Cubic Forms in 14 Variables

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1 Introduction

Let $C(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic form. It was shown by Davenport [5], that there exists a nonzero vector $\mathbf{x} \in \mathbb{Z}^n$ for which $C(\mathbf{x}) = 0$, providing only that $n \geq 16$. The goal of this paper is to extend the admissible range for n as follows.

Theorem 1 *Let $C(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic form with $n \geq 14$. Then there exists $\mathbf{x} \in \mathbb{Z}^n - \{\mathbf{0}\}$ for which $C(\mathbf{x}) = 0$.*

The result can be rephrased in geometric language to say that any projective cubic hypersurface defined over $\mathbb{Q}$, of dimension at least 12, has a $\mathbb{Q}$-point. Davenport's result was extended to arbitrary number fields by Pleasants [9], and it would be interesting to know whether Theorem 1 could similarly be extended.

These results can be seen as an attempt to extend the classical theorem of Meyer (1884) from quadratic forms to cubic forms. Meyer showed that any indefinite quadratic from over $\mathbb{Z}$ in 5 or more variables must represent zero. Indeed Meyer's result was generalized by Minkowski , who showed that a quadratic form over $\mathbb{Z}$, in any number of variables, represents zero if and only if it represents zero over every completion of $\mathbb{Q}$. It is a well-known fact that this local condition is automatically satisfied for $\mathbb{Q}_p$ as soon as $n \geq 5$. Thus Meyer's result requires only the condition of indefiniteness. The analogous fact for cubic forms is that p-adic zeros exist whenever $n \geq 10$, see Davenport [6, Chapter 18] for example. Of course the condition for $\mathbb{R}$ holds for any n in this case. Thus it is natural to conjecture that Theorem 1 should hold as soon as $n \geq 10$. Indeed for smaller values of n one might expect that it should suffice for the form $C(\mathbf{x})$ to represent zero in each field $\mathbb{Q}_p$. However when $n = 3$ or 4 it is possible for a cubic form to have zeros in every completion of $\mathbb{Q}$, without there being a global zero. This is shown by the examples

$$3x_1^3 + 4x_2^3 + 5x_3^3, \quad \text{and} \quad 5x_1^3 + 9x_2^3 + 10x_3^3 + 12x_4^3$$

of Selmer [10] and Cassels and Guy [4] respectively. This phenomenon is explained by the Brauer-Manin obstruction, and it is known that there is no such obstruction for non-singular cubic forms with $n \geq 5$. Thus for cubic forms over $\mathbb{Z}$ we conjecture that there is a non-trivial zero:

(i) whenever $n \geq 10$;

(ii) for non-singular forms, when $5 \leq n \leq 9$ and there is a zero in every completion $\mathbb{Q}_p$; and

(iii) when $n = 3$ or 4 in those cases for which there is a zero in every completion $\mathbb{Q}_p$ and the Brauer-Manin obstruction is empty.

Note that the situation for singular cubic forms is unclear when $5 \le n \le 9$. The author is grateful to Professors Colliot-Thélène and Salberger for alerting him to this area of uncertainty. However Salberger has shown, in unpublished work, that the Bruer-Manin obstruction for singular cubic hypersurfaces is empty, for any n, except possibly when the singular locus has co-dimension 2 or 3.

In the quadratic case one can readily assume that the form is non-singular, or indeed diagonal, but for cubic forms this is a significant issue. Indeed if one is willing to assume that the cubic form is non-singular then substantial further progress is possible. Thus Hooley [8] has shown that a non-singular cubic form in 9 or more variables, defined over $\mathbb{Z}$, has a non-trivial representation of zero if and only if there is a representation in every completion of $\mathbb{Q}$. Moreover Baker [1] has shown that any diagonal cubic form over $\mathbb{Z}$ in 7 or more variables represents zero non-trivially. (For diagonal cubics in 7 or more variables the local conditions hold automatically.) Furthermore there is a conditional result of Swinnerton-Dyer [11], which assumes the finiteness of the Tate-Shafarevich group for elliptic curves over $\mathbb{Q}(\sqrt{-3})$. Under this hypothesis it is shown that a diagonal cubic form in 5 or more variables represents zero if it does so over every $\mathbb{Q}_p$.

The strategy for our proof is similar in several ways to that of Davenport [5], although there will be one major difference. We shall use the circle method, and in suitable circumstances we shall prove an asymptotic formula for the number of zeros of $C(\mathbf{x})$ in an appropriate large cube.

Let $\mathcal{B} \subset \mathbb{R}^n$ be given by

$$\mathcal{B} = \prod_{1 \le i \le n} [\xi_i - \rho, \xi_i + \rho] \subseteq [-1, 1]^n, \tag{1.1}$$

where $C(\boldsymbol{\xi}) = 0$, and let $P\mathcal{B} = \{\mathbf{x} : P^{-1}\mathbf{x} \in \mathcal{B}\}$. The cube $\mathcal{B}$ will be considered fixed throughout. We then define

$$\mathcal{N}(P) = \#\{\mathbf{x} \in \mathbb{Z}^n \cap P\mathcal{B} : C(\mathbf{x}) = 0\}.$$

In general one might hope that $\mathcal{N}(P)$ would grow like P^{n-3}. However there are certain forms $C(\mathbf{x})$ for which this is false. An example is given by

$$C(x_1, \ldots, x_n) = x_1^3 + x_2(x_3^2 + \ldots + x_n^2), \tag{1.2}$$

which vanishes whenever $x_1 = x_2 = 0$, so that $\mathcal{N}(P) \gg P^{n-2}$ for suitable cubes $\mathcal{B}$. As in Davenport [5] we shall therefore consider two alternative cases. In the first the form $C(\mathbf{x})$ has non-trivial zeros for "geometric reasons", and in the second we shall establish an asymptotic formula for $\mathcal{N}(P)$, for suitable $\mathcal{B}$. Thus, in either case, $C(\mathbf{x})$ must represent zero. In the first alternative one shows only that there is at least one zero. Thus it is not clear in this case whether the zeros are Zariski-dense on the variety $C(\mathbf{x}) = 0$. However, on any non-degenerate cubic hypersurface with $n \ge 4$, once one has obtained one rational point one can find infinitely many more.

To describe the two alternatives precisely requires a certain amount of technical detail. We write the form $C(\mathbf{x})$ in the shape

$$C(x_1, \ldots, x_n) = \sum_{i,j,k} c_{ijk} x_i x_j x_k,$$

in which the coefficients c_{ijk} are symmetric in the indices i, j, k. We shall assume, as we may by replacing $C(\mathbf{x})$ by $6C(\mathbf{x})$, that the c_{ijk} are all integral. We define an $n \times n$ matrix $M(\mathbf{x})$ by taking its entries to be

$$M(\mathbf{x})_{jk} = \sum_i c_{ijk} x_i, \tag{1.3}$$

and we write

$$r(\mathbf{x}) := \operatorname{rank}(M(\mathbf{x})). \tag{1.4}$$

We are now ready to describe our two alternatives.

Theorem 2 *Let $\varepsilon > 0$ be given. Then either*

$$\#\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| < H,\, r(\mathbf{x}) = r\} \ll_\varepsilon H^{r+\varepsilon}, \tag{1.5}$$

holds for every non-negative integer $r \le n$, or the equation $C(\mathbf{x}) = 0$ has a non-trivial integral zero (for "geometric reasons").

When the first alternative above holds we use the circle method to produce the following result.

Theorem 3 *Assume that $C(\mathbf{x})$ has no rational linear factor. Suppose that* (1.5) *holds for every r, and every $\varepsilon > 0$. Suppose further that the centre point $\boldsymbol{\xi}$ of the cube $\mathcal{B}$ is a non-singular point of the variety $C(\mathbf{x}) = 0$ such that $\xi_i \ne 0$ for every index i. Then there is a positive $\rho_0(n, \boldsymbol{\xi})$ such that whenever the cube $\mathcal{B}$ has $\rho \le \rho_0(n, \boldsymbol{\xi})$, and whenever $n \ge 14$, we have*

$$\mathcal{N}(P) \sim P^{n-3}\mathfrak{S}J_0 \quad \text{as} \quad P \to \infty,$$

where $\mathfrak{S}$ and J_0 are the usual singular series and singular integral respectively. (Thus $\mathfrak{S}$ depends only on the form $C(\mathbf{x})$, while J_0 depends only on $C(\mathbf{x})$ and the box $\mathcal{B}$.) Both $\mathfrak{S}$ and J_0 are strictly positive.

As Davenport observes [6, Chapter 16], a suitable point $\boldsymbol{\xi}$ always exists under the above hypotheses. (In fact the only point where we use the conditions on $\boldsymbol{\xi}$ is in handling the singular integral.) The above two results therefore suffice for Theorem 1. We remark that our first result is essentially a restatement of Davenport [6, Lemma 14.3], (though the reader should note the switch in notation between r and $n - r$). However the assertion is sufficiently important to warrant formal statement as a theorem. For the precise interpretation of the "geometric reasons" the reader should consult [6, Chapter 14]. As an example of Theorem 2 we observe that, for the form $C(\mathbf{x})$ in (1.2), one finds that $r(\mathbf{x}) \le 3$ whenever $x_2 = 0$. Thus for some $r \le 3$ we will have

$$\#\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| < H,\, r(\mathbf{x}) = r\} \gg H^{n-1} \gg H^{r+1}$$

as soon as $n \ge 5$.

We also remark that the singular integral J_0 is convergent and positive whenever $\boldsymbol{\xi}$ and ρ are chosen as in Theorem 3, providing only that $C(\mathbf{x})$ has no rational linear factor, and $n \ge 2$. This is implicit in the work of Davenport [6, Chapter 16]. As to the singular series we shall prove the following assertion.

Theorem 4 *Suppose that (1.5) holds for every ε and every r. Then the singular series $\mathfrak{S}$ is absolutely convergent providing that $n \ge 11$.*

It is easy to see that our result has a consequence for the representation of rational numbers by cubic forms. Thus, if $C(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ is a non-degenerate cubic form, with 13 or more variables, then $C(\mathbf{x})$ represents every rational number, using rational values for the variables. One might hope to do slightly better, but the obvious line of attrack appears to fail.

Our proofs of Theorems 3 and 4 require substantial new ideas, which we shall explain later. For the time being let it suffice to say that we shall supplement Weyl's inequality with van der Corput's method. This has both advantages and disadvantages, but the former outweigh the latter. Amongst the advantages are the possibility of combining van der Corput's method with an averaging process which leads to additional savings, which prove to be crucial. However the basic form of the method is not quite sufficient even to prove Theorem 3 for $n \geq 15$. Thus our success depends on the use of two additional techniques, each of which provides a small extra saving. As a result this paper is not as elegant as one might like.

A few words about notation are required. We shall use the symbol c to denote various positive real constants depending on the form $C(\mathbf{x})$, not necessarily the same at each occurrence. We shall take the form $C(\mathbf{x})$, the cube $\mathcal{B}$, and the parameter $\varepsilon \in (0,1)$, to be fixed throughout. It will transpire that any sufficiently small $\varepsilon > 0$ will be suitable for our purposes. Any order constants which we write in $O(\ldots)$ or $\ll$ notations will be allowed to depend on $C(\mathbf{x})$, on $\mathcal{B}$, and on ε. It will be convenient to assume that $10 \leq n \leq 16$. We shall write $\mathbb{N}$ for the set of strictly positive integers $\{1, 2, 3, \ldots\}$. We will encounter a number of summations involving vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ etc. These will always be restricted to integer vectors, so that we write $\sum_{\mathbf{x} \in P\mathcal{B}}$ as a shorthand for $\sum_{\mathbf{x} \in P\mathcal{B} \cap \mathbb{Z}^n}$, for example. Finally we shall assume without comment that the parameter P is a sufficiently large integer, at various stages in the argument.

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2 Davenport's Approach

In this section we shall describe the route taken by Davenport. In doing so we shall encounter a number of results which we are able to re-use for our own work. Moreover we hope that the rationale for the approach taken in the present paper will be clearer once it is compared and contrasted with Davenport's method.

We shall base our description on the exposition in [6, Chapters 13–18], rather than [5]. The former establishes a slightly weaker result, in which $C(\mathbf{x})$ has at least 17 variables. One should really think of this as dealing with "$16 + \varepsilon$" variables. In [5] Davenport uses an extra device to reduce "$16+\varepsilon$" to "$16-\delta$", but this particular trick will not be relevant to our discussions.

The underlying approach is based on the circle method, in which one takes the Major Arcs to be of the form

$$\mathfrak{M}'(a,q) = \left[\frac{a}{q} - P^{-3+\Delta}\,,\,\frac{a}{q} + P^{-3+\Delta}\right]$$

for $1 \leq a \leq q$ with $(a,q) = 1$ and $q \leq P^{\Delta}$. Here Δ is a small fixed positive real number to be defined in due course, see Lemma 6.2. From now on any order constants will be allowed to depend on Δ as well as $C(\mathbf{x})$, $\mathcal{B}$ and ε. We write $\mathfrak{M}'$ for the union of the various $\mathfrak{M}'(a,q)$.

We define the generating function

$$S(\alpha) = \sum_{\mathbf{x}\in P\mathcal{B}} e(\alpha C(\mathbf{x})),$$

where $e(\beta) := \exp(2\pi i\beta)$ as usual, so that

$$\mathcal{N}(P) = \int_0^1 S(\alpha)d\alpha.$$

We also define the complete exponential sums

$$S_{a,q} = \sum_{\mathbf{x} \bmod q} e(\frac{a}{q}C(\mathbf{x})),$$

and the singular series, given by

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q \\ (a,q)=1}} q^{-n}S_{a,q}.$$

This may or may not be convergent. Davenport [6, Lemma 15.3] shows under suitable conditions that $S_{a,q} \ll_\varepsilon q^{7n/8+\varepsilon}$ for any $\varepsilon > 0$. This suffices to establish absolute convergence for $n \ge 17$, but is too weak for smaller value of n.

The work of Davenport (see Lemma 15.4, Chapters 16, 17 and 18 of [6]) now suffices to establish the following result.

Lemma 2.1 *Assume that $C(\mathbf{x})$ has no rational linear factor. Let $n \ge 10$ and suppose that the centre point $\boldsymbol{\xi}$ of the cube $\mathcal{B}$ is a non-singular point of the variety $C(\mathbf{x}) = 0$, and satisfies $\xi_i \ne 0$ for each index i. Suppose further that ρ in (1.1) is sufficiently small. Assume that $\Delta < 1/5$ and suppose that the singular series is absolutely convergent. Then the singular series $\mathfrak{S}$ and the singular integral J_0 are both strictly positive, and*

$$\int_{\mathfrak{M}'} S(\alpha)d\alpha = P^{n-3}\mathfrak{S}J_0 + o(P^{n-3})$$

as $P \to \infty$.

We must now consider the minor arcs, as described in [6, Chapter 13]. Here Davenport uses a generalization of Weyl's inequality which passes from the bound

$$|S(\alpha)|^2 \le \sum_{|\mathbf{y}|<P} |\sum_{\mathbf{z}\in\mathcal{R}(\mathbf{y})} e(\alpha(C(\mathbf{y}+\mathbf{z}) - C(\mathbf{z})))| \tag{2.1}$$

via Cauchy's inequality to

$$|S(\alpha)|^4 \ll P^n \sum_{|\mathbf{x}|,|\mathbf{y}|<P} |\sum_{\mathbf{z}\in\mathcal{S}(\mathbf{x},\mathbf{y})} e(\alpha C(\mathbf{x},\mathbf{y},\mathbf{z}))|, \tag{2.2}$$

where

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = C(\mathbf{x}+\mathbf{y}+\mathbf{z}) - C(\mathbf{x}+\mathbf{z}) - C(\mathbf{y}+\mathbf{z}) + C(\mathbf{z}). \tag{2.3}$$

Here $\mathcal{R}(\mathbf{y})$ and $\mathcal{S}(\mathbf{x},\mathbf{y})$ are certain cubes inside $P\mathcal{B}$. Moreover we have written $|\mathbf{x}|$ for the Euclidean length of $\mathbf{x}$, and have taken ρ in (1.1) to be sufficiently small. If we now define the bilinear forms

$$B_i(\mathbf{x};\mathbf{y}) := \sum_{j,k=1}^{n} c_{ijk}x_jy_k \quad (1 \le i \le n)$$

we find that

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = 6\sum_{i=1}^{n} z_iB_i(\mathbf{x};\mathbf{y}) + \psi(\mathbf{x},\mathbf{y}),$$

where $\psi(\mathbf{x},\mathbf{y})$ is independent of $\mathbf{z}$. It therefore follows that

$$\begin{aligned}|S(\alpha)|^4 &\ll P^n \sum_{|\mathbf{x}|,|\mathbf{y}|<P} \Big|\sum_{\mathbf{z}\in\mathcal{S}(\mathbf{x},\mathbf{y})} e(6\alpha\sum_{i=1}^{n} z_iB_i(\mathbf{x};\mathbf{y}))\Big| \\ &\ll P^n \sum_{|\mathbf{x}|,|\mathbf{y}|<P} \prod_{i=1}^{n} \min(P\,,\,||6\alpha B_i(\mathbf{x};\mathbf{y})||^{-1}).\end{aligned}$$

As in the proof of [6, Lemma 13.2] we find that

$$\sum_{|\mathbf{x}|,|\mathbf{y}|<P} \prod_{i=1}^{n} \min(P\,,\,||6\alpha B_i(\mathbf{x};\mathbf{y})||^{-1}) \ll (P\mathcal{L})^n N(\alpha,P),$$

where

$$\mathcal{L} := \log P$$

and

$$N(\alpha,P) := \#\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^{2n} : |\mathbf{x}|,|\mathbf{y}| < P,\ ||6\alpha B_i(\mathbf{x};\mathbf{y})|| < P^{-1}\ \forall i \le n\}. \tag{2.4}$$

It therefore follows that

$$|S(\alpha)|^4 \ll P^{2n}\mathcal{L}^n N(\alpha,P). \tag{2.5}$$

The focus of the investigation now moves to the analysis of $N(\alpha,P)$. This would be straightforward in the case of a diagonal form, since the bilinear form $B_i(\mathbf{x};\mathbf{y})$ would then just be a scalar multiple of x_iy_i. Thus it is exactly at this point that the general shape of $C(\mathbf{x})$ begins to cause difficulties. Davenport's solution is ingenious, but involves a loss relative to the diagonal case. Using the geometry of numbers he proves the following result (see [6, Lemma 12.6]).

Lemma 2.2 *Let $L \in M_n(\mathbb{R})$ be a real symmetric $n \times n$ matrix. Let $a > 0$ be real, and let*

$$N(Z) := \#\{\mathbf{u} \in \mathbb{Z}^n : |\mathbf{u}| < aZ,\ ||(L\mathbf{u})_i|| < a^{-1}Z\ \forall i \le n\}.$$

Then, if $0 < Z_1 \le Z_2 \le 1$, we have

$$N(Z_2) \ll (\frac{Z_2}{Z_1})^n N(Z_1).$$

(Davenport requires that $a > 1$. However if $0 < a \leq 1$ and $0 < Z_1 \leq Z_2 \leq 1$, the only available vector $\mathbf{u}$ will be $\mathbf{u} = \mathbf{0}$. Thus $N(Z_1) = N(Z_2) = 1$, in which case the lemma is trivial.)

Under suitable circumstances, an inequality of the form $||\alpha m|| < P_0^{-1}$ forces m to be zero. Specifically we have the following easy lemma, which we shall prove at the end of this section.

Lemma 2.3 *Let a real number $M \geq 0$ be given and let $\alpha = a/q + \theta$, with $2qM|\theta| \leq 1$. Suppose that $m \in \mathbb{Z}$ is such that $|m| \leq M$ and $||\alpha m|| < P_0^{-1}$ for some $P_0 \geq 2q$. Then $q|m$. In particular we will have $m = 0$ if in addition we have either*

(a) $M < q$; *or*

(b) $|\theta| > (qP_0)^{-1}$.

In some cases it can happen that Lemma 2.3 applies directly to each of the inequalities $||6\alpha B_i(\mathbf{x};\mathbf{y})|| < P^{-1}$. Usually however either the bound M for $B_i(\mathbf{x};\mathbf{y})$ is too large, or $P_0 = P$ is too small. In these case one may apply Lemma 2.2 to good effect. One takes the matrix L to be $6M(\mathbf{x})$, with $M(\mathbf{x})$ given by (1.3). Then $(L\mathbf{y})_i = 6B_i(\mathbf{x};\mathbf{y})$, and if $a = P$ we will have

$$N(1) = \#\{\mathbf{y} \in \mathbb{Z}^n : |\mathbf{y}| < P,\ ||6\alpha B_i(\mathbf{x};\mathbf{y})|| < P^{-1}\ \forall i \leq n\}.$$

Thus $N(1) \ll Z^{-n}N(Z)$ for $0 < Z \leq 1$, whence, on summing over $\mathbf{x}$, we deduce that

$$\begin{aligned} N(\alpha, P) \ \ll\ & Z^{-n}\#\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^{2n} : |\mathbf{x}| < P,\ |\mathbf{y}| < ZP, \\ & \qquad ||6\alpha B_i(\mathbf{x};\mathbf{y})|| < ZP^{-1}\ \forall i \leq n\}. \end{aligned} \tag{2.6}$$

Rather than using the estimate (2.6) directly, Davenport reverses the rôles of $\mathbf{x}$ and $\mathbf{y}$ in (2.6), and uses the matrix $L = 6M(\mathbf{y})$ in Lemma 2.2 with $a = PZ^{-1/2}, Z_1 = Z^{3/2}$ and $Z_2 = Z^{1/2}$ to deduce that

$$\begin{aligned} N(\alpha, P) \ \ll\ & Z^{-2n}\#\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^{2n} : |\mathbf{x}| < ZP,\ |\mathbf{y}| < ZP, \\ & \qquad ||6\alpha B_i(\mathbf{x};\mathbf{y})|| < Z^2P^{-1}\ \forall i \leq n\}. \end{aligned} \tag{2.7}$$

By choosing Z appropriately one can hope to make Lemma 2.3 applicable, and thence to deduce that

$$N(\alpha, P) \ll Z^{-2n}\#\{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^{2n} : |\mathbf{x}| < ZP,\ |\mathbf{y}| < ZP,\ B_i(\mathbf{x};\mathbf{y}) = 0\ \forall i \leq n\}.$$

Specifically, it suffices that Z satisfies each of the conditions

$$0 < Z < 1,\ \ Z^2 < (12cq|\theta|P^2)^{-1},\ \ Z^2 < P/(2q),$$

and

$$Z^2 < \max(\frac{q}{6cP^2}\,,\ qP|\theta|),$$

where $c = \sum |c_{ijk}|$ for the coefficients c_{ijk} of $C(\mathbf{x})$.

This brings us to the study of the density of integer solutions to the system of simultaneous bilinear equations

$$B_i(\mathbf{x};\mathbf{y}) = 0\ \ \forall i \leq n. \tag{2.8}$$

If $r(\mathbf{x}) = r$ the solutions $\mathbf{y}$ belong to a vector space of dimension $n - r$, so that there are $O(Y^{n-r})$ integer solutions $\mathbf{y}$ in the region $|\mathbf{y}| < Y$, for any $Y \geq 1$. If (1.5) holds the total number of solutions with $|\mathbf{x}| < X$ and $r(\mathbf{x}) = r$ is $O(X^{r+\varepsilon}Y^{n-r})$. Thus on summing over $r \leq n$ we see that there are $O(\max(X,Y)^{n+\varepsilon})$ integer solutions with $|\mathbf{x}| < X$ and $|\mathbf{y}| < Y$. This bound is essentially best possible, since if $\mathbf{x} = \mathbf{0}$ then every $\mathbf{y}$ provides a solution, and similarly if $\mathbf{y} = \mathbf{0}$. Hence the most efficient way of using the bound $O(\max(X,Y)^{n+\varepsilon})$ is to have $X = Y$, and it is for this reason that it is natural to use (2.7) rather than (2.6).

On applying the above bound we find that

$$N(\alpha, P) \ll Z^{-2n}.(ZP)^{n+\varepsilon}$$

for appropriate $Z \geq P^{-1}$. The bound is trivially true when $Z \leq P^{-1}$, since we always have $N(\alpha, P) \ll P^{2n}$. It then follows from (2.5) that

$$|S(\alpha)|^4 \ll P^{3n+\varepsilon}\mathcal{L}^n Z^{-n}.$$

We therefore choose Z (essentially) as large as possible, given the constraints above. Thus we take

$$Z = \frac{1}{2}\min\left\{1\,,\ \frac{1}{\sqrt{12cq|\theta|P^2}}\,,\ \frac{\sqrt{P}}{\sqrt{2q}}\,,\ \max(\frac{\sqrt{q}}{\sqrt{6cP^2}}\,,\ \sqrt{qP|\theta|})\right\},$$

which results in the bound

$$\begin{aligned}|S(\alpha)|^4 \quad &\ll \quad P^{3n+\varepsilon}\mathcal{L}^n\{1 + (q|\theta|P^2)^{n/2} + q^{n/2}P^{-n/2} \\ &\qquad\qquad + \min(P^n q^{-n/2}\,,\ (qP|\theta|)^{-n/2})\}. \end{aligned} \tag{2.9}$$

In particular we see that

$$S(\alpha) \ll P^{n+\varepsilon}\{(q|\theta|)^{n/8} + (q|\theta|P^3)^{-n/8}) + P^{-3n/16}\} \quad \text{for} \quad q \leq P^{3/2},$$

and since $A^{1/2} \leq B + AB^{-1}$ for any $A, B > 0$ we deduce that

$$S(\alpha) \ll P^{n+\varepsilon}\{(q|\theta|)^{n/8} + (q|\theta|P^3)^{-n/8})\} \quad \text{for} \quad q \leq P^{3/2}. \tag{2.10}$$

We now consider the effect of applying Dirichlet's Approximation Theorem to a typical α in the minor arcs $\mathfrak{m}$. For a given parameter Q one may write $\alpha = a/q + \theta$, with $|\theta| \leq (qQ)^{-1}$ and $q \leq Q$. Moreover the values of α for which

$$\frac{1}{2qQ} \leq |\theta| \leq \frac{1}{qQ} \quad \text{and} \quad Q/2 \leq q \leq Q$$

will make up a positive proportion of $[0,1]$. For such α the bound (2.9) becomes

$$|S(\alpha)|^4 \ll P^{3n+\varepsilon}\mathcal{L}^n\{1 + P^n Q^{-n/2} + Q^{n/2}P^{-n/2}\},$$

which is optimal for $Q = P^{3/2}$, yielding

$$S(\alpha) \ll P^{13n/16+\varepsilon}. \tag{2.11}$$

Moreover, for this choice of Q, and a general α, we obtain

$$|S(\alpha)|^4 \ll P^{3n+\varepsilon}\mathcal{L}^n \min(P^n q^{-n/2}\,,\ (qP|\theta|)^{-n/2}),$$

and hence

$$S(\alpha) \ll P^{n+\varepsilon} q^{-n/8} \min\{1\,,\,(P^3|\theta|)^{-n/8}\}.$$

This is equivalent to the conclusion given by Davenport [6, (15.1) & (15.2)]. However we have preferred to phrase the result in terms of an upper bound for $S(\alpha)$ subject to approximation properties of α, rather than using the contrapositive of this formulation. By adapting the argument to work with the cube $0 < x_1, \ldots, x_n \le q$ one can obtain similarly the bound

$$S_{a,q} \ll q^{7n/8+\varepsilon}$$

as in Davenport [6, Lemma 15.3]. We shall improve on this in §8.

One now sees that the contribution to the minor arc integral from a range

$$|\theta| \le (qQ)^{-1}, \quad \text{with} \quad Q = P^{3/2}, \tag{2.12}$$

is

$$\ll P^{n-3+\varepsilon} q^{-n/8}$$

as long as $n \ge 9$. Summing for $P^{\Delta} < q \le Q = P^{3/2}$, and for a less than and coprime to q, will produce a bound $o(P^{n-3})$ precisely when $n > 16$. We should point out here that there is a clear sense in which Davenport's bound is better with respect to θ than it is with respect to q, in as much as one needs only $n > 8$ for the θ integration, but $n > 16$ for the q summation. We shall capitalize on this later, using Davenport's result for small θ.

The above estimates suffice to establish a version of Theorem 3 for the case $n > 16$. To handle the case $n = 16$, Davenport [5] slightly improves the treatment of the number of solutions to (2.8), showing under suitable circumstances that the number of solutions with $\mathbf{x}, \mathbf{y} \ne \mathbf{0}$ and $|\mathbf{x}|, |\mathbf{y}| \le X$ is $O(X^{n-\delta})$ for some small positive δ. Subsequent work allows us to reduce the exponent in this result somewhat, but the way in which the bound is used is by no means straightforward, and the improvement seems not to lead to a sharpening of the 16 variable result.

We conclude this section by establishing Lemma 2.3, which is quite elementary. The result is trivial if $M = 0$, so we shall assume that $M > 0$. We shall repeatedly use the fact that $||x + y|| \le ||x|| + ||y||$ for every real $x, y \in \mathbb{R}$. If $||\alpha m|| < P_0^{-1}$ then

$$||\frac{a}{q}m|| \le ||\alpha m|| + ||\theta m|| < P_0^{-1} + (2qM)^{-1}|m| \le (2q)^{-1} + (2qM)^{-1}M = q^{-1}.$$

It follows that $q|m$, and if $|m| \le M < q$ we must have $m = 0$. In the alternative case, in which we have $|\theta| > (qP_0)^{-1}$, we observe that

$$|\theta m| \le (2qM)^{-1}|m| \le (2qM)^{-1}M \le \frac{1}{2},$$

whence

$$|\theta m| = ||\theta m|| \le ||\frac{a}{q}m|| + ||\alpha m|| = ||\alpha m|| \le P_0^{-1},$$

on recalling that $q|m$. Thus

$$|m| \le (P_0|\theta|)^{-1} < q,$$

whence we must have $m = 0$. This completes the proof of the lemma.

3 A Simple Version of van der Corput's Method

Davenport's method, as described in §2, involves the use of Lemma 2.2 two times in order to reduce the size of both the vectors $\mathbf{x}$ and $\mathbf{y}$ occurring in (2.4). The fundamental new idea in this paper is to to use van der Corput's method to reduce the size of one of the variables, and to apply Lemma 2.2 only once. This has a number of advantages but also the disadvantage that the upper bound P^{-1} for $||6\alpha B_i(\mathbf{x};\mathbf{y})||$ is reduced only once. As a result we shall have to make a smaller choice of Q than in (2.12).

We shall use a variant of van der Corput's method, but before looking at this in detail we shall describe the most basic form of the method, to introduce the reader to the fundamental idea. We choose a positive integer $H \le P$ and write, temporarily, $f(\mathbf{x}) = e(\alpha C(\mathbf{x}))$ for $\mathbf{x} \in P\mathcal{B}$ and $f(\mathbf{x}) = 0$ otherwise. Then

$$H^n S(\alpha) = \sum_{\mathbf{h}} \sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{h}),$$

where the sum is for vectors with $1 \le h_i \le H$ for each i. We re-write this as

$$H^n S(\alpha) = \sum_{\mathbf{x}\in\mathbb{Z}^n} \sum_{\mathbf{h}} f(\mathbf{x}+\mathbf{h}).$$

Since $\mathcal{B} \subseteq [-1,1]^n$ in (1.1), and $H \le P$, it follows that $f(\mathbf{x}+\mathbf{h}) = 0$ unless $\max|x_i| \le 2P$. Thus Cauchy's inequality yields

$$H^{2n}|S(\alpha)|^2 \le (2P+1)^n \sum_{\mathbf{x}\in\mathbb{Z}^n} |\sum_{\mathbf{h}} f(\mathbf{x}+\mathbf{h})|^2. \tag{3.1}$$

We expand the square to give

$$\begin{aligned} H^{2n}|S(\alpha)|^2 &\le (2P+1)^n \sum_{\mathbf{h}_1}\sum_{\mathbf{h}_2}\sum_{\mathbf{x}\in\mathbb{Z}^n} f(\mathbf{x}+\mathbf{h}_1)\overline{f(\mathbf{x}+\mathbf{h}_2)} \\ &= (2P+1)^n \sum_{\mathbf{h}_1}\sum_{\mathbf{h}_2}\sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h}_1-\mathbf{h}_2)\overline{f(\mathbf{y})} \\ &= (2P+1)^n \sum_{\mathbf{h}} w(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h})\overline{f(\mathbf{y})}, \end{aligned} \tag{3.2}$$

where, in the final line, the sum over $\mathbf{h}$ is for $\max|h_i| \le H$, and

$$w(\mathbf{h}) := \#\{\mathbf{h}_1,\mathbf{h}_2 : \mathbf{h} = \mathbf{h}_1 - \mathbf{h}_2\} \le H^n. \tag{3.3}$$

We therefore conclude that

$$|S(\alpha)|^2 \ll H^{-n}P^n \sum_{\mathbf{h}} |T(\mathbf{h},\alpha)|, \tag{3.4}$$

where

$$T(\mathbf{h},\alpha) := \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h})\overline{f(\mathbf{y})}. \tag{3.5}$$

A comparison of (3.4) with (2.1) shows that the special case $H = P$ of van der Corput's method reduces to our previous bound. Thus nothing has been lost at this stage, but potential flexibility in the choice of H has been introduced.

As is §2 we now square $|T(\mathbf{h},\alpha)|$ to give

$$|T(\mathbf{h},\alpha)|^2 = \sum_{\mathbf{y},\mathbf{z}} e(\alpha\{C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y}) - C(\mathbf{z}+\mathbf{h}) + C(\mathbf{z})\}),$$

where the summation is subject to the conditions

$$\mathbf{y}+\mathbf{h}, \mathbf{y}, \mathbf{z}+\mathbf{h}, \mathbf{z} \in P\mathcal{B}.$$

If we substitute $\mathbf{y} = \mathbf{z}+\mathbf{w}$ the sum becomes

$$\sum_{\mathbf{z},\mathbf{w}} e(\alpha C(\mathbf{w},\mathbf{h},\mathbf{z})),$$

in the notation of (2.3), where the summation conditions are now

$$\mathbf{z}+\mathbf{w}+\mathbf{h}, \mathbf{z}+\mathbf{w}, \mathbf{z}+\mathbf{h}, \mathbf{z} \in P\mathcal{B}.$$

For given $\mathbf{w}$ and $\mathbf{h}$, the conditions on $\mathbf{z}$ define a box. Hence if ρ in (1.1) is small enough, the argument leading to (2.5) produces

$$|T(\mathbf{h},\alpha)|^2 \ll (P\mathcal{L})^n N(\alpha, P, \mathbf{h}), \tag{3.6}$$

where we have defined

$$N(\alpha, P, \mathbf{h}) := \#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < P,\ ||6\alpha B_i(\mathbf{h};\mathbf{w})|| < P^{-1}\ \forall i \le n\}.$$

We then apply Lemma 2.2 to deduce, as in (2.6), that

$$N(\alpha, P, \mathbf{h}) \ll Z^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < ZP,\ ||6\alpha B_i(\mathbf{h};\mathbf{w})|| < ZP^{-1}\ \forall i \le n\}$$

for any $Z \in (0,1]$.

We have now reached the major point of difference between our approach and Davenport's. In our method, if H is chosen suitably, both variables $\mathbf{h}, \mathbf{w}$ in the bilinear forms B_i above will be suitably small, while Davenport requires a second application of Lemma 2.2 to achieve this. Thus our strategy is to apply Lemma 2.3 at this stage, taking $P_0 = PZ^{-1}$ and $M = cHZP$ with a suitable constant c so that $6|B_i(\mathbf{h};\mathbf{w})| \le cHZP$. The conditions required are then that $Z \le 1$, $2q \le PZ^{-1}$, $|\theta| \le (2cqHZP)^{-1}$ and either $cHZP < q$ or $|\theta| > Z(qP)^{-1}$. We define

$$\psi := |\theta| + \frac{1}{P^2 H}. \tag{3.7}$$

Then it suffices to have

$$Z \le 1, \quad Z \le \{2(c+1)qHP\psi\}^{-1}, \quad \text{and} \quad Z \le \{2(c+1)\}^{-1}qP\psi. \tag{3.8}$$

Providing the above conditions are satisfied Lemma 2.3 will yield

$$N(\alpha, P, \mathbf{h}) \ll Z^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < ZP,\ B_i(\mathbf{h};\mathbf{w}) = 0\ \forall i \le n\}.$$

The values of $\mathbf{w}$ are restricted to a vector space of dimension $n - r(\mathbf{h})$ where $r(\mathbf{h})$ is as in (1.4). Thus, when $ZP \gg 1$ we have

$$N(\alpha, P, \mathbf{h}) \ll Z^{-n}(ZP)^{n-r(\mathbf{h})}.$$

If we insert this bound into (3.6) and (3.4) we find that

$$|S(\alpha)|^2 \ll H^{-n}P^n \sum_{\mathbf{h}} (P\mathcal{L})^{n/2}\{Z^{-n}(ZP)^{n-r(\mathbf{h})}\}^{1/2}.$$

According to (1.5) each rank $r(\mathbf{h}) = r$ occurs $O(H^{r+\varepsilon})$ times, and summing over the possible values of r we conclude that

$$|S(\alpha)|^2 \ll P^{2n+\varepsilon}\{P^{-n/2}Z^{-n/2} + H^{-n}\}\mathcal{L}^{n/2}.$$

Clearly this is trivially true if the condition $ZP \gg 1$ is violated. We choose Z as large as possible within the constraints (3.8), as

$$Z = \min\left(\{2(c+1)qHP\psi\}^{-1}, \{2(c+1)\}^{-1}qP\psi\right). \tag{3.9}$$

Note that this automatically yields $Z \le 1$. We then have

$$|S(\alpha)|^2 \ll P^{2n+\varepsilon}\{(qH\psi)^{n/2} + P^{-n}(q\psi)^{-n/2} + H^{-n}\}\mathcal{L}^{n/2}.$$

We would like to choose H to optimize this, recalling that $H \le P$ must be a positive integer. However this is not straightforward, since ψ depends on H. We shall take

$$H = \begin{cases} [q^{1/3}], & |\theta| \le q^{-1/3}P^{-2}, \\ \max\{[P^{-2}|\theta|^{-1}], [(q|\theta|)^{-1/3}]\}, & |\theta| > q^{-1/3}P^{-2}. \end{cases}$$

Then if $q \le P^3$ we will have $1 \le H \le P$. Moreover we will have

$$H \ll q^{1/3} \ll H, \qquad \psi \ll (P^2H)^{-1} \ll \psi$$

and

$$H \ll \max\{P^{-2}|\theta|^{-1}, (q|\theta|)^{-1/3}\} \ll H, \qquad \psi \ll |\theta| \ll \psi$$

respectively in the two cases. Under the additional assumption that $|\theta| \le q^{-2}$ this leads to

$$|S(\alpha)|^2 \ll P^{2n+\varepsilon}\{P^{-n}q^{n/2} + q^{-n/3}\}\mathcal{L}^{n/2} \tag{3.10}$$

and

$$|S(\alpha)|^2 \ll P^{2n+\varepsilon}\{(q|\theta|)^{n/3} + P^{-n}(q|\theta|)^{-n/2}\}\mathcal{L}^{n/2} \tag{3.11}$$

in the two cases respectively.

As is §2 we proceed to consider the effect of these bounds when α is approximated via Dirichlet's Theorem with

$$|\theta| \le \frac{1}{qQ} \quad \text{and} \quad q \le Q.$$

The values of α for which

$$\frac{1}{2qQ} \le |\theta| \le \frac{1}{qQ} \quad \text{and} \quad Q/2 \le q \le Q$$

will make up a positive proportion of $[0, 1]$. For such α the above bounds reduce to

$$S(\alpha) \ll \begin{cases} P^{n+\varepsilon}Q^{-n/6}, & Q \le P^{6/5}, \\ P^{n/2+\varepsilon}Q^{n/4}, & Q \ge P^{6/5}. \end{cases}$$

Thus the optimal choice is $Q = P^{6/5}$, for which our estimate becomes

$$S(\alpha) \ll P^{4n/5+\varepsilon}. \tag{3.12}$$

This is clearly superior to (2.11). However, even if (3.12) were to hold for all α, it would not suffice to establish Theorem 3 for $n \geq 15$, let alone for $n \geq 14$. Indeed the bounds (3.10) and (3.11), even taken in conjunction with (2.9), are not quite enough to handle the case $n = 16$, so that further savings are essential. The author is grateful to Dr Browning for this observation.

4 Averaged Versions of van der Corput's Method

It is well known in the context of one-dimensional exponential sums that the inequality on which van der Corput's method is based can be interpreted as arising from a mean square average of the sum over a short range. Thus, for example, the classical bound

$$\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{1/6+\varepsilon}, \quad (t \geq 1)$$

for the Riemann Zeta-function, valid for any $\varepsilon > 0$, corresponds to a mean-value estimate

$$\int_T^{T+T^{1/3}} |\zeta(\frac{1}{2}+it)|^2 dt \ll_\varepsilon T^{1/3+\varepsilon} \quad (T \geq 1)$$

(see Heath-Brown [7]), or indeed to the bound $E(T) \ll_\varepsilon T^{1/3+\varepsilon}$ for the error term in the asymptotic formula

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = T(\log\frac{T}{2\pi} + 2\gamma - 1) + E(T),$$

(see Balasubramanian [2]). In general one can expect a better bound for the mean square of an exponential sum than can be obtained by applying van der Corput's method pointwise. In effect, the averaging process automatically produces a shortened variable, corresponding to the reduction from P to H which one sees on comparing (3.4) with (2.1). Unfortunately, when one has n-dimensional exponential sums, a 1-dimensional averaging corresponds to the shortening of only one variable, rather than n variables.

We proceed to investigate the mean-square

$$M(\alpha, H) := \int_{\alpha-(P^2H)^{-1}}^{\alpha+(P^2H)^{-1}} |S(\beta)|^2 d\beta.$$

We required the centre point $\boldsymbol{\xi}$ of the box $\mathcal{B}$ to be non-singular. We may therefore re-order the indices so that $G > 0$ where we write, temporarily,

$$G := \left|\frac{\partial C(\boldsymbol{\xi})}{\partial \xi_1}\right|. \tag{4.1}$$

We now run through the argument of the previous section, up to (3.2), but with α replaced by β and $\mathbf{h}$ restricted by the conditions $1 \le h_1 \le P$ and $1 \le h_2, h_3, \ldots, h_n \le H$. This yields the inequality

$$P^2 H^{2n-2} |S(\beta)|^2 \le (2P+1)^n \sum_{\mathbf{h}} w(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} f(\mathbf{y}+\mathbf{h})\overline{f(\mathbf{y})},$$

where now the summation condition on $\mathbf{h}$ is $|h_1| \le P$ and $\max_{2\le i\le n} |h_i| \le H$. Moreover, the weight $w(\mathbf{h})$ is still given by (3.3), but with the new restrictions on $\mathbf{h}_1, \mathbf{h}_2$. We now have

$$\begin{aligned} M(\alpha, H) &\le e\int_{-\infty}^{\infty} \exp\{-H^2P^4(\beta-\alpha)^2\}|S(\beta)|^2 d\beta \\ &\le e(2P+1)^n P^{-2} H^{2-2n} \sum_{\mathbf{h}} w(\mathbf{h}) \sum_{\mathbf{y}\in\mathbb{Z}^n} I(\mathbf{h},\mathbf{y}), \qquad (4.2)\end{aligned}$$

where

$$I(\mathbf{h},\mathbf{y}) := \int_{-\infty}^{\infty} \exp\{-H^2P^4(\beta-\alpha)^2\} e(\beta\{C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y})\})d\beta, \qquad (4.3)$$

and where the sums over $\mathbf{h}$ and $\mathbf{y}$ are restricted by the condition that $\mathbf{y}+\mathbf{h}$ and $\mathbf{y}$ belong to $P\mathcal{B}$. We may alternatively write

$$I(\mathbf{h},\mathbf{y}) = \frac{\sqrt{\pi}}{HP^2}\exp(-\pi^2\{\frac{C(\mathbf{y}+\mathbf{h})-C(\mathbf{y})}{HP^2}\}^2) e(\alpha\{C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y})\}),$$

and since $w(\mathbf{h}) \ll PH^{n-1}$ it is clear that terms with

$$|C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y})| \ge HP^2\mathcal{L} \qquad (4.4)$$

make a total contribution $O(1)$ to $M(\alpha, H)$. However

$$C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y}) = h_1\frac{\partial C(\mathbf{y})}{\partial y_1} + O(HP^2) + O(h_1^2 P),$$

the order constants depending on the form $C(\mathbf{x})$ alone, and not on the box $\mathcal{B}$. If we choose ρ small enough in (1.1) we will have

$$|\frac{\partial C(\mathbf{y})}{\partial y_1}| \ge \frac{1}{2}P^2 G,$$

since we must have $\mathbf{y}+\mathbf{h}, \mathbf{y} \in P\mathcal{B}$. Moreover, again by choosing ρ small enough, we can force the error term $O(h_1^2 P)$ above to be at most $\frac{1}{4}G|h_1|P^2$ in absolute magnitude. It then follows that

$$|C(\mathbf{y}+\mathbf{h}) - C(\mathbf{y})| \ge \frac{1}{4}G|h_1|P^2 + O(HP^2).$$

Thus, for large enough P, the condition (4.4) will be met unless $|h_1| \le 5G^{-1}H\mathcal{L}$, say, or indeed unless $|h_1| \le H\mathcal{L}^2$. We therefore conclude that the contribution to (4.2), arising from those terms with $|h_1| > H\mathcal{L}^2$, is $O(1)$.

We may now deduce from (4.2) that

$$M(\alpha, H) \ll P^{n-2}H^{2-2n} \sum_{|h_1|\le H\mathcal{L}^2} \sum_{|h_2|\le H} \cdots \sum_{|h_n|\le H} w(\mathbf{h}) \left|\sum_{\mathbf{y}\in\mathbb{Z}^n} I(\mathbf{h},\mathbf{y})\right|.$$

In (4.3) the range $|\beta-\alpha| \geq H^{-1}P^{-2}\mathcal{L}$ trivially contributes $O(1)$ in total, whence

$$M(\alpha, H) \ll 1 + P^{n-1}H^{1-n} \sum_{|h_i| \leq H\mathcal{L}^2} \int_{\alpha-(P^2H)^{-1}\mathcal{L}}^{\alpha+(P^2H)^{-1}\mathcal{L}} |T(\mathbf{h}, \beta)| d\beta,$$

where $T(\mathbf{h}, \beta)$ is given by (3.5). Roughly speaking, this bound says that a suitable mean-square average of $S(\beta)$ is bounded by the corresponding average of

$$P^{n-1}H^{1-n} \sum_{\mathbf{h}} |T(\mathbf{h}, \beta)|.$$

If we compare this with (3.4) we see that we have gained a factor HP^{-1}.

We conclude our initial treatment of $M(\alpha, H)$ by estimating $T(\mathbf{h}, \beta)$ as in (3.6), to deduce that

$$M(\alpha, H) \ll 1 + P^{3n/2-3}H^{-n}\mathcal{L}^{n/2+1} \sum_{|h_i| \leq H\mathcal{L}^2} \max_{\beta \in \mathcal{I}} N(\beta, P, \mathbf{h})^{1/2}, \tag{4.5}$$

where

$$\mathcal{I} = \{\beta : |\beta - \alpha| \leq H^{-1}P^{-2}\mathcal{L}\}. \tag{4.6}$$

5 Averages of $N(\beta, P, \mathbf{h})$

Our next task is to consider the size of $N(\beta, P, \mathbf{h})$. The average we will consider is

$$A(\theta, R, H, P) := \sum_{R<q\leq 2R} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{|h_i| \leq H\mathcal{L}^2} \max_{\beta} N(\beta, P, \mathbf{h})^{1/2}, \tag{5.1}$$

where β is in the range (4.6) with $\alpha = a/q + \theta$. We begin by observing that if $|h_i| \leq H\mathcal{L}^2$ and $|\mathbf{w}| < P$, then $|B_i(\mathbf{h}; \mathbf{w})| \leq cHP\mathcal{L}^2$ for some constant c. Thus, if β lies in the range (4.6), the inequality $||6\beta B_i(\mathbf{h}; \mathbf{w})|| < P^{-1}$ implies

$$||6\alpha B_i(\mathbf{h}; \mathbf{w})|| < P^{-1} + 6cP^{-1}\mathcal{L}^3 \leq (1 + 6c)P^{-1}\mathcal{L}^3.$$

Now, if we set $\widetilde{P} = \{(1 + 6c)\mathcal{L}^3\}^{-1}P$ we will have

$$\max_{\beta} N(\beta, P, \mathbf{h}) \leq \#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < P, \ ||6\alpha B_i(\mathbf{h}; \mathbf{w})|| < \widetilde{P}^{-1} \ \forall i \leq n\}.$$

Write $\widehat{P} = \widetilde{P}/2$. We proceed to cover the region $|\mathbf{w}| < P$ by $O(P^n\widehat{P}^{-n})$ balls $|\mathbf{w}| < \widehat{P}/2$. If $\mathbf{w}_0, \mathbf{w}_1$ are two integer vectors in the same ball, both counted in the set above, then $|\mathbf{w}_1 - \mathbf{w}_0| < \widehat{P}$ and

$$||6\alpha B_i(\mathbf{h}; \mathbf{w}_1 - \mathbf{w}_0)|| < 2\widetilde{P}^{-1} = \widehat{P}^{-1} \ \ \forall i \leq n.$$

It follows that

$$\begin{aligned} &\max_{\beta} N(\beta, P, \mathbf{h}) \\ &\qquad \ll \ P^n\widehat{P}^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < \widehat{P}, \ ||6\alpha B_i(\mathbf{h}; \mathbf{w})|| < \widehat{P}^{-1} \ \forall i \leq n\} \\ &\qquad \ll \ \mathcal{L}^{3n}N(\alpha, \widehat{P}, \mathbf{h}). \end{aligned}$$

We proceed to use Lemma 2.2 in two different ways, taking $a = \widehat{P}$ so that $N(1) \ll Z^{-n}N(Z)$ for any $Z \in (0,1]$. In the first application we will choose Z just small enough to ensure that $m = 6B_i(\mathbf{h};\mathbf{w}) = 0$, while in the second we shall only require that $q|m$. The condition $|m| \le M$ will hold with $M = cH\widehat{P}Z\mathcal{L}^2$ for a suitable constant c. The first choice needs $0 < Z \le 1$, $|\theta| \le (2qcH\widehat{P}Z\mathcal{L}^2)^{-1}$, $P_0 = \widehat{P}Z^{-1} \ge 2q$, and either

$$cH\widehat{P}Z\mathcal{L}^2 < q \text{ or } |\theta| > (q\widehat{P})^{-1}Z.$$

When $R < q \le 2R$ it therefore suffices to take

$$Z = Z_1 := c\mathcal{L}^{-3}\min\{(RHP\psi)^{-1}\,,\, RP\psi\},$$

for a suitable constant $c > 0$, with ψ given by (3.7) as before. In particular

$$Z_1 \le c\mathcal{L}^{-3}(RHP\psi)^{-1/2}(RP\psi)^{1/2} \le 1$$

for sufficiently small c. For the second application it suffices similarly to take

$$Z = Z_2 := c\mathcal{L}^{-3}\min\{1, (RHP\psi)^{-1}\}. \tag{5.2}$$

Thus our first choice leads to

$$\begin{aligned}\max_{\beta} N(\beta, P, \mathbf{h}) &\ll \mathcal{L}^{3n}N(\alpha, \widehat{P}, \mathbf{h}) \\ &\ll \mathcal{L}^{3n}Z_1^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < Z_1\widehat{P},\, B_i(\mathbf{h};\mathbf{w}) = 0\ \forall i \le n\} \\ &\ll \mathcal{L}^{3n}Z_1^{-n}\{1 + (Z_1\widehat{P})^{n-r}\},\end{aligned}$$

where $r = r(\mathbf{h})$. Since $P \ge H \ge 1$ we trivially have $N(\beta, P, \mathbf{h}) \ll P^n \ll \mathcal{L}^{3n}\widehat{P}^n$, whence

$$N(\beta, P, \mathbf{h}) \ll \mathcal{L}^{3n}\min\left(\widehat{P}^n\,,\, Z_1^{-n}\{1 + (Z_1\widehat{P})^{n-r}\}\right).$$

However the minimum above is clearly $O(Z_1^{-n}(Z_1\widehat{P})^{n-r})$, whether $Z_1\widehat{P} \ge 1$ or not. Thus

$$\begin{aligned}\max_{\beta} N(\beta, P, \mathbf{h}) &\ll \mathcal{L}^{3n}Z_1^{-n}(Z_1\widehat{P})^{n-r} \\ &\ll \mathcal{L}^{6n}P^n\left(\min\{(RH\psi)^{-1}\,,\, RP^2\psi\}\right)^{-r} \\ &\ll \mathcal{L}^{6n}P^n\{(RH\psi)^r + (RP^2\psi)^{-r}\}. \end{aligned} \tag{5.3}$$

Our second choice for Z yields

$$\max_{\beta} N(\beta, P, \mathbf{h}) \ll$$

$$\mathcal{L}^{3n}Z_2^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < Z_2\widehat{P},\, q|B_i(\mathbf{h};\mathbf{w})\ \forall i \le n\}. \tag{5.4}$$

In order to count vectors $\mathbf{w}$ with $q|B_i(\mathbf{h};\mathbf{w})$ for every i we shall decompose q into different types of prime factors. For given $\mathbf{h}$, with $r(\mathbf{h}) = r$, we say that p is of type I if p divides every $r \times r$ minor of the matrix $M(\mathbf{h})$. If p is not of type I we shall say that p is of type II. We then decompose q as $q = q_1q_2$, where q_1 is a product of type I primes, and q_2 a product of type II primes. For any integer m we write

$$\Lambda(m) = \{\mathbf{w} \in \mathbb{Z}^n : m|B_i(\mathbf{h};\mathbf{w})\,\forall i\}$$

and we set $\Lambda(\mathbf{h}) = \Lambda(q_2)$. Thus the condition that $q|B_i(\mathbf{h};\mathbf{w})$ for every i then restricts $\mathbf{w}$ to the lattice $\Lambda(\mathbf{h})$, whose determinant $d(\Lambda(\mathbf{h}))$ will be a product of prime powers for primes $p|q_2$. We proceed to estimate this determinant, using the fact that $d(\Lambda(m))$ is the index $[\mathbb{Z}^n : \Lambda(m)]$. Then

$$[\mathbb{Z}^n : \Lambda(p^e)] = [\Lambda(p^0) : \Lambda(p^e)] = \prod_{f=1}^{e} [\Lambda(p^{f-1}) : \Lambda(p^f)]. \tag{5.5}$$

Moreover the map

$$\theta : \mathbb{Z}^n/\Lambda(p) \to \Lambda(p^{f-1})/\Lambda(p^f)$$

given by

$$\mathbf{w} + \Lambda(p) \mapsto p^{f-1}\mathbf{w} + \Lambda(p^f)$$

is an injective homomorphism, so that $[\mathbb{Z}^n : \Lambda(p)]$ divides $[\Lambda(p^{f-1}) : \Lambda(p^f)]$. However if p is a type II prime, then the conditions $p|B_i(\mathbf{h};\mathbf{w})$ define a lattice of determinant p^r, whence (5.5) implies that $p^{er}|d(\Lambda(\mathbf{h}))$ whenever $p^e||q_2$. It follows that $q_2^r|d(\Lambda(\mathbf{h}))$.

We shall require some further facts about lattices and their successive minima, which we summarize in the following lemma.

Lemma 5.1 *Let $\Lambda \subseteq \mathbb{Z}^n$ be an n-dimensional lattice of determinant $d(\Lambda)$ and with successive minima $\lambda_1 \leq \ldots \leq \lambda_n$. Then $d(\Lambda) \leq \prod_{i=1}^n \lambda_i$ and*

$$\#\{\mathbf{x} \in \Lambda : |\mathbf{x}| \leq B\} \ll \prod_{i\leq n}(1 + B/\lambda_i).$$

For a proof, see (2.2) and Lemma 4 of Browning and Heath-Brown [3], for example.

We shall take $\Lambda = \Lambda(\mathbf{h})$. We observe that

$$q_2\mathbb{Z}^n \subseteq \Lambda \subseteq \mathbb{Z}^n$$

whence $1 \ll \lambda_i \ll q_2$ for every index i. Moreover Lemma 5.1 implies that $\prod \lambda_i \geq q_2^r$ and

$$\#\{\mathbf{x} \in \Lambda : |\mathbf{x}| \leq B\} \ll \prod_{i\leq n}(1 + B/\lambda_i).$$

If we maximize the right hand side with respect to the various λ_i, subject to the above constraints, we find that the extremum occurs when r of the λ_i are of order q_2 and the remainder are of order 1. If $B \geq 1$ this leads to an estimate of the form

$$\#\{\mathbf{x} \in \Lambda : |\mathbf{x}| \leq B\} \ll (1 + \frac{B}{q_2})^r B^{n-r}.$$

We now apply this bound in (5.4), with Z_2 as in (5.2), and $B = 1 + Z_2\widehat{P}$. Then if $HR\psi \leq 1$ we have

$$Z_2^{-1}B = Z_2^{-1} + \widehat{P} \ll \mathcal{L}^3(1 + HRP\psi + P) \ll \mathcal{L}^3 P$$

and

$$B^{-1} \ll Z_2^{-1}\widehat{P}^{-1} \ll \mathcal{L}^6(P^{-1} + HR\psi).$$

It follows that

$$\begin{aligned}
\max_{\beta} N(\beta, P, \mathbf{h}) &\ll \mathcal{L}^{3n} Z_2^{-n} (1 + \frac{B}{q_2})^r B^{n-r} \\
&= \mathcal{L}^{3n} \{B^{-1} + q_2^{-1}\}^r (Z_2^{-1} B)^n \\
&\ll \mathcal{L}^{12n} \{P^{-r} + (HR\psi)^r + q_2^{-r}\} P^n
\end{aligned}$$

when $HR\psi \le 1$. Moreover the bound is trivial when $HR\psi \ge 1$, since we always have $N(\beta, P, \mathbf{h}) \ll P^n$.

We may now combine the above result with (5.3) to deduce that

$$\begin{aligned}
&\max_{\beta} N(\beta, P, \mathbf{h}) \\
&\qquad \ll \mathcal{L}^{12n} P^n \left((RH\psi)^{r(\mathbf{h})} + \min\{(RP^2\psi)^{-r(\mathbf{h})}\,,\, P^{-r(\mathbf{h})} + q_2^{-r(\mathbf{h})}\} \right) \\
&\qquad \ll \mathcal{L}^{12n} P^n \left((RH\psi)^{r(\mathbf{h})} + P^{-r(\mathbf{h})} + \min\{(RP^2\psi)^{-r(\mathbf{h})}\,,\, q_2^{-r(\mathbf{h})}\} \right).
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
A(\theta, R, H, P) \quad &\ll \quad R\mathcal{L}^{6n} P^{n/2} \sum_{|h_i| \le H\mathcal{L}^2} \sum_{R < q \le 2R} \Big((RH\psi)^{r(\mathbf{h})/2} + P^{-r(\mathbf{h})/2} \\
&\qquad + \min\{(RP^2\psi)^{-r(\mathbf{h})/2}\,,\, q_2^{-r(\mathbf{h})/2}\} \Big). \qquad (5.6)
\end{aligned}$$

We therefore proceed to consider

$$V(\mathbf{h}, R, \psi) := \sum_{R < q \le 2R} \min\{(RP^2\psi)^{-r/2}\,,\, q_2^{-r/2}\}$$

for a given vector $\mathbf{h}$, with $r(\mathbf{h}) = r$. Recall that $q = q_1 q_2$, with q_1 composed of powers of primes each of which divides every $r \times r$ minor of the matrix $M(\mathbf{h})$. However, since $r(\mathbf{h}) = r$, there is some such minor $M_0 = M_0(\mathbf{h})$, say, which is non-zero. It follows that we may write

$$\begin{aligned}
V(\mathbf{h}, R, \psi) \quad &\ll \quad \sum_{q_1 \le 2R} \sum_{R/q_1 < q_2 \le 2R/q_1} \min\{(RP^2\psi)^{-r/2}\,,\, q_2^{-r/2}\} \\
&\ll \quad \sum_{q_1 \le 2R} Rq_1^{-1} \min\{(RP^2\psi)^{-r/2}\,,\, (R/q_1)^{-r/2}\},
\end{aligned}$$

where q_1 runs over integers all of whose prime factors divide M_0. We split the available range for q_1 into dyadic intervals $S \le q_1 < 2S$, whence

$$V(\mathbf{h}, R, \psi) \ll \mathcal{L} RS^{-1} \min \left((RP^2\psi)^{-r/2}\,,\, (R/S)^{-r/2} \right) \#\{q_1 \le 2S\}$$

for some positive integer $S \le R$. We estimate the number of admissible values of q_1 by the well-known method of Rankin. We have

$$\#\{q_1 \le 2S\} \ll S^{\varepsilon} \sum_{q_1} q_1^{-\varepsilon}$$

where the sum on the right runs over all integers q_1 composed only of prime factors dividing M_0. Thus

$$\sum_{q_1} q_1^{-\varepsilon} = \prod_{p | M_0} (1 - p^{-\varepsilon})^{-1} \ll d(|M_0|) \ll |M_0|^{\varepsilon/n},$$

where $d(|M_0|)$ is the divisor function. Since $|M_0| \ll H^r \mathcal{L}^{2r}$ we conclude that

$$\#\{q_1 \leq 2S\} \ll (HS)^\varepsilon \mathcal{L},$$

and hence that

$$V(\mathbf{h}, R, \psi) \ll \mathcal{L}^2 (HR)^\varepsilon \max_{S \geq 1} RS^{-1} \min\{(RP^2\psi)^{-r/2}, (R/S)^{-r/2}\}.$$

When $P^2\psi \geq 1$ and $r \geq 2$ the value $S = 1$ is maximal, while if $P^2\psi \leq 1$ and $r \geq 2$ the worst S is $(P^2\psi)^{-1}$. Hence

$$V(\mathbf{h}, R, \psi) \ll \mathcal{L}^2 (HR)^\varepsilon R (RP^2\psi)^{-r/2} \min\{1, P^2\psi\}$$

if $r \geq 2$. On the other hand, when $r \leq 1$, the maximum occurs at $S = 1$, so that

$$V(\mathbf{h}, R, \psi) \ll \mathcal{L}^2 (HR)^\varepsilon R (RP^2\psi)^{-r/2} \min\{1, (P^2\psi)^{r/2}\}.$$

We combine our two estimates by writing

$$V(\mathbf{h}, R, \psi) \ll \mathcal{L}^2 (HR)^\varepsilon R (RP^2\psi)^{-r/2} \min\{1, (P^2\psi)^{e(r)}\}$$

with $e(r) = r/2$ for $r \leq 1$ and $e(r) = 1$ otherwise.

We now see from (5.6) and (1.5) that

$$\begin{aligned}
A(\theta, R, H, P) &\ll R^2 \mathcal{L}^{6n} P^{n/2} \sum_{|h_i| \leq H\mathcal{L}^2} \Big((RH\psi)^{r(\mathbf{h})/2} + P^{-r(\mathbf{h})/2} \\
&\qquad + R^{-1} V(\mathbf{h}, R, \psi)\Big) \\
&\ll R^{2+\varepsilon} \mathcal{L}^{6n+2} P^{n/2+\varepsilon} \sum_{|h_i| \leq H\mathcal{L}^2} \Big((RH\psi)^{r(\mathbf{h})/2} + P^{-r(\mathbf{h})/2} \\
&\qquad + (RP^2\psi)^{-r(\mathbf{h})/2} \min\{1, (P^2\psi)^{e(r(\mathbf{h}))}\}\Big) \\
&\ll R^{2+\varepsilon} P^{n/2+3\varepsilon} \sum_{r=0}^{n} H^r \Big((RH\psi)^{r/2} + P^{-r/2} \\
&\qquad + (RP^2\psi)^{-r/2} \min\{1, (P^2\psi)^{e(r)}\}\Big) \\
&\ll R^{2+\varepsilon} P^{n/2+3\varepsilon} \Big(1 + (RH^3\psi)^{n/2} + H^n P^{-n/2} \\
&\qquad + (H^2 R^{-1} P^{-2} \psi^{-1})^{n/2} \min\{1, P^2\psi\} \\
&\qquad + H^2 R^{-1} P^{-2} \psi^{-1} \min\{1, P^2\psi\} \\
&\qquad + (H^2 R^{-1} P^{-2} \psi^{-1})^{1/2} \min\{1, (P^2\psi)^{1/2}\}\Big).
\end{aligned}$$

The first, fourth, fifth, and sixth terms in the brackets have the form

$$1 + A^n m + A^2 m + A m^{1/2}$$

with $m \leq 1$. However it is clear that $Am^{1/2} \leq \max(1, A^2 m)$, and that $A^2 m \leq \max(1, A^n m)$. Thus the final two terms are redundant and we may conclude that

$$\begin{aligned}
A(\theta, R, H, P) &\ll R^{2+\varepsilon} P^{n/2+3\varepsilon} \Big(1 + (RH^3\psi)^{n/2} + H^n P^{-n/2} \\
&\qquad + (H^2 R^{-1} P^{-2} \psi^{-1})^{n/2} \min\{1, P^2\psi\}\Big).
\end{aligned}$$

Finally we note that

$$\begin{aligned}
H^n P^{-n/2} &= \{(RH^3\psi)^{n/2}.(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}\}^{1/2}H^{-n/4} \\
&\leq \max\{(RH^3\psi)^{n/2}\,,\,(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}\}H^{-n/4} \\
&\leq \{(RH^3\psi)^{n/2}+(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}\}H^{-1} \\
&\leq (RH^3\psi)^{n/2}+(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}\min\{1\,,\,P^2\psi\},
\end{aligned}$$

since $P^2\psi \geq H^{-1}$. Thus

$$\begin{aligned}
A(\theta,R,H,P) \ll\ & R^{2+\varepsilon}P^{n/2+3\varepsilon}\Big(1+(RH^3\psi)^{n/2} \\
&+(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}\min\{1\,,\,P^2\psi\}\Big). \qquad (5.7)
\end{aligned}$$

It turns out that it suffices to use the term $P^2\psi$ from the minimum occurring here.

6 Bounding the Minor Arc Integral

Our goal in this section is to estimate the contribution to the minor arc integral arising from those regions where $\alpha = a/q+\theta$ and either q or θ is "large" (or both). We assume that the minor arcs are defined via Dirichlet's Approximation Theorem with

$$|\alpha-\frac{a}{q}|\leq\frac{1}{qQ},\quad q\leq Q$$

for some Q in the range $P \leq Q \leq P^{5/4}$. Suppose that $H = H(R,\phi,P)$ is a positive integer with $H \leq P$, and consider

$$\Sigma(R,\phi,\pm) := \sum_{R<q\leq 2R}\ \sum_{\substack{a\leq q\\(a,q)=1}}\int_{\phi}^{2\phi}|S(\frac{a}{q}\pm\nu)|d\nu$$

with $\phi \leq (RQ)^{-1}$ and $R \leq Q$. Our aim in this section is to achieve a bound $\Sigma(R,\phi,\pm) \ll P^{n-3-\varepsilon}$.

The simplest procedure is to apply the bound (2.10) directly, leading to

$$\Sigma(R,\phi,\pm) \ll R^2\phi P^{n+\varepsilon}\{(R\phi)^{n/8}+(R\phi P^3)^{-n/8})\}.$$

We therefore obtain our first result, as follows.

Lemma 6.1 *We have*

$$\Sigma(R,\phi,\pm) \ll P^{n-3-\varepsilon}$$

providing that $\phi \leq (RQ)^{-1}$ *and*

$$R^{(16-n)/(n-8)}P^{-3+8\varepsilon} \leq \phi \leq R^{-(n+16)/(n+8)}P^{-24/(n+8)-\varepsilon}.$$

Here we use the assumption that $n \geq 10$ to help in controlling the ε terms in the exponents.

To use our version of van der Corput's method we begin by employing Cauchy's inequality to deduce that

$$\Sigma(R,\phi,\pm) \ll \phi^{1/2}R\left\{\sum_{R<q\le 2R}\sum_{\substack{a\le q\\(a,q)=1}}\int_{\phi}^{2\phi}|S(\frac{a}{q}\pm\nu)|^2 d\nu\right\}^{1/2}.$$

We may cover the range $[\phi, 2\phi]$ with $O(1+P^2H\phi)$ intervals of the form

$$[\theta - P^{-2}H^{-1}, \theta + P^{-2}H^{-1}]$$

with $\phi \le \theta \le 2\phi$, whence

$$\Sigma(R,\phi,\pm) \ll \phi^{1/2}\psi^{1/2}R\left\{P^2H\sum_{R<q\le 2R}\sum_{\substack{a\le q\\(a,q)=1}} M(\frac{a}{q}+\theta, H)\right\}^{1/2}$$

for some θ in the range $\phi \le |\theta| \le 2\phi$, with ψ as in (3.7) as before. We may then use (4.5) and (5.1) to deduce that

$$\Sigma(R,\phi,\pm) \ll \phi^{1/2}\psi^{1/2}R\left\{R^2P^2H + P^{3n/2-1}H^{1-n}\mathcal{L}^{n/2+1}A(\theta,R,H,P)\right\}^{1/2},$$

whence (5.7) produces

$$\Sigma(R,\phi,\pm) \ll \phi^{1/2}\psi^{1/2}R^2\left\{P^2H + P^{2n-1+5\varepsilon}H^{1-n}E\right\}^{1/2}, \tag{6.1}$$

where

$$E = 1 + (RH^3\psi)^{n/2} + (H^2R^{-1}P^{-2}\psi^{-1})^{n/2}P^2\psi. \tag{6.2}$$

The term P^2H in the braces in (6.1) will turn out to be negligible.

We proceed to show that if $n \ge 14$ then

$$\Sigma(R,\phi,\pm) \ll P^{n-3-\varepsilon}, \tag{6.3}$$

if Q is suitably chosen. We shall also show that if $n \le 13$, and if $Q \ll R \ll Q$ and $Q^{-2} \ll \phi \ll Q^{-2}$, then there is no choice of Q for which our bounds will suffice to prove (6.3).

We begin by supposing that $n \ge 14$. In fact we shall present the calculations for $n = 14$, larger values being handled similarly. In the course of our analysis we shall assume that ε is sufficiently small whenever it is necessary. We will have

$$\phi^{1/2}\psi^{1/2}R^2\{P^{2n-1+5\varepsilon}H^{1-n}E\}^{1/2} \ll P^{n-3-\varepsilon}$$

providing that

$$\phi^{1/2}\psi^{1/2}R^2\{P^{2n-1+5\varepsilon}H^{1-n}\}^{1/2} \ll P^{n-3-\varepsilon}. \tag{6.4}$$

and

$$E \ll 1. \tag{6.5}$$

We begin by choosing H so that (6.4) holds. For this latter condition it is sufficient to have

$$\phi\psi R^4 P^{5+7\varepsilon} \ll H^{n-1}.$$

In view of (3.7) this is equivalent to

$$H^{n-1} \gg \phi^2 R^4 P^{5+7\varepsilon} \quad \text{and} \quad H^n \gg \phi R^4 P^{3+7\varepsilon}.$$

When $n = 14$ we may therefore take

$$H = \left[P^{\varepsilon} \max\{(\phi^2 R^4 P^5)^{1/13}, (\phi R^4 P^3)^{1/14}, 1\}\right].$$

We will then have $H \le P$, since $\phi \le (RQ)^{-1}$ and $R \le Q \le P^{5/4}$. Moreover we find that

$$\phi^{1/2}\psi^{1/2}R^{1/2}\{P^2 H\}^2 \ll R^2\{P^3\}^{1/2} \ll P^{5/2}.P^{3/2} \ll P^{n-3-\varepsilon},$$

so that the term $P^2 H$ in the braces in (6.1) produces a satisfactory contribution.

We now need to investigate whether or not our choice of H ensures that (6.5) holds. We first consider the term $(RH^3\psi)^{n/2}$ in E. It is convenient to note at this point that if we set

$$\phi_0 = (R^4 P^{31})^{-1/15},$$

then

$$H = \left[P^{\varepsilon} \max\{(\phi R^4 P^3)^{1/14}, 1\}\right], \quad \psi \ll P^{\varepsilon}(P^2 H)^{-1} \quad \text{for} \quad \phi \le \phi_0, \qquad (6.6)$$

and

$$H = \left[P^{\varepsilon} \max\{(\phi^2 R^4 P^5)^{1/13}, 1\}\right], \quad \psi \ll \phi \quad \text{for} \quad \phi \ge \phi_0. \qquad (6.7)$$

Now, when $\phi \le \phi_0$ we calculate that

$$\begin{aligned} RH^3\psi &\ll RH^2 P^{\varepsilon-2} \\ &\ll RP^{3\varepsilon-2}\{1 + (\phi R^4 P^3)^{1/7}\} \\ &\ll RP^{3\varepsilon-2}\{1 + (Q^{-1}R^3 P^3)^{1/7}\} \\ &\ll QP^{3\varepsilon-2} + P^{3\varepsilon-11/7}Q^{9/7} \\ &\ll 1 \end{aligned}$$

providing that

$$Q \le P^{11/9-3\varepsilon}.$$

Similarly when $\phi \ge \phi_0$ we have

$$\begin{aligned} RH^3\psi &\ll RH^3\phi \\ &\ll R\phi P^{3\varepsilon}\{1 + (\phi^2 R^4 P^5)^{3/13}\} \\ &\ll Q^{-1}P^{3\varepsilon}\{1 + (Q^{-2}R^2 P^5)^{3/13}\} \\ &\ll Q^{-1}P^{15/13+3\varepsilon} \\ &\ll 1 \end{aligned}$$

providing that

$$Q \ge P^{15/13+3\varepsilon}.$$

We therefore choose

$$Q = P^{13/11}$$

which then suffices to ensure that $RH^3\psi \ll 1$ if ε is small enough.

We turn now to the condition

$$(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}P^2\psi \ll 1, \tag{6.8}$$

which is also necessary for (6.5). When $\phi \le \phi_0$ we have $\psi \ge (P^2H)^{-1}$, whence

$$(H^2R^{-1}P^{-2}\psi^{-1})^{n/2}P^2\psi \ll (H^3R^{-1})^{n/2}H^{-1}.$$

Thus, for $n = 14$, it is enough to have $H \le R^{7/20}$. We therefore see from (6.6) that it suffices to have $R \ge P^{3\varepsilon}$ and $\phi \le \min(\phi_0, \phi_1)$, where

$$\phi_1 = R^{9/10}P^{-3-14\varepsilon}.$$

Similarly when $\phi \ge \phi_0$ we have $\psi \ge \phi$, whence (6.8) holds for $n = 14$ if $H^{14} \ll R^7P^{12}\phi^6$. In view of (6.7) it suffices to have

$$P^{14\varepsilon} \le R^7P^{12}\phi^6$$

and

$$P^{14\varepsilon}(\phi^2R^4P^5)^{14/13} \le R^7P^{12}\phi^6.$$

If we set

$$\phi_2 = P^{4\varepsilon}\max\{R^{-7/6}P^{-2}\,,\,R^{-7/10}P^{-43/25}\} = R^{-7/10}P^{4\varepsilon-43/25}$$

it follows that (6.8) holds for $\phi \ge \max(\phi_0, \phi_2)$.

A further calculation shows that we will have $\phi_2 \le \phi_0 \le \phi_1$ whenever $R \ge P^{12\varepsilon+4/5}$. Thus (6.8) is always true in this case. For the remaining range $P^{3\varepsilon} \le R \le P^{12\varepsilon+4/5}$ we have $\phi_1 \le \phi_0$ and $\phi_2 \ge P^{-2\varepsilon}\phi_0$. It therefore follows for this case that (6.8) holds unless $\phi_1 \le \phi \le P^{2\varepsilon}\phi_2$. For this intermediate range we call on Lemma 6.1, which gives a satisfactory result when

$$R^{1/3}P^{-3+8\varepsilon} \le \phi \le R^{-15/11}P^{-12/11-\varepsilon}. \tag{6.9}$$

It therefore suffices to note that

$$R^{1/3}P^{-3+8\varepsilon} \le \phi_1 = R^{9/10}P^{-3-14\varepsilon}$$

for $R \ge P^{40\varepsilon}$, and that

$$R^{-15/11}P^{-12/11-\varepsilon} \ge P^{2\varepsilon}\phi_2 = R^{-7/10}P^{6\varepsilon-43/25}$$

for

$$R^{365} \le P^{346-3850\varepsilon}.$$

Since we are assuming that $R \le P^{4/5+12\varepsilon}$ this last condition holds providing that ε is small enough.

We have therefore shown that for $n = 14$ the bound (6.3) holds, for $\phi \le (RQ)^{-1}$ and $P^{40\varepsilon} \le R \le Q$, providing we choose $Q = P^{13/11}$. We also need to handle the case in which $R \leq P^{40\varepsilon}$ and ϕ is not too small. We shall write $\Delta = 40\varepsilon$. Then if $R \leq P^{\Delta}$ and $P^{-3+\Delta} \leq \phi \leq (RQ)^{-1}$ we have

$$\phi \geq P^{-3+40\varepsilon} \geq R^{1/3}P^{-3+8\varepsilon}$$

and

$$\phi \leq (RQ)^{-1} = R^{-1}P^{-13/11} \leq R^{-15/11}P^{-12/11-\varepsilon}.$$

Thus ϕ is in the range (6.9) when $R \leq P^{\Delta}$ and $P^{-3+\Delta} \leq \phi \leq (RQ)^{-1}$, so that (6.3) holds, by Lemma 6.1.

We are now able to conclude as follows.

Lemma 6.2 *Suppose that $n = 14$, that ε is sufficiently small, and that $\Delta = 40\varepsilon$. Then if $\phi \leq (RQ)^{-1}$ and $R \leq Q = P^{13/11}$, we will have*

$$\Sigma(R, \phi, \pm) \ll P^{n-3-\varepsilon}$$

unless

$$R \leq P^{\Delta}, \quad \text{and} \quad \phi \leq P^{-3+\Delta}.$$

We proceed to cover the bulk of the minor arcs by sets of the form

$$I(a,q) = \{\alpha = \frac{a}{q} + \nu : \phi_0 \leq |\nu| \leq \frac{1}{qQ}\},$$

where $\phi_0 = P^{-3+\Delta}$ if $q \leq P^{\Delta}$ and $\phi_0 = P^{-n}$ otherwise. By removing the small intervals $[a/q - P^{-n}, a/q + P^{-n}]$ we are able to use a dyadic subdivision of the remaining range $I(a,q)$ into $O(\mathcal{L})$ subintervals. It is clear that the intervals $[a/q - P^{-n}, a/q + P^{-n}]$ contribute $\ll R^2 \ll P^{n-3-\varepsilon}$ to the minor arc integral. We also note that

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \int_{I(a,q)} |S(\alpha)| d\alpha \ll \mathcal{L}^2 \Sigma(R, \phi, \pm)$$

for some $R \leq Q$, some $\phi \leq (RQ)^{-1}$, and some choice of $\pm$, as a double dyadic subdivision shows. Thus the minor arc integral is $O(P^{n-3-\varepsilon/2})$, and hence Theorem 3 follows from Lemma 2.1, once we have established the convergence of the singular series, as in Theorem 4.

We conclude this section by discussing the case $n = 13$, with a view to showing that the bounds we have established are not sufficient to handle the case in which R is of order Q and ϕ is of order Q^{-2}. It is not possible for ϕ to be in the range covered by Lemma 6.1, since that would imply

$$Q^{3/5}P^{-3} \ll Q^{-2} \ll Q^{-29/21}P^{-24/21},$$

and hence

$$Q \ll P^{15/13} \quad \text{and} \quad Q \gg P^{24/13},$$

which is impossible, since $\frac{15}{13} < \frac{24}{13}$. If (6.1) were to apply we would have

$$\phi^{1/2}\psi^{1/2}R^2\{P^{2n-1}H^{1-n}(1 + (RH^3\psi)^{n/2})\}^{1/2} \ll P^{n-3},$$

whence

$$Q^2\psi P^5 \ll H^{12} \quad \text{and} \quad Q^{17}P^{10}\psi^{15}H^{15} \ll 1.$$

Since $\psi \geq \phi \gg Q^{-2}$ we would deduce that

$$P^5 \ll H^{12} \quad \text{and} \quad P^{10}H^{15} \ll Q^{13}.$$

Similarly, since $\psi \gg (P^2H)^{-1}$ we would find that

$$Q^2P^3 \ll H^{13} \quad \text{and} \quad Q^{17} \ll P^{20}.$$

We would therefore have $H \gg P^{5/12}$, whence

$$Q^{13} \gg P^{10}H^{15} \gg P^{65/4}.$$

However this is incompatible with $Q^{17} \ll P^{20}$. Thus we see that it is not possible to show even that

$$\Sigma(R, \phi, \pm) \ll P^{n-3}$$

for $n = 13$, $Q \ll R \ll Q$ and $Q^{-2} \ll \phi \ll Q^{-2}$.

7 The Singular Series

It remains to establish Theorem 4. We must begin by relating $S_{a,q}$ to $S(a/q)$. The analysis of §3 goes though unchanged, with the cube $\mathcal{B}$ in (1.1) replaced by

$$\mathcal{B}' = \prod_{1 \leq i \leq n} (-\frac{1}{2}, \frac{1}{2}],$$

in which case $S_{a,q} = S(a/q)$ with $P = q$. In particular we see from (3.10) that

$$S_{a,q} \ll q^{5n/6+\varepsilon}. \tag{7.1}$$

This would suffice to prove the convergence of the singular series for $n \geq 13$, and so is enough for Theorem 3. However for Theorem 4 we must work a little harder.

We begin by deducing from (3.4) and (3.6), with $P = q$, that

$$|S_{a,q}|^2 \ll H^{-n}q^n(q\log q)^{n/2} \sum_{\mathbf{h}} N(a/q, q, \mathbf{h})^{1/2}.$$

We shall need to consider only prime values $p = q \geq 5$. We divide the available vectors $\mathbf{h}$ into two types. If $r(\mathbf{h}) = r$ we shall say that $\mathbf{h}$ is "bad" if p divides at least one of the $r \times r$ minors of $M(\mathbf{h})$. Otherwise we shall say that $\mathbf{h}$ is "good". If $\mathbf{h}$ is good we merely observe that if $||6aB_i(\mathbf{h};\mathbf{w})/p|| < p^{-1}$ for every i then $p|B_i(\mathbf{h};\mathbf{w})$ for every i. Thus $\mathbf{w}$ is restricted to an $(n-r)$-dimensional vector space modulo p, whence

$$N(a/p, p, \mathbf{h}) \leq p^{n-r}$$

in this case. When $\mathbf{h}$ is bad we apply Lemma 2.2 with $a = p$, $Z_2 = 1$ and $Z_1 = cH^{-1}$, in which $c \in (0,1)$ is chosen so as to make $|B_i(\mathbf{h};\mathbf{w})| < p$ whenever $|\mathbf{w}| < pZ_1$. Then $||6aB_i(\mathbf{h};\mathbf{w})/p|| < p^{-1}$ implies $B_i(\mathbf{h};\mathbf{w}) = 0$, so that

$$\begin{aligned} N(a/p, p, \mathbf{h}) &\ll Z_1^{-n}\#\{\mathbf{w} \in \mathbb{Z}^n : |\mathbf{w}| < cpH^{-1},\, B_i(\mathbf{h};\mathbf{w}) = 0\ \forall i \leq n\} \\ &\ll Z_1^{-n}(pH^{-1})^{n-r} \\ &\ll H^r p^{n-r}. \end{aligned}$$

We now consider

$$\Sigma(R) := \sum_{R<p\le 2R} \sum_{1\le a\le p-1} |S_{a,p}|^2.$$

From the above analysis we see that

$$\Sigma(R) \ll H^{-n}R^{2n+1}(\log R)^{n/2}\sum_{\mathbf{h}}\sum_{R<p\le 2R} p^{-r/2}\delta(\mathbf{h},p),$$

where $\delta(\mathbf{h},p) = H^{r/2}$ if $\mathbf{h}$ is bad for p, and $\delta(\mathbf{h},p) = 1$ otherwise. suppose firstly that $r \ge 1$. Since $r = r(\mathbf{h})$ it follows that not all the $r \times r$ minors of $M(\mathbf{h})$ can vanish, so that there can be at most $\ll \log H \ll \log R$ primes p which divide all such minors. Thus

$$\sum_{R<p\le 2R} p^{-r/2}\delta(\mathbf{h},p) \ll R^{-r/2}\{R + H^{r/2}\log R\}.$$

However we can have $r(\mathbf{h}) = 0$ only for $\mathbf{h} = \mathbf{0}$, so the above estimate holds also for $r = 0$. It then follows from (1.5) that

$$\begin{aligned}\Sigma(R) &\ll H^{-n}R^{2n+1}(\log R)^{1+n/2}\sum_{r\le n} H^{r+\varepsilon}R^{-r/2}\{R + H^{r/2}\}\\ &\ll H^{-n}R^{2n+1+2\varepsilon}\{R + R^{1-n/2}H^n + R^{-n/2}H^{3n/2}\}.\end{aligned}$$

We therefore choose $H = [R^{(n+2)/(3n)}]$, which leads to

$$\Sigma(R) \ll R^{(5n+4)/3+2\varepsilon}. \tag{7.2}$$

We are now ready to complete the proof of Theorem 4. The singular series is absolutely convergent if and only if the singular product

$$\prod_p\{1 + \sum_{e=1}^{\infty}\sum_{\substack{a\le p^e\\(a,p)=1}} p^{-en}|S_{a,p^e}|\}$$

is also absolutely convergent. Let

$$a_p := \sum_{e=1}^{\infty}\sum_{\substack{a\le p^e\\(a,p)=1}} p^{-en}|S_{a,p^e}|.$$

Then the product $\prod(1 + a_p)$ will be absolutely convergent if and only if the sum $\sum a_p$ is absolutely convergent. The bound (7.1) suffices to handle terms involving p^e for $e \ge 2$, as soon as $n > 9$, so that it remains to consider the convergence of

$$\sum_p \sum_{\substack{a\le p\\(a,p)=1}} p^{-n}|S_{a,p}|.$$

However each dyadic range $R < p \le 2R$ contributes $O(R^{(10-n)/6+3\varepsilon})$, by (7.2) in conjunction with Cauchy's inequality. Thus we have absolute convergence as soon as $n > 10$, as required.

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