and there can't be a computational procedure for testing theoremhood in T + ¬γ. Contradiction.