The Fundamental Theorem of Calculus:

1. If $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

2.If f does not have a jump discontinuity at x, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Vito Volterra, 1881: There exists a function, $F(x)$, whose derivative, $F'(x)$, exists and is bounded for all x, but the derivative, $F'(x)$, cannot be integrated.

Create a new set like the Cantor set except

the first middle piece only has length 1/4

each of the next two middle pieces only have length 1/16

the next four pieces each have length 1/64, etc.

The amount left has size $1 - \left(\frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \frac{2^3}{4^4} + \cdots\right)$

$$= 1 - \frac{1}{4}\left(1 + \left(\frac{2}{4}\right) + \left(\frac{2}{4}\right)^2 + \left(\frac{2}{4}\right)^3 + \cdots\right)$$

$$= 1 - \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}$$

We'll call this set SVC (for Smith-Volterra-Cantor).

It has some surprising characteristics:

1. SVC contains no intervals - no matter how small a subinterval of [0,1] we take, there will be points in that subinterval that are not in SVC. SVC is **nowhere dense.**

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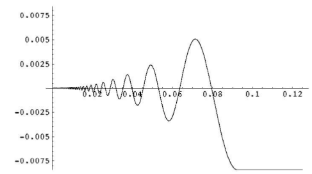
3. Given *any* partition of [0,1] into subintervals, the sum of the lengths of the intervals that contain points in SVC will always be at least 1/2. SVC has **measure 1/2**.

Volterra’s construction:

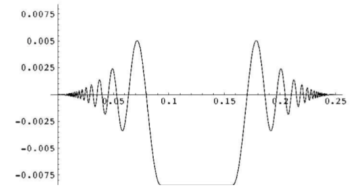
Start with the function $F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$

Restrict to the interval [0,1/8], except find the largest value of x on this interval at which $F'(x) = 0$, and keep F constant from this value all the way to $x = 1/8$.



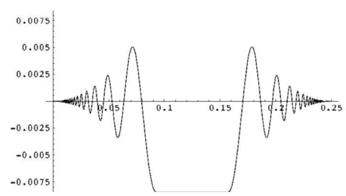
Volterra’s construction:

To the right of $x = 1/8$, take the mirror image of this function: for $1/8 < x < 1/4$, and outside of [0,1/4], define this function to be 0. Call this function $f_1(x)$.

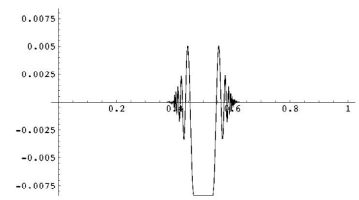


Volterra’s construction:

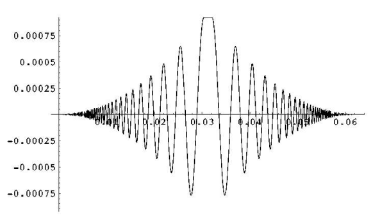
$f_1(x)$ is a differentiable function for all values of x, but $\lim_{x\to 0^+} f_1'(x)$ and $\lim_{x\to \frac{1}{4}^-} f_1'(x)$ do not exist



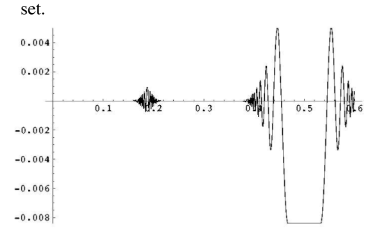
Now we slide this function over so that the portion that is not identically 0 is in the interval [3/8,5/8], that middle piece of length 1/4 taken out of the SVC set.



We follow the same procedure to create a new function, $f_2(x)$, that occupies the interval [0,1/16] and is 0 outside this interval.



We slide one copy of $f_2(x)$ into each interval of length 1/16 that was removed from the SVC set.



Volterra's function, $V(x)$, is what we obtain in the limit as we do this for *every* interval removed from the SVC set. It has the following properties:

1. V is differentiable at every value of x, and its derivative is bounded (below by –1.013 and above by 1.023).

Volterra's function, $V(x)$, is what we obtain in the limit as we do this for *every* interval removed from the SVC set. It has the following properties:

1. V is differentiable at every value of x, and its derivative is bounded (below by –1.013 and above by 1.023).

2. If a is a left or right endpoint of one of the removed intervals, then the derivative of V at a exists (and equals 0), but we can find points arbitrarily close to a where the derivative is +1, and points arbitrarily close to a where the derivative is –1.

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No matter how we partition [0,1], the pieces that contain endpoints of removed intervals must have lengths that add up to at least 1/2.

The pieces on which the variation of V' is at least 2 must have lengths that add up to at least 1/2.

Bernhard Riemann (1852, 1867) *On the representation of a function as a trigonometric series*

Defined definite integral as limit of $\sum f(x_i^*)(x_i - x_{i-1})$

Key to convergence: on each interval, look at the **variation** of the function

$$V_i = \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)$$

Integral exists if and only if $\sum V_i(x_i - x_{i-1})$ can be made as small as we wish by taking sufficiently small intervals.

Conclusion: Volterra's function V can be differentiated and has a bounded derivative, but its derivative, V', cannot be integrated:

$$\frac{d}{dx}V(x) = v(x), \text{ but } \int_0^x v(t)\,dt \neq V(x) - V(0).$$

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$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$