Journal club notes on the cluster-state model of quantum computation

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(Dated: November 29, 2003)

I. INTRODUCTION

The following journal club talk is based on a sequence of papers by Raussendorf and Briegel [1, 2, 3], especially [1].

Consider a standard quantum circuit. The number of qubits involved in the circuit is known as the *breadth* of the circuit. The number of distinct timesteps at which a gate may be applied is known as the *depth* of the circuit. The product of these two numbers — effectively, the maximal number of gates that may be applied in the circuit — is sometimes called the *size* of the circuit.

The main claim of [1] is as follows:

> For a given breadth m and depth n there exists a quantum state $\psi(m, n)$, known as the *cluster state*, depending only on m and n, such that:
>
> - $\psi(m, n)$ involves $O(mn)$ qubits.
> - $\psi(m, n)$ can be prepared using poly(mn) physical resources (time, space, energy, etcetera).
> - By doing single-qubit measurements on $\psi(m, n)$, in appropriate bases, and with feed-forward of the measurement results to modify later measurement bases, we may simulate any quantum circuit of breadth m and depth n, with at most a polynomial overhead in physical resources.

This model, which I'll refer to as the *cluster-state* model of quantum computation, is interesting for several reasons:

- It may suggest new approaches for the construction of quantum computers.
- By studying it we may learn more about what resources are universal for quantum computation.
- It may suggest new approaches to quantum algorithms.
- It may have significance for fundamental reasons, showing that essentially all quantum dynamics can be reduced to measurement.

II. BASIC CIRCUIT NOTATION

In this section I introduce some circuit notation useful in later sections. I have deferred the introduction of some more complex pieces of notation until later, when the motivation is clearer.

X, Y and Z denote the Pauli matrices, as usual. Fig. 1 depicts our circuit notation for a single-qubit quantum gate, U. Special cases of significance are the rotations about the respective axes of the Bloch sphere, $X_x \equiv \exp(-ixX/2), Y_y \equiv \exp(-iyY/2), Z_z \equiv \exp(-izZ/2)$.

FIG. 1: Our notation for the single-qubit gate U.

Fig. 2 depicts our circuit notation for measurement of a single qubit in the computational basis. Fig. 3 depicts our circuit notation for measurement of a single qubit in a rotated basis.

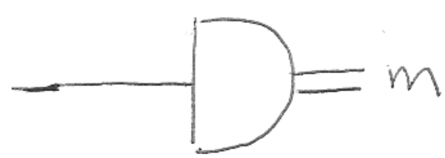

FIG. 2: A measurement in the computational basis of a single qubit, with output the classical bit $m = 0$ or 1.

FIG. 3: A single-qubit measurement in a rotated basis.

We often have occasion to use a two-qubit gate known as the *controlled-phase* gate. Our notation for the controlled-phase gate is illustrated in Fig. 4. This gate is well-known to be locally equivalent to the controlled-

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NOT gate, and thus is universal for quantum computation, when assisted by local unitaries.

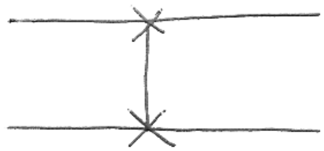

FIG. 4: The controlled-phase gate, whose action on the computational basis takes $|z_1, z_2\rangle \rightarrow (-1)^{z_1 z_2}|z_1, z_2\rangle$.

An important point about the controlled-phase gate is that it commutes with Z rotations, and also with other controlled-phase gates. In particular, this means that if multiple controlled-phase gates are applied to a set of qubits, then it does not matter in what order the gates are applied. This allows us to use, for example, notations like that in Fig. 5 to indicate two controlled-phase gates, one between qubits one and two, and one between qubits two and three.

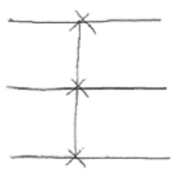

FIG. 5: Multiple controlled-phase gates commute with one another, and thus may be represented in this fashion, without worrying about the order in which they are applied.

III. ELEMENTARY CIRCUIT IDENTITIES

To understand the cluster-state model of quantum computation, it helps to first prove three quantum circuit identities. It may not be immediately obvious how these help in developing the cluster-state model of quantum computation, but the identities are, in any case, of considerable interest in their own right.

A. The transport circuit

The most important circuit identity used in the cluster-state model of computation is illustrated in Fig. 6[1].

To see that the output of the transport circuit is as claimed, note first that the transport circuit is equivalent to the circuit in Fig. 7. But the Z_θ rotation commutes

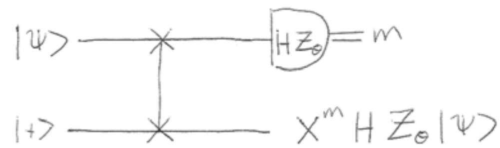

FIG. 6: The *transport circuit*. We show in the text that the output is as claimed.

with the phase gate, and so the output of the circuit is the same as if we had input $Z_\theta|\psi\rangle$ to the circuit in Fig. 8.

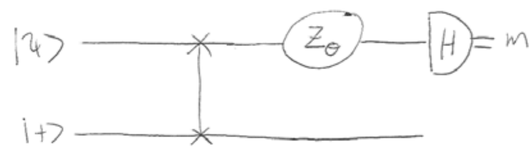

FIG. 7: A circuit equivalent to the transport circuit, Fig. 6.

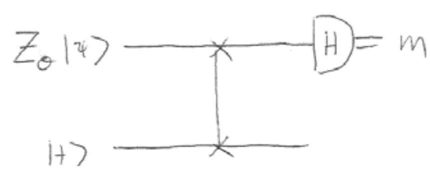

FIG. 8: Another circuit whose output is the same as that of the transport circuit, Fig. 6.

It follows that to verify the output of the transport circuit is as illustrated in Fig. 6, we need only verify it for the case $\theta = 0$, i.e., we need only verify the circuit identity of Fig. 9.

To verify the result of Fig. 9, express $|\psi\rangle$ as $\alpha|0\rangle + \beta|1\rangle$, so the input to the circuit is $\alpha|0\rangle|+\rangle + \beta|1\rangle|-\rangle$. After the controlled-phase gate the state is then $\alpha|0\rangle|+\rangle + \beta|1\rangle|-\rangle$. Applying a Hadamard gate to the first qubit gives the state

$$\frac{|0\rangle(\alpha|+\rangle + \beta|-\rangle) + |1\rangle(\alpha|+\rangle - \beta|-\rangle)}{\sqrt{2}}. \tag{1}$$

But $\alpha|+\rangle + \beta|-\rangle = H|\psi\rangle$, and $\alpha|+\rangle - \beta|-\rangle$ may be obtained by applying an extra X, so the outuptu state, before measurement of the first qubit in the computational basis, is:

$$\frac{|0\rangle H|\psi\rangle + |1\rangle XH|\psi\rangle}{\sqrt{2}}. \tag{2}$$

We see that measuring $m = 0$ gives a state $H|\psi\rangle$ output from the transport circuit, while measuring $m = 1$ gives the state $XH|\psi\rangle$, as was required to prove.

[1] This identity is also useful in arriving at fault-tolerant gate constructions [4]. I'm struck by the idea that [4] may stand in relation to [5] somewhat as [1] does to [6].

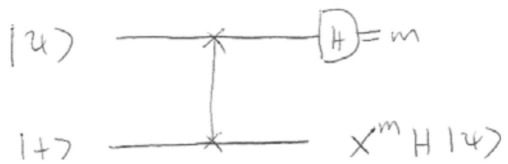

FIG. 9: The circuit identity of Fig. 6 will follow from this circuit identity.

B. The discard circuit

Our next circuit identity is the claim that the output of the *discard circuit* in Fig. 10 is identical to the output of the circuit in Fig. 11. Note an interesting thing about this circuit identity: the "circuit" in Fig. 11 is not, strictly speaking, a quantum circuit at all, since the origin of m is undefined. Instead, it's simply a convenient shorthand for describing the state output from the quantum circuit of Fig. 10. The beauty of the circuit notation is that we can still apply all our usual tricks for manipulating circuits to the circuit in Fig. 11.

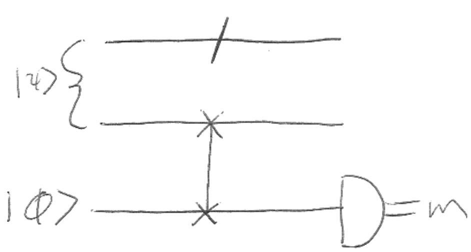

FIG. 10: The *discard circuit*. Note that $|\psi\rangle$ is an abritrary state of $k + 1$ qubits; the bar in the top line in the circuit indicates a bundle of k qubits. $|\phi\rangle$ is an arbitrary state of a single qubit. We show in the text that the output of this circuit is identical to the output of Fig. 11.

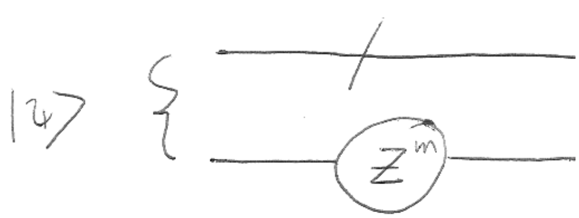

FIG. 11: The output of this circuit is identical to the output of the discard circuit in Fig. 10.

To verify that the output of Fig. 10 is the same as the output of Fig. 11, note that the output of Fig. 10 is certainly the same as that of Fig. 12. Now we commute the controlled-NOT gate in Fig. 12 through the controlled-phase gate, so Fig. 10 has the same output as Fig. 13. But applying a controlled-NOT to $|\phi\rangle$ and then measuring the output, m, has the same effect as preparing $|0\rangle$ or $|1\rangle$, depending on the value of m, as illustrated in Fig. 14. But the output of this circuit is the same as that of Fig. 11, by definition of the controlled-phase gate.

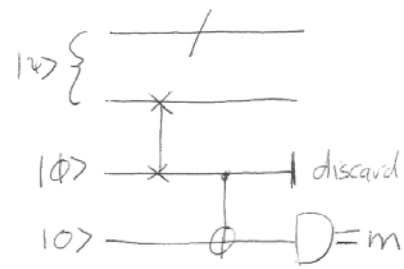

FIG. 12: The output of this circuit is identical to the output of Fig. 10.

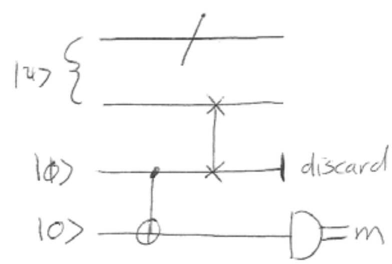

FIG. 13: The output of this circuit is identical to the output of Fig. 12.

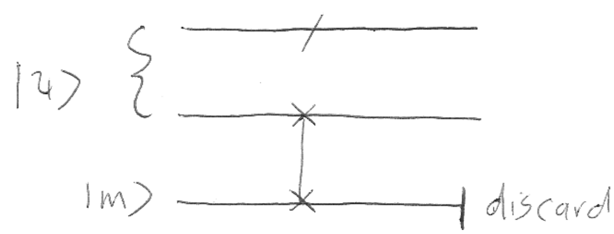

FIG. 14: The output of this circuit is identical to the output of Fig. 13, by definition of the controlled-phase gate.

We now show how to generalize the identification of the output of Fig. 10 with the output of Fig. 11. For example, essentially the same argument as just given shows that the output of the circuit in Fig. 15 is the same as that of Fig. 16.

More generally, if we take an arbitrary state $|\psi\rangle$ and apply as many controlled-phase gates as we like to an ancilla in the state $|\phi\rangle$, and then measure the ancilla to obtain the result m, this is equivalent to simply applying Z^m to those qubits of $|\psi\rangle$ which participate in a controlled-phase gate an odd number of times.

C. Indirect entangling gate

The final circuit identity we need is an indirect way of applying an entangling gate to two qubits, prepared in

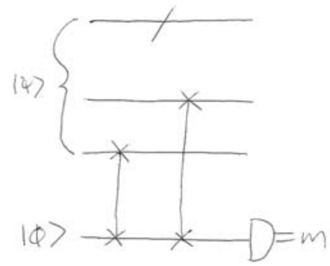

FIG. 15: A more complex version of the discard circuit. The output of this circuit is identical to the output of Fig. 16.

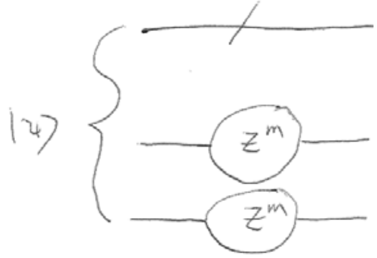

FIG. 16: The output of this circuit is identical to the output of Fig. 15.

arbitrary states $|\psi\rangle$ and $|\phi\rangle$, via two intermediate qubits prepared in fixed states $|+\rangle$. The identity we need is that the output on the top and bottom lines of Fig. 17 is the same as the output of Fig. 18, up to an irrelevant global phase.

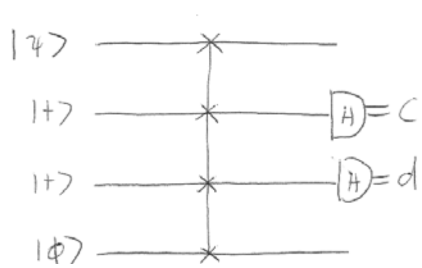

FIG. 17: This circuit indirectly entangles the top and the bottom qubit. The output on the top and the bottom line is the same as the output of Fig. 18.

To prove this, note that the initial state of the four qubits in Fig. 17 is

$$\frac{\sum_{ab} |\psi\rangle|a\rangle|b\rangle|\phi\rangle}{2}. \tag{3}$$

Applying the three controlled-phase gates results in the state

$$\frac{\sum_{ab}(-1)^{ab} Z^a|\psi\rangle|a\rangle|b\rangle Z^b|\phi\rangle}{2}. \tag{4}$$

Recall that the action of the Hadamard gate is $H|a\rangle = \sum_c (-1)^{ac}|c\rangle/\sqrt{2}$. As a result, after applying Hadamard

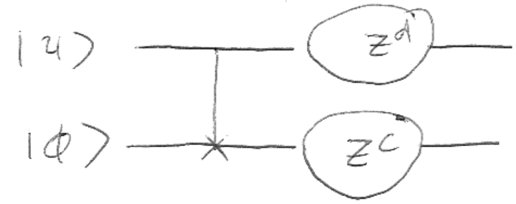

FIG. 18: The output of this circuit is the same as the output on the top and the bottom line of Fig. 17.

gates to the second and third qubits we have

$$\frac{\sum_{abcd}(-1)^{ab+ac+bd} Z^a|\psi\rangle|c\rangle|d\rangle Z^b|\phi\rangle}{4}. \tag{5}$$

It will be convenient to treat the addition in the exponent of -1 as addition modulo two. Suppose the measurement results in Fig. 17 are c and d. Then the resulting conditional state of the first and the fourth qubits is

$$\frac{\sum_{ab}(-1)^{ab+ac+bd} Z^a|\psi\rangle Z^b|\phi\rangle}{2}. \tag{6}$$

Rewriting $ab+ac+bd = (a+d)(b+c)+cd$ (recall that we're using modulo two addition) we see that the conditional state is, up to an unimportant phase factor,

$$\frac{\sum_{ab}(-1)^{(a+d)(b+c)} Z^a|\psi\rangle Z^b|\phi\rangle}{2}. \tag{7}$$

This can be rewritten

$$(Z^d \otimes Z^c)\frac{\sum_{ab}(-1)^{(a+d)(b+c)} Z^{a+d}|\psi\rangle Z^{b+c}|\phi\rangle}{2}. \tag{8}$$

Changing variables in the sum to $a' = a+d$ and $b' = b+c$ we obtain

$$(Z^d \otimes Z^c)\frac{\sum_{a'b'}(-1)^{a'b'} Z^{a'}|\psi\rangle Z^{b'}|\phi\rangle}{2}. \tag{9}$$

It is straightforward to verify directly that the controlled-phase gate may be written $\sum_{a'b'}(-1)^{a'b'} Z^{a'} \otimes Z^{b'}$, and thus the output on the top and bottom lines of Fig. 17 is the same as the output of Fig. 18, up to an irrelevant global phase.

IV. THE CLUSTER-STATE MODEL OF QUANTUM COMPUTATION

A. The cluster state

To describe the cluster-state model of quantum computation it is extremely helpful to introduce some new notation. In particular, we denote a basic cluster state as shown in Fig. 19. This represents an array of 3×4 qubits. The joint state of the qubits may be described by specifying a procedure to prepare the state [2]: (a) prepare each qubit in the state $|+\rangle$; (b) apply a controlled-phase gate between each pair of neighbouring qubits.

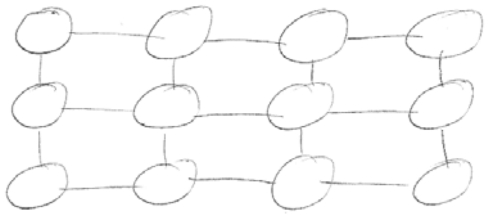

FIG. 19: A 3×4 cluster state.

Obviously, this notation for the cluster state differs quite a bit from the standard circuit model. In particular, the notation of Fig. 19 represents a *static* state, with no dynamics.

We have defined the cluster state for a square lattice, but in fact a cluster state may be defined for any graph. For example, the cluster state illustrated in Fig. 20 is prepared by initializing all the qubits in the state $|+\rangle$, and then applying controlled-phase gates between those qubits connected by the graph.

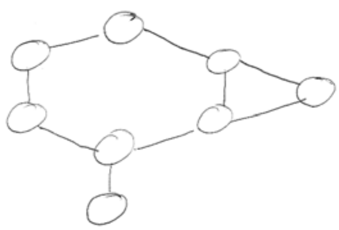

FIG. 20: Notation for a cluster state on a graph.

The way a cluster-state computation works is to apply a series of single-qubit measurements to the qubits. To indicate this process we use a notation like that in Fig. 21. This indicates doing first a measurement on the qubit marked with a "1", by rotating the qubit using H and then measuring in the computational basis. That is, we measure the top left qubit in the X basis. Then we measure the qubit marked with a "2" in the same way. Finally, we measure the qubit marked "3" by applying HZ_θ and measuring in the computational basis.

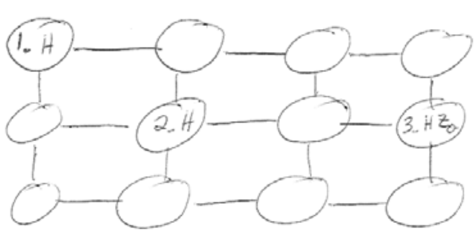

FIG. 21: The notation used for measurement.

This notation has several noteworthy points. First, the temporal order of the measurements is made clear by the numbers attached to the qubits. Second, the results of earlier measurements can be fed forward to influence later measurements. So, for example, we could have used the results of measurements 1 and 2 to influence the choice of angle θ used for measurement 3.

Indeed, since measurements on different qubits commute with one another, the point in introducing a temporal order is usually for one of the following three reasons: (a) for pedagogical clarity; (b) for ease of analysis; or — and this is the really critical reason — (c) when feed-forward is used so that later measurement bases are influenced by earlier measurement results. It is sometimes convenient to emphasize this fact by using the notation of Fig. 22 to denote measurements. Here, the measurements 1 and $1'$ are done first, but can be regarded as being done in any order, or in parallel, since the choice of α and β are made independently. However, the measurements 2 and $2'$ are done later (again, in either order or in parallel), since the values of γ and δ depend on the outcomes of the measurements 1 and $1'$. Note that we will usually omit the primes, using them only when it is helpful to refer to specific qubits in the circuit.

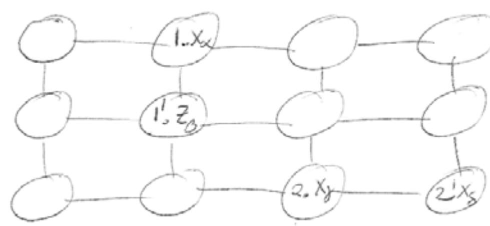

FIG. 22: The notation used for parallel measurements.

A typical cluster-state quantum computation thus looks something like Fig. 22. The *output* of the computation is defined to consist of those qubits which have not yet been measured, i.e., the blank qubits in the figure.

B. A single-qubit unitary gate

We begin our explanation of the cluster-state model of quantum computation by showing how to simulate the simple quantum circuit in Fig. 23. The simulation procedure is sufficiently useful that we also introduce a notation for the gate we're simulating, in Fig. 24. Note that the gate introduced, $U_{\alpha,\beta}$, is universal for computation, together with the controlled-phase gate. It is also useful to note that by choosing $\alpha = \beta = 0$ we can simulate the identity gate, i.e., a wire.

The cluster-state computation we use to simulate the circuit of Fig. 23 is shown in Fig. 25. Note that α in this figure is the same as α in Fig. 23, but that β' is not

[2] Of course, other preparation procedures are possible.

FIG. 23: A simple quantum circuit that we'll simulate in the cluster-state model.

FIG. 24: A useful gate notation.

necessarily the same; it will depend on the result of the first measurement.

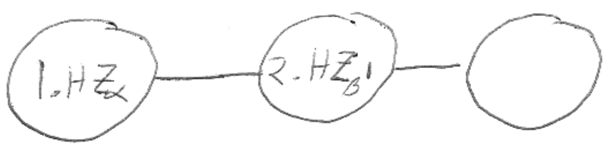

FIG. 25: The cluster-state circuit used to simulate the circuit of Fig. 23

To analyse the output of Fig. 25, note that it is equivalent to the quantum circuit shown in Fig. 26. This, in turn, is equivalent to the quantum circuit shown in Fig. 27, just by delaying the operations on the second and third qubits. But the circuit in Fig. 27 is just two copies of the transport circuit, Fig. 6, with the output of the first used as input to the second. The output of the first transport circuit is

$$X^{m_1} H Z_\alpha |+\rangle, \tag{10}$$

and thus the output of the two combined transport circuits is

$$X^{m_2} H Z_{\beta'} X^{m_1} H Z_\alpha |+\rangle. \tag{11}$$

We choose $\beta' = \pm\beta$ so that $Z'_\beta X^{m_1} = X^{m_1} Z_\beta$; this can be done since β' is chosen after m_1 is known. Furthermore, since H is in the Clifford group, it follows that $HX^{m_1} = \sigma^{m_1} H$, where by σ^{m_1} we just mean some known Pauli matrix (in this case, Z), without worrying too much about keeping track of the details for the purposes of analysis. (In practice, you would need to keep track, of course!) It follows that the output of the circuit in Fig. 27 is

$$\sigma^{m_1,m_2} H Z_\beta H Z_\alpha |+\rangle, \tag{12}$$

which is just

$$\sigma^{m_1,m_2} X_\beta Z_\alpha |+\rangle. \tag{13}$$

That is, the output of Fig. 23 is the same as the final state of the cluster-state computation in Fig. 25, up to a known Pauli error σ^{m_1,m_2}. Assuming that the computation is read out in the computational basis, this known Pauli error can be compensated by classical post-processing of the readout result from Fig. 25, and thus the two methods of computation are equivalent.

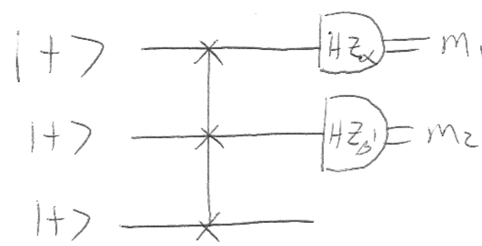

FIG. 26: A quantum circuit equivalent to the cluster-state circuit in Fig. 25.

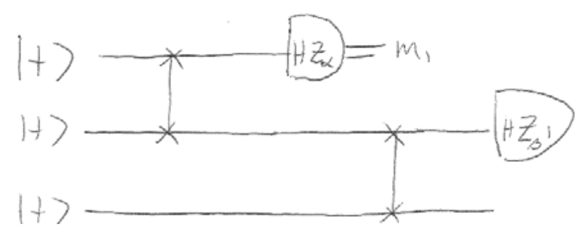

FIG. 27: This quantum circuit is equivalent to the circuit in Fig. 26, but is easier to analyse.

C. Two consecutive single-qubit unitary gates

Suppose next that we want to simulate a circuit containing two consecutive $U_{\alpha,\beta}$-type gates, as illustrated in Fig. 28.

FIG. 28: A simple circuit containing two consecutive single-qubit gates.

This can be done using the cluster-state computation depicted in Fig. 29. In this computation, α is the same as in Fig. 28, but $\beta' = \pm\beta, \gamma' = \pm\gamma, \delta' = \pm\delta$, with the specific value of the sign depending on the values of the earlier measurement outcomes. Note that β' is chosen as in the previous section; γ' and δ' will also be chosen in a similar way.

To see that this simulation works, note that the cluster-state computation of Fig. 29 is equivalent to the quantum

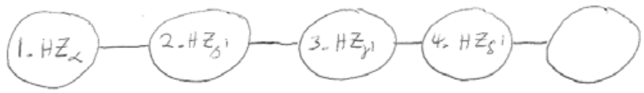

FIG. 29: The cluster-state computation used to simulate the circuit of Fig. 28.

circuit depicted in Fig. 30.

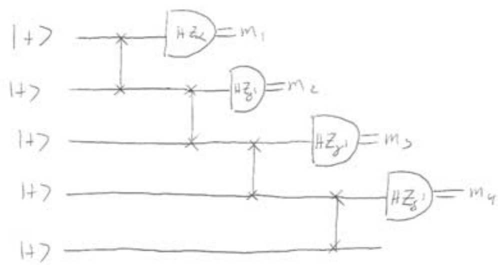

FIG. 30: The cluster-state computation of Fig. 29 is equivalent to this quantum circuit.

To understand the output of the circuit in Fig. 30, note that it consists of four consecutive transport circuits. We already know that the output of the first two transport circuits is

$$\sigma^{m_1,m_2} U_{\alpha,\beta}|+\rangle, \tag{14}$$

provided β' is chosen as described in the previous section. This is then used as input to the next two transport circuits, giving as output

$$X^{m_4} H Z_{\delta'} X^{m_3} H Z_{\gamma'} \sigma^{m_1,m_2} U_{\alpha,\beta}|+\rangle. \tag{15}$$

We can commute the known Pauli error σ^{m_1,m_2} through $Z_{\gamma'}$ to obtain Z_γ (provided γ' is chosen appropriately), and then through H, to obtain

$$X^{m_4} H Z_{\delta'} \sigma^{m_1,m_2,m_3} H Z_\gamma U_{\alpha,\beta}|+\rangle, \tag{16}$$

where σ^{m_1,m_2,m_3} is a known Pauli error depending on the first three measurement results. This can then be commuted through $Z_{\delta'}$, making it Z_δ, provided δ' is chosen appropriately, and through H, giving

$$\sigma^{m_1,m_2,m_3,m_4} U_{\gamma,\delta} U_{\alpha,\beta}|+\rangle, \tag{17}$$

which, up to a known Pauli error, is the same as the output of Fig. 28.

There are two important general lessons to be learnt from this example. The first lesson is that the Pauli errors which result from the unpredictability of the measurement results are not serious. We can always propragate them through to the end of the computation, simply by making appropriate choices of the measurement bases.

The second lesson relates to how one concatenates cluster-state computations. Consider the cluster-state computation of Fig. 29. This is equivalent to first preparing the cluster state on the first three qubits, and then doing measurements 1 and 2. As we saw in the previous section, this leads to the state $U_{\alpha,\beta}|+\rangle$ being output on the third qubit (up to a Pauli error). The remaining parts of the cluster are then prepared with controlled-phase gates, followed by the measurements 3 and 4, which results in the gate $U_{\gamma,\delta}$ being applied, and the result output on the final qubit.

D. A small quantum circuit

As our final example, consider the quantum circuit in Fig. 31. The simulation of this circuit in the cluster-state model illustrates essentially all the general principles of cluster-state quantum computation.

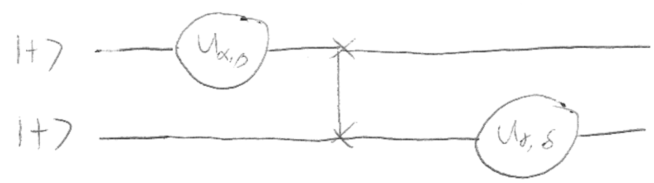

FIG. 31: A simple quantum circuit.

The quantum circuit of Fig. 31 may be simulated using a 4×7 cluster state. The first step in the simulation is to measure the qubits as shown in Fig. 32, all in the computational basis. There is no need to put a temporal label on the measurements, since they are all in a fixed basis.

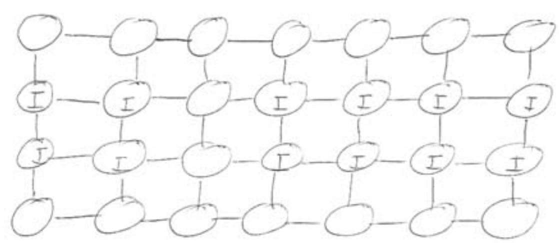

FIG. 32: The first step in the cluster-state simulation of the quantum circuit in Fig. 31.

How does this sequence of measurements affect the remainder of the cluster state? Imagine, without loss of generality, that the first qubit measured is the second qubit in the first column. By our results on the discard circuit, Fig. 10, and its generalizations, we see that the effect of this measurement is to remove the qubit from the cluster, and apply a Z^m operation to the neighbouring qubits, where m was the measurement outcome.

The resulting state is thus, up to the Z^m operations, exactly the same as would have resulted by preparing a cluster-state on the original lattice, but with the measured qubit deleted, as illustrated in Fig. 33. However, such factors of Z^m can be compensated for in subsequent single-qubit measurements, since m is known. Thus, we may as well regard the resulting state as being the cluster state of Fig. 33.

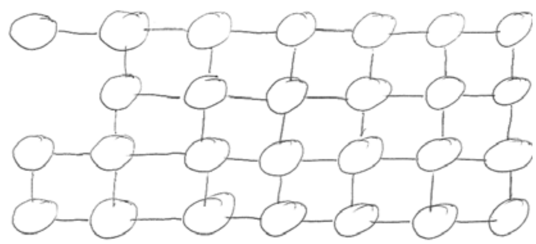

FIG. 33: The state that results after the measurement on the second row of the first column in Fig. 32 can be treated as though it is this cluster state.

Repeated applications of this argument show that the state output from the cluster-state computation of Fig. 32 may be regarded as the cluster state of Fig. 34.

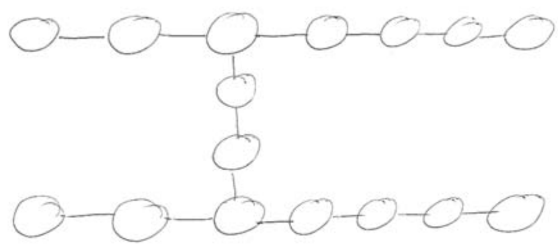

FIG. 34: The result of the cluster-state computation of Fig. 32 is this new cluster state.

The next step in the computation is to perform the measurement prescribed in Fig. 35. Suppose we regard these measurements as being performed *before* the controlled-phase gates setting up the cluster state on the rest of the graph. The effect is to prepare the qubits immediately to the right of those labelled 2 in the states $U_{\alpha,\beta}|+\rangle$, and $|+\rangle$, up to known Pauli errors.

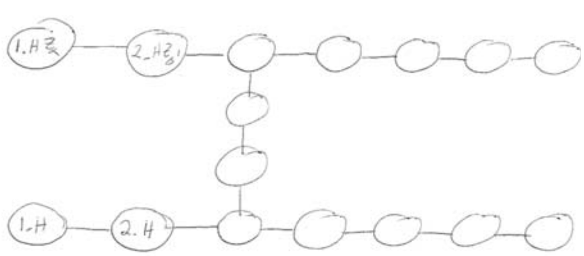

FIG. 35: These measurements simulate the first single-qubit gates in the computation of Fig. 31.

The next step is to perform the measurements prescribed in Fig. 36. By the equivalence of Figs. 17 and 18, the output at the qubits immediately above and below those marked 3 is the result of letting controlled-phase act on $(U_{\alpha,\beta} \otimes I)|+\rangle|+\rangle$, again up to a known Pauli error. Note that in order for this to work, it is crucial that the controlled-phase gate be in the Clifford group. If it were not the Pauli errors from earlier stages of the computation would not propagate through the computation to become Pauli errors, and the whole scheme would break down.

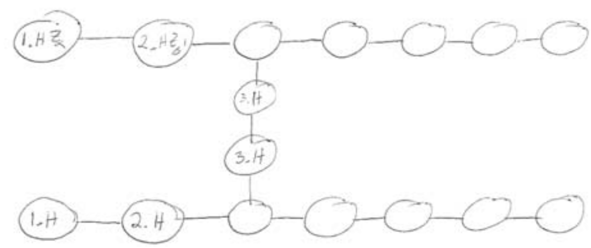

FIG. 36: The measurements labelled 3 simulate the controlled-phase gate in the circuit of Fig. 31.

It should now be clear how to complete the simulation of Fig. 31. The next step is to do measurements 4 and 5, as indicated in Fig. 37. These measurements simply make the output of the previous computation propagate further along the chain; we regard them as being a trvial part of the controlled-phase gate. Our reason for doing this is so that the controlled-phase gate and single-qubit unitaries are always the same "breadth", taking up two columns of cluster-state computation. In actual fact, a little more thought shows that this extra step is not necessary, but from our point of view this is a detail.

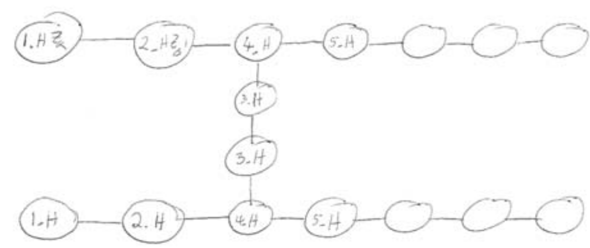

FIG. 37: The extra measurements, 4 and 5, cause the quantum information to propagate further along the chain, without changing it, except possibly to change the known Pauli errors.

An interesting point about Fig. 37 is that the measurements labelled $3, 4$ and 5 could all be done at any time whatsoever in the computation, since they don't depend on the previous measurement results. Thus, they could be done even before the measurements labelled 1 and 2. It is a remarkable fact about the cluster-state model that interactions in the quantum circuits model of computation can all be effectively performed in parallel at the beginning of a cluster-state computation. It is only the single-qubit gates that require careful temporal ordering.

The cluster-state computation is concluded by doing the measurements in Fig. 38, in order to simulate the final single-qubit gates in Fig. 31. The output on the final two qubits of the cluster is identical, up to known Pauli errors, to the output of the circuit in Fig. 31.

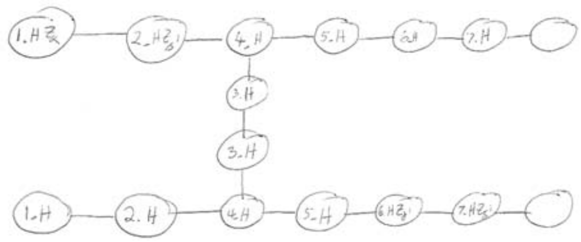

FIG. 38: The extra measurements conclude the simulation of Fig. 31.

E. Summary

It should now be clear how to simulate an arbitrary quantum circuit in the cluster-state model of quantum computation. With the cluster states I've described there are some obvious limitations — e.g., we can only simlulate nearest-neighbour controlled-phase gates. But this is universal for quantum computation, and it is obvious that by starting with cluster states on more complex graphs we could simulate an arbitrary circuit in the standard model, with only a small overhead.

Obviously, it would be very interesting to play further with this model. Lots of ideas suggest themselves; too many to enumerate now. In particular, it'd be a lot of fun to try to understand cluster-state computations that get away from the standard quantum circuit model, especially doing interesting things with the temporal ordering. There are all kinds of possibilities for encoding interesting problems in cluster states, and then trying to process them! Maybe by so doing we'll be inspired to think of new algorithms? I'm fairly convinced that one of the keys to making progress in this way is to think rather carefully about notation. The notation I've introduced can be improved quite a bit further, I think, and this may be very useful for doing computations in the cluster-state model. I think the notation used for concatenation, in particular, could be considerably improved, perhaps by introducing some way of describing the "effective output" of a partial cluster-state computation (i.e., the resulting state of those unemasured qubits adjacent to measured qubits), which is then used as the "effective input" when further measurements are added.

Acknowledgments

Thanks to Andrew Doherty, and, especially, Chris Dawson, for many helpful discussions of the cluster-state model of quantum computation.

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