CHAPTER 1

Basic properties of holomorphic functions
Preview of differences between one and several variables

For any $n \geq 1$, the holomorphy or complex differentiability of a function on a domain in $\mathbf{C}^n$ implies its analyticity: a holomorphic function has local representations by convergent power series. This amazing fact was discovered by Cauchy in the years 1830-1840 and it helps to explain the nice properties of holomorphic functions. On the other hand, when it comes to integral representations of holomorphic functions, the situation for $n \geq 2$ is much more complicated than for $n = 1$: simple integral formulas in terms of boundary values exist only for $\mathbf{C}^n$ domains that are products of $\mathbf{C}^1$ domains. It turns out that function theory for a ball in $\mathbf{C}^n$ is different from function theory for a "polydisc", a product of discs.

The foregoing illustrates a constant theme: there are similarities between complex analysis in several variables and in one variable, but also differences and some of the differences are very striking. Thus the subject of analytic continuation presents entirely new phenomena for $n \geq 2$. Whereas every $\mathbf{C}^1$ domain carries noncontinuable holomorphic functions, there are $\mathbf{C}^n$ domains for which all holomorphic functions can be continued analytically across a certain part of the boundary (Section 1.9). The problems in $\mathbf{C}^n$ require a variety of new techniques which yield a rich theory.

Sections 1.1 – 1.8 deal with simple basic facts, while Sections 1.9 and 1.10 contain previews of things to come.

NOTATION. The points or vectors of $\mathbf{C}^n$ are denoted by

$$z = (z_1, \ldots, z_n) = x + iy = (x_1 + iy_1, \ldots, x_n + iy_n).$$

For vectors z and w in $\mathbf{C}^n$ we use the standard 'Euclidean' *norm* or length and *inner product*,

$$|z| = \|z\| = \left(|z_1|^2 + \ldots + |z_n|^2\right)^{\frac{1}{2}},$$
$$(z, w) = \langle z, \overline{w} \rangle = z \cdot \overline{w} = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n.$$

Subsets of $\mathbf{C}^n$ may be considered as subsets of $\mathbf{R}^{2n}$ through the correspondence

$$(x_1 + iy_1, \ldots, x_n + iy_n) \leftrightarrow (x_1, y_1, \ldots, x_n, y_n).$$

Ω will always denote a (nonempty) *open subset* of the basic underlying space, here $\mathbf{C}^n$. We also speak of a *domain* Ω in $\mathbf{C}^n$, whether it is connected or not. A connected domain will often be denoted by D if that letter is not required for a derivative.

1.1 Holomorphic functions. Later on we will use the terms 'analytic' and 'holomorphic' interchangeably, but for the moment we will distinguish between them. According to

Weierstrass's definition (about 1870), analytic functions on domains Ω in $\mathbf{C}^n$ are locally equal to sum functions of (multiple) power series [cf. Definition 1.51]. Here we will discuss holomorphy.

In order to establish notation, we first review the case of one complex variable. Let Ω be a domain in $\mathbf{C} \sim \mathbf{R}^2$. For Riemann (about 1850), as earlier for Cauchy, a complex-valued function

$$f(x,y) = u(x,y) + iv(x,y) \;\text{ on }\; \Omega$$

provided a convenient way to combine two real-valued functions u and v that occur together in applications. [For example, a flow potential and a stream function.] Geometrically, $f = u + iv$ defines a map from one planar domain, Ω, to another. Let us think of a differentiable map (see below) or of a *smooth* map (u and v at least of class $\mathbf{C^1}$). We fix $a \in \Omega$ and write

$$\begin{aligned}
&z = x + iy, \quad \overline{z} = x - iy,\\
&z - a = \Delta z = \Delta x + i\Delta y, \quad \overline{z} - \overline{a} = \Delta\overline{z} = \Delta x - i\Delta y.
\end{aligned}$$

Then the *differential* or *linear part* of f at a is given by

$$\begin{aligned}
df = df(a) &= \frac{\partial f}{\partial x}(a)\Delta x + \frac{\partial f}{\partial y}(a)\Delta y\\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)\Delta z + \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)\Delta\overline{z}.
\end{aligned}$$

In particular $dz = \Delta z$, $d\overline{z} = \Delta\overline{z}$. It is now natural to introduce the following symbolic notation:

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z}, \quad \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial \overline{z}}$$

since it leads to the nice formula

$$df(a) = \frac{\partial f}{\partial z}(a)\Delta z + \frac{\partial f}{\partial \overline{z}}(a)\Delta\overline{z} = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

[Observe that $\partial f/\partial z$ and $\partial f/\partial\overline{z}$ are not partial derivatives in the ordinary sense – here one does not differentiate with respect to one variable, while keeping the other variable(s) fixed. However, in calculations, $\partial f/\partial z$ and $\partial f/\partial\overline{z}$ do behave like partial derivatives. Their definition is in accordance with the chain rule if one formally replaces the independent variables x and y by z and $\overline{z}$. For a historical remark on the notation, see [Remmert].]

We switch now to complex notation for the independent variables, writing $f((z+\overline{z})/2, (z-\overline{z})/2i)$ simply as $f(z)$. By definition, the *differentiability* of the map f at a (in the *real* sense) means that for all small complex numbers $\Delta z = z - a = \rho e^{i\theta}$ we have

$$\Delta f(a) \stackrel{\text{def}}{=} f(a + \Delta z) - f(a) = df(a) + o(|\Delta z|) \;\text{ as }\; \Delta z \to 0. \tag{1a}$$

Complex differentiability of such a function f at a requires the existence of

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z}=\lim\left\{\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \overline{z}}\frac{\Delta\overline{z}}{\Delta z}+o(1)\right\}. \tag{1b}$$

Note that $\Delta\overline{z}/\Delta z=e^{-2i\theta}$. Thus for a differentiable map, one has complex differentiability at a precisely when the *Cauchy-Riemann condition* holds at a :

$$\frac{\partial f}{\partial\overline{z}}(a)=0 \quad \text{or} \quad \frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y}.$$

[If $\partial f/\partial\overline{z}\neq 0$, the limit $(1b)$ as $\Delta z\to 0$ can not exist.] The representation $f=u+iv$ gives the familiar real Cauchy-Riemann conditions $u_x=v_y,\ u_y=-v_x$. For the complex derivative one now obtains the formulas

$$f'(a)=\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z}=\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y}=u_x+iv_x=u_x-iu_y. \tag{1c}$$

Observe that complex differentiability implies differentiability in the real sense.

Functions f which possess a complex derivative at every point of a planar domain Ω are called *holomorphic*. In particular, analytic functions in $\mathbf{C}$ are holomorphic since sum functions of power series in $z-a$ are differentiable in the complex sense. On the other hand, by Cauchy's integral formula for a disc and series expansion, holomorphy implies analyticity, cf. also Section 1.6.

HOLOMORPHY IN THE CASE OF $\mathbf{C}^n$. Let Ω be a domain in $\mathbf{C}^n\sim\mathbf{R}^{2n}$ and let $f=f(z)=f(z_1,\ldots,z_n)$ be a complex-valued function on Ω:

$$f=u+iv:\Omega\to\mathbf{C}. \tag{1d}$$

Suppose for a moment that f is analytic in each complex variable z_j separately, so that f has a complex derivative with respect to z_j when the other variables are kept fixed. Then f will satisfy the following *Cauchy-Riemann conditions* on Ω :

$$\frac{\partial f}{\partial\overline{z}_j}\overset{\text{def}}{=}\frac{1}{2}\left(\frac{\partial f}{\partial x_j}-\frac{1}{i}\frac{\partial f}{\partial y_j}\right)=0,\qquad j=1,\ldots,n. \tag{1e}$$

Moreover, the complex partial derivatives $\partial f/\partial z_j$ will be equal to the corresponding formal derivatives, given by

$$\frac{\partial f}{\partial z_j}\overset{\text{def}}{=}\frac{1}{2}\left(\frac{\partial f}{\partial x_j}+\frac{1}{i}\frac{\partial f}{\partial y_j}\right), \tag{1f}$$

cf. $(1c)$.

Suppose now that the map $f=u+iv$ of $(1d)$ is just differentiable in the real sense. [This is certainly the case if f is of class C^1.] Then the increment $\Delta f(a)$ can be written in the form $(1a)$, but this time $\Delta z=(\Delta z_1,\ldots,\Delta z_n)$ and the differential of f at a is given by

$$df(a) = \sum_1^n \left(\frac{\partial f}{\partial x_j}(a)\Delta x_j + \frac{\partial f}{\partial y_j}(a)\Delta y_j \right) = \sum_1^n \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \right).$$

Thus

$$df = \partial f + \overline{\partial} f$$

[*del f* and *del-bar* or *d-bar f*], where

$$\partial f \stackrel{\text{def}}{=} \sum_1^n \frac{\partial f}{\partial z_j} dz_j, \quad \overline{\partial} f \stackrel{\text{def}}{=} \sum_1^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

With this notation, the Cauchy-Riemann conditions ($1e$) may be summarized by the single equation

$$\overline{\partial} f = 0.$$

Definition 1.11. *A function f on $\Omega \subset \mathbf{C}^n$ to $\mathbf{C}$ is called* HOLOMORPHIC *if the map f is "differentiable in the complex sense":*

$$\Delta f(a) = \partial f(a) + o(|\Delta z|) \quad \text{as} \quad \Delta z \to 0$$

at every point $a \in \Omega$. In particular a function $f \in \mathbf{C}^1(\Omega)$ is holomorphic precisely when it satisfies the Cauchy-Riemann conditions.

More generally, a function f defined on an arbitrary nonempty set $E \subset \mathbf{C}^n$ is called holomorphic, notation

$$f \in \mathcal{O}(E), \quad \text{(also for open } E = \Omega!) \tag{1g}$$

if f has a holomorphic extension to some open set containing E .

The notation $\mathcal{O}(E)$ for the class or ring of holomorphic functions on E goes back to a standard notation for rings, cf. [Van der Waerden] Section 16. The letter $\mathcal{O}$ is also appropriate as a tribute to the Japanese mathematician Oka, who has made fundamental contributions to complex analysis in several variables, beginning about 1935, cf. [Oka].

A function $f \in \mathcal{O}(\Omega)$ will have a complex derivative with respect to each variable z_j at every point of Ω, hence by Cauchy's theory for a disc, f will be analytic in each complex variable z_j separately. A corresponding Cauchy theory for so-called polydiscs will show that every holomorphic function is analytic in the sense of Weierstrass, see Sections 1.3 and 1.6. Thus in the end, holomorphy and analyticity will come to the same thing.

REMARK. The expressions for df, ∂f and $\overline{\partial} f$ (with variable a) have the appearance of *differential forms*. First order differential forms

$$p_1 dx_1 + q_1 dy_1 + \ldots + q_n dy_n \quad \text{or} \quad u_1 dz_1 + v_1 d\overline{z}_1 + \ldots + v_n d\overline{z}_n,$$

with $dx_1, \ dy_1, \ldots, dy_n$ or $dz_1, d\overline{z}_1, \ldots, d\overline{z}_n$ as basis forms (!), will frequently be used as a notational device. Later on we will also need higher order differential forms, cf. Chapter 10 for a systematic discussion.

1.2 Complex affine subspaces. Ball and polydisc. A single complex linear equation

$$(2a) \qquad c \cdot (z-a) \stackrel{\text{def}}{=} c_1(z_1-a_1)+\ldots+c_n(z_n-a_n)=0 \qquad (c \neq 0)$$

over $\mathbf{C}^n$ defines a *complex hyperplane* V through the point a, just as a single real linear equation over $\mathbf{R}^n$ defines a real hyperplane.

EXAMPLE 1.21 (*Tangent hyperplanes*). Let f be a *real* C^1 function on a domain Ω in $\mathbf{C}^n \sim \mathbf{R}^{2n}$, let $a = a' + ia''$ be a point in Ω and $\operatorname{grad} f|_a \neq 0$. Then the equation $\Delta f(a) = f(z) - f(a) = 0$ will locally define a *real hypersurface* S through a. The linearized equation $df(a) = 0$ with $\Delta z_j = z_j - a_j$ represents the (*real*) *tangent hyperplane* to S at a:

$$0 = df(a) = \sum_j \left\{ \frac{\partial f}{\partial x_j}(a)(x_j - a_j') + \frac{\partial f}{\partial y_j}(a)(y_j - a_j'') \right\} = 2 \operatorname{Re} \sum_j \frac{\partial f}{\partial z_j}(z_j - a_j).$$

The real tangent hyperplane contains a (unique) complex hyperplane through a, the "*complex tangent hyperplane*" to S at a:

$$0 = \partial f(a) = \sum_j \frac{\partial f}{\partial z_j}(a)(z_j - a_j),$$

cf. exercises 1.4 and 2.9.

A set of k complex linear equations of the form

$$c^{(j)} \cdot (z-a) = 0, \quad j = 1, \ldots, k$$

defines a *complex affine subspace* W of $\mathbf{C}^n$, or a complex linear subspace if it passes through the origin. Assuming that the vectors $c^{(j)}$ are linearly independent in $\mathbf{C}^n$, W will have complex dimension $n-k$. In the case $k = n-1$ one obtains a *complex line* L (an ordinary complex plane, complex dimension 1). Complex lines are usually given in equivalent parametric form as

$$(2b) \qquad z = a + wb, \quad \text{or} \quad z_j = a_j + wb_j, \;\; j = 1, \ldots, n,$$

where a and b are fixed elements of $\mathbf{C}^n$ ($b \neq 0$) and w runs over all of $\mathbf{C}$.

If $f \in \mathcal{O}(\Omega)$ and L is a complex line that meets Ω, the restriction of f to $\Omega \cap L$ can be considered as a holomorphic function of one complex variable. Indeed, if $a \in \Omega \cap L$ and we represent L in the form (2b), then $f(a + wb)$ will be defined and holomorphic on a certain domain in $\mathbf{C}$. [Compositions of holomorphic functions are holomorphic, cf. exercise 1.5.] Similarly, if V is a complex hyperplane that meets Ω, the restriction of f to $\Omega \cap V$ can be considered as a holomorphic function on a domain in $\mathbf{C}^{n-1}$.

Open discs in $\mathbf{C}$ will be denoted by $B(a, r)$ or $\Delta(a, r)$, circles by $C(a, r)$. There are two kinds of domains in $\mathbf{C}^n$ that correspond to discs in $\mathbf{C}$, namely, *balls*

$$B(a,r) \stackrel{\text{def}}{=} \{z \in \mathbf{C}^n : |z-a| < r\}$$

and *polydiscs* (or polycylinders):

$$\begin{aligned}\Delta(a,r) = \Delta_n(a,r) &= \Delta(a_1,\ldots,a_n; r_1,\ldots,r_n) \\ &\stackrel{\text{def}}{=} \{z \in \mathbf{C}^n : |z_1-a_1| < r_1,\ldots,|z_n-a_n| < r_n\} \\ &= \Delta_1(a_1,r_1) \times \ldots \times \Delta_1(a_n,r_n).\end{aligned}$$

Polyradii $r = (r_1,\ldots,r_n)$ must be strictly positive: $r_j > 0$, $\forall j$. Cartesian products $D_1 \times \ldots \times D_n$ of domains in $\mathbf{C}$ are sometimes called *polydomains*.

Figures 1.1 and 1.2 illustrate the ball $B(0,r)$ and the polydisc $\Delta(0,r)$ for the case of $\mathbf{C}^2$ in the *plane of absolute values* $|z_1|, |z_2|$. Every point in the first quadrant represents the product of two circles. Thus $r = (r_1,r_2)$ represents the *"torus"*

$$T(0,r) = C(0,r_1) \times C(0,r_2).$$

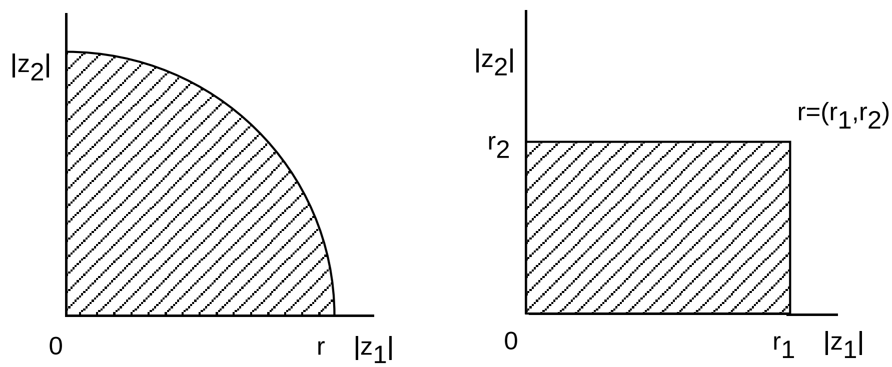


fig 1.1 and 1.2

The actual domains lie in complex 2-dimensional or real 4-dimensional space. The boundary of the ball $B(0,r)$ is the sphere $S(0,r)$, the boundary of the "bidisc" $\Delta = \Delta(0,r)$ is the disjoint union

$$\{C(0,r_1) \times \Delta_1(0,r_2)\} \cup \{\Delta_1(0,r_1) \times C(0,r_2)\} \cup \{C(0,r_1) \times C(0,r_2)\}.$$

Observe that the boundary $\partial\Delta(0,r)$ may also be described as the union of closed discs in certain complex lines $z_1 = c_1$ and $z_2 = c_2$ such that the circumferences of those discs belong to the torus $T(0,r)$. This fact will imply a very strong maximum principle for holomorphic functions f on the closed bidisc $\overline{\Delta}(0,r)$. First of all, the absolute value $|f|$ of such a function must assume its maximum on the boundary $\partial\Delta$. This follows readily from the maximum principle for holomorphic functions of one variable: just consider the restrictions of f to complex lines z_2=constant. By the same maximum principle, the absolute value of f on the boundary discs of Δ will be majorized by max $|f|$ on the torus $T(0,r)$. Thus the maximum of $|f|$ on $\overline{\Delta}(0,r)$ is always assumed on the torus $T(0,r)$.

By similar considerations, all *holomorphic functions* on $\overline{\Delta}(0,r) = \overline{\Delta}_n(0,r) \subset \mathbf{C}^n$ assume their *maximum absolute value* on the "torus"

$$T(0,r) = T_n(0,r) = C(0,r_1) \times \ldots \times C(0,r_n),$$

a relatively small part (real dimension n) of the whole boundary $\partial\Delta(0,r)$ (real dimension $2n-1$). In the language of function algebras, the torus is the distinguished or *Shilov boundary* of $\Delta(0,r)$. [It is the smallest closed subset of the topological boundary on which all f under consideration assume their maximum absolute value.] As a result, a holomorphic function f on $\overline{\Delta}(0,r)$ will be determined by its values on $T(0,r)$. [If $f_1 = f_2$ on T, then ... .] Thus mathematical folklore [or functional analysis!] suggests that one can express such a function in terms of its values on $T(0,r)$. We will see below that there is a Cauchy integral formula which does just that.

For the ball $B(0,r)$ there is no "small" distinguished boundary: all boundary points are equivalent. To every point $b \in S(0,r)$ there is a holomorphic function f on $\overline{B}(0,r)$ such that $|f(b)| > |f(z)|$ for all points $z \in \overline{B}(0,r)$ different from b, cf. exercise 1.9. Integral representations for holomorphic functions on $\overline{B}(0,r)$ will therefore involve all boundary values, cf. exercise 1.24 and Chapter 10.

Function theory for a ball in $\mathbf{C}^n$ ($n \geq 2$) is different from function theory for a polydisc, cf. also [Rudin3,Rudin5]. Indeed, ball and polydisc are *holomorphically inequivalent* in the following sense: there is no $1-1$ holomorphic map

$$w_j = f_j(z_1, \ldots, z_n), \quad j = 1, \ldots, n \qquad (\text{each } f_j \text{ holomorphic})$$

of one onto the other [Chapter 5]. This is in sharp contrast to the situation in $\mathbf{C}$, where all simply connected domains (different from $\mathbf{C}$ itself) are holomorphically equivalent [Riemann mapping theorem]. In $\mathbf{C}$, function theory is essentially the same for all bounded simply connected domains.

1.3 Cauchy integral formula for a polydisc. For functions f that are holomorphic on a closed polydisc $\overline{\Delta}(a,r)$, there is an integral representation of Cauchy which extends the well-known one-variable formula. We will actually assume a little less than holomorphy:

Theorem 1.31. *Let $f(z) = f(z_1, \ldots, z_n)$ be continuous on $\Omega \subset \mathbf{C}^n$ and differentiable in the complex sense with respect to each of the variables z_j separately. Then for every closed polydisc $\overline{\Delta}(a,r) \subset \Omega$,*

$$(3a) \qquad f(z) = \frac{1}{(2\pi i)^n} \int_{T(a,r)} \frac{f(\zeta)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n, \qquad \forall z \in \Delta(a,r)$$

where $T(a,r)$ is the torus $C(a_1,r_1) \times \ldots \times C(a_n,r_n)$, with positive orientation of the circles $C(a_j,r_j)$.

PROOF. We write out a proof for $n = 2$. In the first part we only use the complex differentiability of f with respect to each variable z_j, not the continuity of f.

Fix z in $\Delta(a,r) = \Delta_1(a_1,r_1) \times \Delta_1(a_2,r_2)$ where $\overline{\Delta}(a,r) \subset \Omega$. Then $g(w) = f(w,z_2)$ has a complex derivative with respect to w throughout a neighbourhood of the closed disc $\overline{\Delta}_1(a_1,r_1)$ in $\mathbf{C}$. The one-variable Cauchy integral formula thus gives

$$f(z_1,z_2) = g(z_1) = \frac{1}{2\pi i}\int_{C(a_1,r_1)} \frac{g(w)}{w - z_1}dw = \frac{1}{2\pi i}\int_{C(a_1,r_1)} \frac{f(\zeta_1,z_2)}{\zeta_1 - z_1}d\zeta_1.$$

For fixed $\zeta_1 \in C(a_1,r_1)$, the function $h(w) = f(\zeta_1,w)$ has a complex derivative throughout a neighbourhood of $\overline{\Delta}_1(a_2,r_2)$ in $\mathbf{C}$. Hence

$$f(\zeta_1,z_2) = h(z_2) = \frac{1}{2\pi i}\int_{C(a_2,r_2)} \frac{h(w)}{w - z_2}dw = \frac{1}{2\pi i}\int_{C(a_2,r_2)} \frac{f(\zeta_1,\zeta_2)}{\zeta_2 - z_2}d\zeta_2.$$

Substituting this result into the first formula, we obtain for $f(z_1,z_2)$ the repeated integral

$$f(z_1,z_2) = \frac{1}{(2\pi i)^2}\int_{C(a_1,r_1)} \frac{d\zeta_1}{\zeta_1 - z_1}\int_{C(a_2,r_2)} \frac{f(\zeta_1,\zeta_2)}{\zeta_2 - z_2}d\zeta_2. \tag{3b}$$

If we would have started by varying the second variable instead of the first, we would have wound up with a repeated integral for $f(z_1,z_2)$ in which the order of integration is the reverse. For the applications it is convenient to introduce the (explicit) assumption that f is continuous, cf. Section 1.6. This makes it possible to rewrite the repeated integral in (3b) as a double integral:

$$f(z_1,z_2) = \frac{1}{(2\pi i)^2}\int_{C(a_1,r_1)\times C(a_2,r_2)} \frac{f(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}d\zeta_1 d\zeta_2. \tag{3c}$$

Indeed, setting $\zeta_1 = a_1 + r_1 e^{it_1}$, $\zeta_2 = a_2 + r_2 e^{it_2}$ and

$$\frac{f(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}d\zeta_1 d\zeta_2 = F(t_1,t_2)dt_1 dt_2,$$

we obtain a continuous function F on the closed square region $Q = I_1 \times I_2$, where I_j is the closed interval $-\pi \le t_j \le \pi$. The integral in (3c) now reduces to the double integral of F over Q. Since F is continuous on Q, one has the elementary "Fubini" reduction formula

$$\int_Q F(t_1,t_2)dt_1 dt_2 = \int_{I_1} dt_1 \int_{I_2} F(t_1,t_2)dt_2$$

which implies the equality of the integrals in (3c) and (3b).

REMARKS 1.32. In the Theorem, the continuity of f does not have to be postulated explicitly. Indeed, in his basic paper of 1906, Hartogs proved that the continuity of f follows from its complex differentiability with respect to each of the variables z_j. Since we will not need this rather technical result, we refer to other books for a proof, for example [Hörmander 1].

Cauchy's integral formula for polydiscs (and polydomains) goes back to about 1840. It then took nearly a hundred years before integral representations for holomorphic functions on general $\mathbf{C}^n$ domains with (piecewise) smooth boundary began to make their appearance, cf. Chapter 10. Integral representations and their applications continue to be an active area of research.

In Section 1.6 we will show that functions as in Theorem 1.31 are locally equal to sum functions of power series.

1.4 Multiple power series. The general power series in $\mathbf{C}^n$ with center a has the form

$$\sum_{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0} c_{\alpha_1 \ldots \alpha_n} (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n}. \tag{4a}$$

Here the α_j's are nonnegative integers and the c's are complex constants. We will see that multiple power series have properties similar to those of power series in one complex variable.

Before we start it is convenient to introduce abbreviated notation. We write α for the *multi-index* or ordered n-tuple $(\alpha_1, \ldots, \alpha_n)$ of integers. Such n-tuples are added in the usual way; the inequality $\alpha \geq \beta$ will mean $\alpha_j \geq \beta_j, \ \forall j$. In the case $\alpha \geq 0$ [that is, $\alpha_j \geq 0$, $\forall j$], we also write

$$\alpha \in \mathbf{N}_0^n, \quad \alpha! = \alpha_1! \ldots \alpha_n!, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n \quad (\textit{height} \text{ of } \alpha).$$

One sets

$$z_1^{\alpha_1} \ldots z_n^{\alpha_n} = z^\alpha, \quad (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n} = (z - a)^\alpha,$$

so that the multiple sum (4a) becomes simply

$$\sum_{\alpha \geq 0} c_\alpha (z - a)^\alpha. \tag{4a$'$}$$

We will do something similar for derivatives, writing

$$\frac{\partial}{\partial z_j} = D_j, \quad \frac{\partial^{\beta_1 + \ldots + \beta_n}}{\partial z_1^{\beta_1} \ldots \partial z_n^{\beta_n}} = D_1^{\beta_1} \ldots D_n^{\beta_n} = D^\beta, \quad \frac{\partial}{\partial \overline{z}_j} = \overline{D}_j.$$

Returning to (4a), suppose for a moment that the series converges at some point z with $|z_j - a_j| = r_j > 0, \ \forall j$ for *some* (total) ordering of its terms. Then the terms will form a bounded sequence at the given point z [and hence at all points z with $|z_j - a_j| = r_j$]:

$$|c_\alpha| \, r_1^{\alpha_n} \ldots r_n^{\alpha_n} \leq M < +\infty, \ \forall \alpha \in \mathbf{N}_0^n. \tag{4b}$$

We will show that under the latter condition, the series (4a) is absolutely convergent throughout the polydisc $\Delta(a, r)$ [for every total ordering of its terms]. The same will be true for the differentiated series $\sum c_\alpha D^\beta (z - a)^\alpha$. Thus all these series will have well-defined sum functions on the polydisc: the sums are independent of the order of the terms.

For the proofs it will be sufficient to consider power series with center 0:

$$\sum_{\alpha\geq 0} c_\alpha z^\alpha = \sum c_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1}\ldots z_n^{\alpha_n}. \tag{4c}$$

Lemma 1.41. *Suppose that the terms $c_\alpha z^\alpha$ form a bounded sequence at the point $z = r > 0$ (4b). Then the power series (4c) is absolutely convergent throughout the polydisc $\Delta(0, r)$. The convergence is uniform on every smaller polydisc $\Delta(0, \lambda r)$ with $0 < \lambda < 1$, no matter in what order the terms are arranged. For every multi-index $\beta \in \mathbf{N}_0^n$ and $D^\beta = D_1^{\beta_1}\ldots D_n^{\beta_n}$, the termwise differentiated series $\sum c_\alpha D^\beta z^\alpha$ is also absolutely convergent on $\Delta(0, r)$ and uniformly convergent on $\Delta(0, \lambda r)$.*

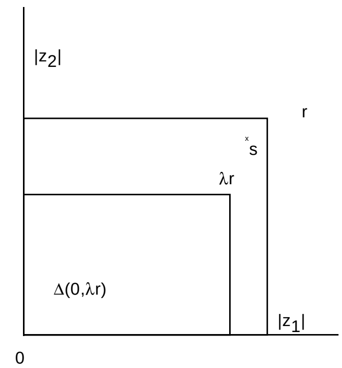


fig 1.3

PROOF. For $z \in \Delta(0, \lambda r)$ we have $|z_j| < \lambda r_j,\ \forall j$ so that by (4b)

$$|c_\alpha z^\alpha| = |c_\alpha||z_1^{\alpha_1}|\ldots|z_n^{\alpha_n}| \leq |c_\alpha|\lambda^{\alpha_1} r_1^{\alpha_1}\ldots\lambda^{\alpha_n} r_n^{\alpha_n} \leq M\lambda^{\alpha_1}\ldots\lambda^{\alpha_n}.$$

On $\Delta(0, \lambda r)$ the series (4c) is thus (termwise) *majorized* by the following convergent (multiple) series of positive constants:

$$\sum_{\alpha\geq 0} M\lambda^{\alpha_1}\ldots\lambda^{\alpha_n} = M\sum_{\alpha_1\geq 0}\lambda^{\alpha_1}\ldots\sum_{\alpha_n\geq 0}\lambda^{\alpha_n} = M\frac{1}{1-\lambda}\cdots\frac{1}{1-\lambda} = \frac{M}{(1-\lambda)^n}.$$

It follows that the power series (4c) is absolutely convergent [for every total ordering of its terms] at each point of $\Delta(0, \lambda r)$ and finally, at each point of $\Delta(0, r)$. Moreover, by Weierstrass's criterion for uniform convergence, the series will be uniformly convergent on $\Delta(0, \lambda r)$ for any given order of the terms. [The remainders are dominated by those of the majorizing series of constants.]

We now turn to the final statement in the Lemma. To show the method of proof, it will be sufficient to consider the simple differential operator D_1. It follows from (4b) that the differentiated series

$$\sum c_\alpha D_1 z^\alpha = \sum c_\alpha \alpha_1 z_1^{\alpha_1-1} z_2^{\alpha_2}\ldots z_n^{\alpha_n}$$

is also majorized by a convergent series of constants on $\Delta(0,\lambda r)$, namely, by the series

$$\sum_{\alpha\geq 0}\frac{M}{r_1}\alpha_1\lambda^{\alpha_1-1}\lambda^{\alpha_2}\ldots\lambda^{\alpha_n}=\frac{M}{r_1}\left(\frac{d}{d\lambda}\sum\lambda^{\alpha_1}\right)\sum\lambda^{\alpha_2}\ldots\sum\lambda^{\alpha_n}=\frac{M/r_1}{(1-\lambda)^{n+1}}.$$

Thus the differentiated series converges absolutely and uniformly on $\Delta(0,\lambda r)$ for each $\lambda\in(0,1)$.

Proposition 1.42. *Let $\sum c_\alpha z^\alpha$ be a power series (4c) whose terms are uniformly bounded at $z=r>0$, or suppose only that the series converges throughout the polydisc $\Delta(0,r)$ for some total ordering of the terms, or at least suppose that the terms $c_\alpha z^\alpha$ form a bounded sequence at certain points z arbitrarily close to r. Then the series converges absolutely throughout $\Delta(0,r)$, so that the sum*

$$f(z)=\sum_{\alpha\geq 0}c_\alpha z^\alpha,\quad z\in\Delta(0,r)$$

is well-defined (the sum is independent of the order of the terms). The sum function f will be continuous on $\Delta(0,r)$ and infinitely differentiable (in the complex sense) with respect to each of the variables $z_1,\ldots,z_n$; similarly for the derivatives. The derivative $D^\beta f(z)$ will be equal to the sum of the differentiated series $\sum c_\alpha D^\beta z^\alpha$.

PROOF. Choose any λ in $(0,1)$. Either one of the hypotheses in the Proposition implies that the terms $c_\alpha z^\alpha$ form a bounded sequence at some point $z=s>\lambda r$ (fig 1.3). Thus we may apply Lemma 1.41 with s instead of r to obtain absolute and uniform convergence of the series on $\Delta(0,\lambda r)$. It follows in particular that the sum function f is well-defined and continuous on $\Delta(0,\lambda r)$ and finally, on $\Delta(0,r)$.

We now prove the complex differentiability of f with respect to z_1. Fix $z_2=b_2,\ldots,$ $z_n=b_n$ ($|b_j|<r_j$). By suitable rearrangement of the terms in our absolutely convergent series (4c) we obtain

$$f(z_1,b_2,\ldots,b_n)=\sum_{\alpha_1}\left(\sum_{\alpha_2,\ldots,\alpha_n}c_\alpha b_2^{\alpha_2}\ldots b_n^{\alpha_n}\right)z_1^{\alpha_1},\quad |z_1|<r_1.$$

[In an absolutely convergent multiple series we may first sum over some of the indices, then over the others, cf. Fubini's theorem for multiple integrals.] A well-known differentiation theorem for power series in one variable now shows that $f(z_1,b_2,\ldots,b_n)$ has a complex derivative $D_1 f$ for $|z_1|<r_1$ which can be obtained by termwise differentiation. The resulting series for $D_1 f$ may be rewritten as an absolutely convergent multiple series:

$$\sum_{\alpha_1}\left(\sum_{\alpha_2,\ldots,\alpha_n}c_\alpha b_2^{\alpha_2}\ldots b_n^{\alpha_n}\right)D_1 z_1^{\alpha_1}=\sum_{\alpha}c_\alpha D_1\left(z_1^{\alpha_1}b_2^{\alpha_2}\ldots b_n^{\alpha_n}\right),$$

cf. Lemma 1.41. Conclusion: $D_1 f$ exists throughout $\Delta(0,r)$ and $D_1 f(z)=\sum c_\alpha D_1 z^\alpha$; similarly for each D_j. Since the new power series converge throughout $\Delta(0,r)$, one can repeat the argument to obtain higher order derivatives.

1.5 Analytic functions. Sets of uniqueness. We formalize our earlier rough description of analytic functions:

Definition 1.51. *A function f on $\Omega \subset \mathbf{C}^n$* to $\mathbf{C}$ *is called* ANALYTIC *if for every point $a \in \Omega$, there is a polydisc $\Delta(a, r)$ in Ω and a multiple power series $\sum c_\alpha (z-a)^\alpha$ which converges to $f(z)$ on $\Delta(a, r)$ for some total ordering of its terms.*

It follows from Proposition 1.42 that a power series ($4a$) for f on Δ is absolutely convergent, hence the order of the terms is immaterial. Proposition 1.42 also implies the following important

Theorem 1.52. *Let $f(z)$ be analytic on $\Omega \subset \mathbf{C}^n$. Then f is continuous on Ω and infinitely differentiable (in the complex sense) with respect to the variables $z_1, \ldots, z_n$; the partial derivatives $D^\beta f$ are likewise analytic on Ω. If $f(z) = \sum c_\alpha (z-a)^\alpha$ on $\Delta(a, r) \subset \Omega$, then*

$$D^\beta f(z) = \sum_{\alpha \geq 0} c_\alpha D^\beta (z-a)^\alpha = \sum_{\alpha \geq \beta} c_\alpha \frac{\alpha!}{(\alpha-\beta)!}(z-a)^{\alpha-\beta}, \quad \forall z \in \Delta(a, r).$$

In particular $D^\beta f(a) = c_\beta \beta!$. Replacing β by α, one obtains the coefficient formula

$$(5a) \qquad c_\alpha = \frac{1}{\alpha!} D^\alpha f(a) = \frac{1}{\alpha_1! \ldots \alpha_n!} D_1^{\alpha_1} \ldots D_n^{\alpha_n} f(a).$$

COROLLARIES 1.53. An analytic function f on a domain Ω in $\mathbf{C}^n$ has only one (locally) representing power series with center $a \in \Omega$. It is the *Taylor series*, the coefficients are the *Taylor coefficients* ($5a$) of f at a.

Analytic functions are *holomorphic* in the sense of Definition 1.11. [For analytic f one has $\partial f / \partial x_j = \partial f / \partial z_j$ and $\partial f / \partial y_j = i \partial f / \partial z_j$, cf. ($1c$), hence the map f is of class $\mathbf{C}^1$ and $\overline{\partial} f = 0$.]

Theorem 1.54 (UNIQUENESS THEOREM). *Let f_1 and f_2 be analytic on a connected domain $\Omega \subset \mathbf{C}^n$ and suppose that $f_1 = f_2$ throughout a nonempty open subset $U \subset \Omega$. (This will in particular be the case if f_1 and f_2 have the same power series at some point $a \in \Omega$.) Then $f_1 = f_2$ throughout Ω.*

PROOF. Define $f = f_1 - f_2$. We introduce the set

$$E = \{z \in \Omega : D^\alpha f(z) = 0, \ \forall \alpha \in \mathbf{N}_0^n\}.$$

E is *open*. For suppose $a \in E$. There will be a polydisc $\Delta = \Delta(a, r) \subset \Omega$ on which $f(z)$ is equal to the sum of its Taylor series $\sum D^\alpha f(a) \cdot (z-a)^\alpha / \alpha!$. Hence by the hypothesis, $f = 0$ throughout Δ. It follows that also $D^\alpha f = 0$ throughout Δ for every α, so that $\Delta \subset E$.

The complement $\Omega - E$ is also *open*. Indeed, if $b \in \Omega - E$ then $D^\beta f(b) \neq 0$ for some β. By the continuity of $D^\beta f$, it follows that $D^\beta f(z) \neq 0$ throughout a neighbourhood of b. Now Ω is connected, hence it is not the union of two disjoint nonempty open sets. Since E contains U it is nonempty. Thus $\Omega - E$ must be empty or $\Omega = E$, so that $f \equiv 0$.

DEFINITION 1.55. A subset $E \subset \Omega$ in $\mathbf{C}^n$ is called a *set of uniqueness* for Ω [or better, for the class of analytic functions $\mathcal{A}(\Omega)$] if the condition "$f = 0$ throughout E" for analytic f on Ω implies that $f \equiv 0$ on Ω.

For a connected domain $D \subset \mathbf{C}$, every infinite subset E with a limit point in D is a set of uniqueness. [Why? Cf. exercises 1.15, 1.16.] For a connected domain $D \subset \mathbf{C}^n$ with $n \geq 2$, every ball $B(a,r) \subset D$ is a set of uniqueness, but the intersection of D with a complex hyperplane $c \cdot (z-a) = 0 \quad (c \neq 0)$ is not a set of uniqueness: think of $f(z) = c \cdot (z-a)$! One may use the maximum principle for a polydisc [Section 1.2] to show that if $\overline{\Delta}(a,r) \subset D$, then the torus $T(a,r)$ is a set of uniqueness for D. It is not so much the size of a subset $E \subset D$ which makes it a set of uniqueness, as well as the way in which it is situated in $\mathbf{C^n}$, cf. also exercise 1.17.

The counterpart to sets of uniqueness is formed by the zero sets of analytic functions, cf. Section 1.10. Sets of uniqueness (or zero sets) for subclasses of $\mathcal{A}(\Omega)$, for example, the bounded analytic functions, are not yet well understood, except in very special cases, cf. [Rudin5] for references. Discrete sets of uniqueness for subclasses of $\mathcal{A}(\Omega)$ are important for certain approximation problems, cf. [Korevaar1983].

1.6 Analyticity of the Cauchy integral and consequences. Under the conditions of Theorem 1.31 the function f represented by the Cauchy integral ($3a$) will turn out to be analytic on $\Delta(a,r)$. More generally we prove

Theorem 1.61. *Let $g(\zeta) = g(\zeta_1, \ldots, \zeta_n)$ be defined and continuous on the torus $T(a,r) = C(a_1,r_1) \times \ldots \times C(a_n,r_n)$. Then the* CAUCHY TRANSFORM

$$f(z) = \hat{g}(z) \stackrel{\text{def}}{=} \frac{1}{(2\pi i)^n} \int_{T(a,r)} \frac{g(\zeta)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n \tag{6a}$$

[*where we use positive orientation of the generating circles $C(a_j,r_j)$ of $T(a,r)$*] *is* ANALYTIC *on the polydisc $\Delta(a,r)$.*

PROOF. By translation we may assume that $a = 0$. Now taking an arbitrary point b in $\Delta(0,r)$: $|b_j| < r_j$, $\forall j$, we have to show that $f(z)$ is equal to the sum of a convergent power series with center b on some polydisc around b. In a situation like the present one, where $f(z)$ is given by an integral with respect to ζ in which z occurs as a parameter, it is standard procedure to expand the integrand in a power series of the form $\sum d_\alpha(\zeta)(z-b)^\alpha$ and to integrate term by term.

In order to obtain a suitable series for the integrand, we begin by expanding each factor $1/(\zeta_j - z_j)$ around $z_j = b_j$:

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j - b_j - (z_j - b_j)} = \frac{1}{\zeta_j - b_j} \, \frac{1}{1 - \frac{z_j - b_j}{\zeta_j - b_j}} = \sum_{p=0}^{\infty} \frac{(z_j - b_j)^p}{(\zeta_j - b_j)^{p+1}}. \tag{6b}$$

When does this series converge? We must make sure that the ratio $|z_j - b_j|/|\zeta_j - b_j|$ remains less than 1 as ζ_j runs over the circle $C(0,r_j)$. To that end we *fix* z such that $|z_j - b_j| < r_j - |b_j|$, $\forall j$ (fig 1.4). Then

$$\frac{|z_j - b_j|}{|\zeta_j - b_j|} \leq \frac{|z_j - b_j|}{r_j - |b_j|} \stackrel{\text{def}}{=} \lambda_j < 1, \quad \forall \zeta_j \in C(0,r_j). \tag{6c}$$

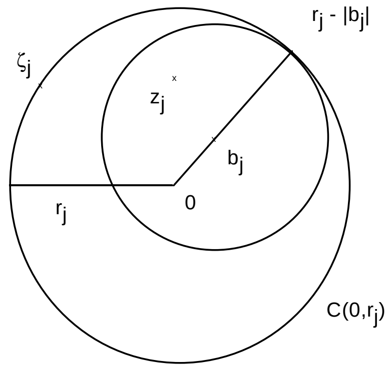


fig 1.4

Thus for ζ_j running over $C(0, r_j)$ the series in $(6b)$ is termwise majorized by the convergent series of constants

$$\sum_p M_j \lambda_j^p = \sum_{\alpha_j} \frac{1}{r_j - |b_j|} \lambda_j^{\alpha_j}.$$

There is such a result for each j. Forming the termwise product of the series in $(6b)$ for $j = 1, \ldots, n$, we obtain a multiple series for our integrand:

$$(6d) \qquad \begin{aligned} &\frac{g(\zeta)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} \\ &= \sum_{\alpha \geq 0} \frac{g(\zeta)}{(\zeta_1 - b_1)^{\alpha_1 + 1} \ldots (\zeta_n - b_n)^{\alpha_n + 1}} (z_1 - b_1)^{\alpha_1} \ldots (z_n - b_n)^{\alpha_n}. \end{aligned}$$

By $(6c)$ and using the boundedness of $g(\zeta)$ on $T(0, r)$, the expansion $(6d)$ is termwise majorized on $T(0, r)$ by a convergent multiple series of constants $\sum_\alpha M \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$. Hence the series in $(6d)$ is absolutely and uniformly convergent (for any given order of the terms) as ζ runs over $T(0, r)$, so that we may integrate term by term. Thus we obtain a representation for the value $f(z)$ in $(6a)$ by a convergent multiple power series:

$$(6e) \qquad f(z) = \sum_{\alpha \geq 0} c_\alpha (z - b)^\alpha.$$

Here the coefficients c_α [which must also be equal to the Taylor coefficients for f at b] are given by the following integrals:

$$(6f) \qquad c_\alpha = \frac{1}{\alpha!} D^\alpha f(b) = \frac{1}{(2\pi i)^n} \int_{T(0,r)} \frac{g(\zeta)}{(\zeta_1 - b_1)^{\alpha_1 + 1} \ldots (\zeta_n - b_n)^{\alpha_n + 1}} d\zeta_1 \ldots d\zeta_n.$$

The representation will be valid for every z in the polydisc

$$(6g) \qquad \Delta(b_1, \ldots, b_n; \, r_1 - |b_1|, \ldots, r_n - |b_n|).$$

COROLLARY 1.62 (*Osgood's Lemma*). Let $f(z) = f(z_1, \ldots, z_n)$ be continuous on $\Omega \subset \mathbf{C^n}$ and differentiable in the complex sense on Ω with respect to each variable z_j separately. Then f is analytic on Ω.

[By Theorem 1.31, the function f is locally representable as a Cauchy transform. Now apply Theorem 1.61. Actually, the continuity of f need not be postulated, cf. Remarks 1.32.]

Osgood's lemma shows, in particular, that **every holomorphic function is analytic**. Thus the class of analytic functions on a domain Ω is the same as the class of holomorphic functions, $\mathcal{A}(\Omega) = \mathcal{O}(\Omega)$. From here on, we will not distinguish between the terms analytic and holomorphic; we usually speak of holomorphic functions.

COROLLARY 1.63 (*Convergence of power series throughout polydiscs of holomorphy*). Let f be holomorphic on $\Delta(a, r)$. Then the power series for f with center a converges to f throughout $\Delta(a, r)$.

[We may take $a = 0$. If f is holomorphic on (a neighbourhood of) $\overline{\Delta}(0, r)$, it may be represented on $\Delta(0, r)$ by a Cauchy transform over $T(0, r)$. The proof of Theorem 1.61 now shows that the (unique) power series for f with center $b = 0$ converges to f throughout $\Delta(0, r)$, see ($6e$–g). If f is only known to be holomorphic on $\Delta(0, r)$, the preceding argument may be applied to $\overline{\Delta}(0, \lambda r)$, $0 < \lambda < 1$.]

COROLLARY 1.64 (*Cauchy integrals for derivatives*). Let f be holomorphic on $\overline{\Delta}(a, r)$. Then

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{T(a,r)} \frac{f(\zeta)}{(\zeta_1 - z_1)^{\alpha_1 + 1} \ldots (\zeta_n - z_n)^{\alpha_n + 1}} d\zeta_1 \ldots d\zeta_n, \quad \forall z \in \Delta(a, r).$$

[By Theorem 1.61, $f(z)$ is equal to a Cauchy transform ($6a$) on $\Delta(a, r)$, with $g(\zeta) = f(\zeta)$ on $T(a, r)$. Taking $a = 0$ as we may, the result now follows from ($6f$) with $b = z$. Observe that the result corresponds to differentiation under the integral sign in the Cauchy integral for f ($3a$). Such differentiation is thus permitted.]

COROLLARY 1.65 (*Cauchy inequalities*). Let f be holomorphic on $\overline{\Delta}(a, r)$, $f(z) = \sum c_\alpha (z - a)^\alpha$. Then

$$|c_\alpha| = \frac{|D^\alpha f(a)|}{\alpha!} \leq \frac{M}{r^\alpha} = \frac{M}{r_1^{\alpha_1} \ldots r_n^{\alpha_n}},$$

where $M = \sup |f(\zeta)|$ on $T(a, r)$.

[Use Corollary 1.64 with $z = a$. Set $\zeta_j = a_j + r_j e^{it_j}$, $j = 1, \ldots, n$ to obtain a bound for the integral.]

1.7 Limits of holomorphic functions. We will often use yet another consequence of Theorems 1.31 and 1.61:

Theorem 1.71 (WEIERSTRASS). *Let $\{f_\lambda\}$, $\lambda \in \Lambda$ be an indexed family of holomorphic functions on $\Omega \subset \mathbf{C^n}$ which converges uniformly on every compact subset of Ω as $\lambda \to \lambda_0$. Then the limit function f is holomorphic on Ω. Furthermore, for every multi-index $\alpha \in \mathbf{N}_0^n$,*

$$D^\alpha f_\lambda \to D^\alpha f \quad as \quad \lambda \to \lambda_0,$$

uniformly on every compact subset of Ω.

In particular, uniformly convergent sequences and series of analytic functions on a domain *"may be differentiated term by term"*.

PROOF. Choose a closed polydisc $\overline{\Delta}(a,r)$ in Ω. For convenience we write the Cauchy integral $(3a)$ for f_λ in abbreviated form as follows:

$$f_\lambda(z) = (2\pi i)^{-n} \int_{T(a,r)} \frac{f_\lambda(\zeta)}{\zeta - z} d\zeta, \quad z \in \Delta(a,r). \tag{7}$$

Keeping z fixed, we let $\lambda \to \lambda_0$. Then

$$\frac{f_\lambda(\zeta)}{\zeta - z} \to \frac{f(\zeta)}{\zeta - z}, \text{ uniformly for } \zeta \in T(a,r).$$

[The denominator stays away from 0.] Integrating, we conclude that the right-hand side of (7) tends to the corresponding expression with f instead of f_λ. The left-hand side tends to $f(z)$, hence the Cauchy integral representation is valid for the limit function f just as for f_λ $(3a)$. Theorem 1.61 now implies the analyticity of f on $\Delta(a,r)$. Varying $\overline{\Delta}(a,r)$ over Ω, we conclude that $f \in \mathcal{O}(\Omega)$.

Again fixing $\overline{\Delta}(a,r)$ in Ω, we next apply the Cauchy formula for derivatives to $f - f_\lambda$ [Corollary 1.64]. Fixing α and letting $\lambda \to \lambda_0$, we may conclude that $D^\alpha(f - f_\lambda) \to 0$ uniformly on $\Delta(a, \frac{1}{2}r)$. Since a given compact subset $E \subset \Omega$ can be covered by a finite number of polydiscs $\Delta(a, \frac{1}{2}r)$ with $a \in E$ and $\overline{\Delta}(a,r) \subset \Omega$, it follows that $D^\alpha f_\lambda \to D^\alpha f$ uniformly on E.

COROLLARY 1.72 (*Holomorphy theorem for integrals*). Let Ω be an open set in $\mathbf{C}^n$ and let I be a compact interval in $\mathbf{R}$, or a product of m such intervals in $\mathbf{R}^m$. Suppose that the "kernel" $K(z,t)$ is defined and continuous on $\Omega \times I$ and that it is holomorphic on Ω for every $t \in I$. Then the integral

$$f(z) = \int_I K(z,t)dt = \lim \sum_{j=1}^{s} K(z,\tau_j)m(I_j)$$

defines a holomorphic function f on Ω. Furthermore, $D_z^\alpha K(z,t)$ will be continuous on $\Omega \times I$ and

$$D^\alpha f(z) = \int_I D_z^\alpha K(z,t)dt.$$

Thus, *"one may differentiate under the integral sign"* here.

For the proof, one may observe the following:
(i) The Riemann sums

$$\sigma(z,P,\tau) = \sum_{j=1}^{s} K(z,\tau_j)m(I_j), \quad \tau_j \in I_j$$

corresponding to partitionings P of I into appropriate subsets I_j, are holomorphic in z on Ω;

(ii) For a suitable sequence of partitionings, the Riemann sums converge to the integral $f(z)$, uniformly for z varying over any given compact subset $E \subset \Omega$.

Indeed, $K(z,t)$ will be uniformly continuous on $E \times I$. We now write the integral as a sum of integrals over the parts I_j of small (diameter and) size $m(I_j)$. It is then easy to show that the difference between the integral and the approximating sum will be small.

The continuity of $D_z^\alpha K(z,t)$ on $\Omega \times I$ may be obtained from the Cauchy integral for a derivative [Corollary 1.64]. The integral formula for $D^\alpha f$ then follows by differentiation of the limit formula for $f(z)$:

$$D^\alpha f(z) = \lim \sum_{j=1}^{s} D_z^\alpha K(z, \tau_j) m(I_j).$$

The following two convergence theorems for $\mathbf{C}^n$ are sometimes useful. We do not include the proofs which are similar to those for the case $n = 1$, cf. [Narasimhan] or [Rudin2].

THEOREM 1.73 (*Montel*). A locally bounded family $\mathcal{F}$ of holomorphic functions on $\Omega \subset \mathbf{C}^n$ is *normal*, that is, every infinite sequence $\{f_k\}$ chosen from $\mathcal{F}$ contains a subsequence which converges throughout Ω and uniformly on every compact subset.

The key observation in the proof is that a locally bounded family of holomorphic functions is locally equicontinuous, cf. exercise 1.28. A subsequence $\{\tilde{f}_k\}$ which converges on a countable dense subset of Ω will then converge uniformly on every compact subset.

THEOREM 1.74 (*Stieltjes-Vitali-Osgood*). Let $\{f_k\}$ be a locally bounded sequence of holomorphic functions on Ω which converges at every point of a set of uniqueness E for $\mathcal{O}(\Omega)$. Then the sequence $\{f_k\}$ converges throughout Ω and uniformly on every compact subset.

Certain useful approximation theorems for $\mathbf{C}$ do not readily extend to $\mathbf{C}^n$. In this connection we mention Runge's theorem on *polynomial approximation* in $\mathbf{C}$. One may call $\Omega \subset \mathbf{C}^n$ a *Runge domain* if every function $f \in \mathcal{O}(\Omega)$ is the limit of a sequence of polynomials in $z_1, \ldots, z_n$ which converges uniformly on every compact subset of Ω.

More generally, let $V \subset W \subset \mathbf{C}$ be two domains. Then V is called Runge in W if every function $f \in \mathcal{O}(V)$ is the limit of a sequence of functions $f_k \in \mathcal{O}(W)$ which converges uniformly on every compact subset of V.

THEOREM 1.75 (cf. [Runge] 1885). The Runge domains in $\mathbf{C}$ are precisely those open sets, whose complement relative to the extended plane $\mathbf{C}_e = \mathbf{C} \cup \{\infty\}$ is connected.

There are several results on Runge domains in $\mathbf{C}^n$, but also open problems, cf. [Hörmander1, Range] and especially [Fornæss-Stensønes]. The one-variable theorem provides an extremely useful tool for the construction of counterexamples in complex analysis.

1.8 Open mapping theorem and maximum principle.

Theorem 1.81. *Let $D \subset \mathbf{C}^n$ be a connected domain, $f \in \mathcal{O}(D)$ nonconstant. Then the range $f(D)$ is open* [hence $f(D) \subset \mathbf{C}$ is a connected domain].

This result follows easily from the special case $n = 1$ by restricting f to a suitable complex line. We include a detailed proof because parts of it will be useful later on. The situation is more complicated in the case of *holomorphic mappings*

$$\zeta_j = f_j(z), \;\; j = 1, \ldots, p, \;\; f_j \in \mathcal{O}(D)$$

from a connected domain $D \subset \mathbf{C}^n$ to $\mathbf{C}^p$ with $p \geq 2$. The range of such a map will be open only in special cases, cf. exercise 1.29 and Section 5.2.

PROOF of Theorem 1.81. It is sufficient to show that for any point $a \in D$ and for small balls $B = B(a, r) \subset D$, the range $f(B)$ contains a neighbourhood of $f(a)$ in $\mathbf{C}$. By translation we may assume that $a = 0$ and $f(a) = 0$.

(i) *The case* $n = 1$. Since $f \not\equiv 0$, the origin is a zero of f of some finite order s, hence it is not a limit point of zeros of f. Choose $r > 0$ such that $\overline{B}(0, r) = \overline{\Delta}(0, r)$ belongs to D and $f(z) \neq 0$ on $C(0, r)$. Set $\min|f(z)|$ on $C(0, r)$ equal to m, so that $m > 0$. We will show that for any number c in the disc $\Delta(0, m)$, the equation $f(z) = c$ has the same number of roots in $B(0, r)$ as the equation $f(z) = 0$, counting multiplicities.

Indeed, by the residue theorem, the number of zeros of f in $B(0, r)$ is equal to

$$N(f) = \frac{1}{2\pi i} \int_{C(0,r)^+} \frac{f'(z)}{f(z)} dz.$$

[Around a zero z_0 of f of multiplicity μ, the quotient $f'(z)/f(z)$ behaves like $\mu/(z - z_0)$.] We now calculate the number of zeros of $f - c$ in $B(0, r)$:

$$N(f - c) = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f'(z)}{f(z) - c} \, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f'(re^{it})}{f(re^{it}) - c} \, re^{it} dt.$$

By the holomorphy theorem for integrals [Corollary 1.72], $N(f - c)$ will be holomorphic in c on $\Delta(0, m)$. Indeed, the final integrand is continuous in (c, t) on $\Delta(0, m) \times [-\pi, \pi]$ and it is holomorphic in c on $\Delta(0, m)$ for every $t \in [-\pi, \pi]$. Thus since $N(f - c)$ is integer-valued, it must be constant and equal to $N(f) \geq 1$.

Final conclusion: $f(B)$ contains the whole disc $\Delta(0, m)$.

(ii) *The case* $n \geq 2$. Choose $\overline{B}(0, r) \subset D$. By the uniqueness theorem, $f \not\equiv 0$ in B or else $f \equiv 0$ in D. Choose $b \in B(0, r)$ such that $f(b) \neq 0$ and consider the restriction of f to the intersection Δ of B with the complex line $z = wb$, $w \in \mathbf{C}$. The image $f(\Delta)$ is the same as the range of the function

$$h(w) = f(wb), \;\; |w| < r/|b|.$$

That function is holomorphic and nonconstant: $h(0) = 0 \neq h(1) = f(b)$, hence by part (i), the range of h contains a neighbourhood of the origin in $\mathbf{C}$. The same holds a fortiori for the image $f(B)$.

For functions f as in the Theorem, the absolute value $|f|$ and the real part $\mathrm{Re}\, f$ can not have a relative maximum at a point $a \in D$. Indeed, any neighbourhood of the point $f(a)$ in $\mathbf{C}$ must contain points $f(z)$ of larger absolute value and of larger real part. One may thus obtain upper bounds for $|f|$ and $\operatorname{Re} f$ on D in terms of the boundary values of those functions.

Let us define the extended boundary $\partial_e\Omega$ by $\partial\Omega$ if Ω is bounded and by $\partial\Omega \cup \{\infty\}$ otherwise; $z \to \infty$ will mean $|z| \to \infty$.

COROLLARY 1.82 (*Maximum principle* or *maximum modulus theorem*). Let Ω be any domain in $\mathbf{C}^n$, $f \in \mathcal{O}(\Omega)$. Suppose that there is a constant M such that

$$\limsup_{z\to\zeta,\, z\in\Omega} |f(z)| \le M, \quad \forall\zeta \in \partial_e\Omega.$$

Then $|f(z)| \le M$ throughout Ω. If Ω is connected and f is nonconstant, one has $|f(z)| < M$ throughout Ω.

Indeed, if $\mu = \sup_D |f|$ would be larger than M for some connected component D of Ω, then f would be nonconstant on D and μ would be equal to $\lim |f(z_\nu)|$ for some sequence $\{z_\nu\} \subset D$ that can not tend to $\partial_e\Omega$. Taking a convergent subsequence we would find that $\mu = |f(a)|$ for some point $a \in D$, contradicting the open mapping theorem.

In $\mathbf{C}$, more refined ways of estimating $|f|$ from above depend on the fact that $\log|f|$ is a subharmonic function - such functions are majorized by harmonic functions with the same boundary values. For holomorphic functions f in $\mathbf{C}^n$, $\log|f|$ is a so-called *plurisubharmonic* function: its restrictions to complex lines are subharmonic. Plurisubharmonic functions play an important role in n-dimensional complex analysis, cf. Chapter 8; their theory is an active subject of research.

1.9 Preview: analytic continuation, domains of holomorphy, the Levi problem and the $\overline{\partial}$ equation. Given an analytic function f on a domain $\Omega \subset \mathbf{C}^n$, we can choose any point a in Ω and form the power series for f with center a, using the Taylor coefficients (5a). Let U denote the union of all polydiscs $\Delta(a, r)$ on which the Taylor series converges. The sum function g of the series will be analytic on U [see Osgood's criterion 1.62] and it coincides with f around a. Suppose now that U extends across a boundary point b of Ω (fig 1.5). Then g will provide an ANALYTIC CONTINUATION of f. It is not required that such a continuation coincide with f on all components of $U \cap \Omega$.

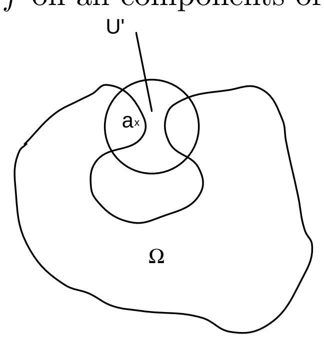
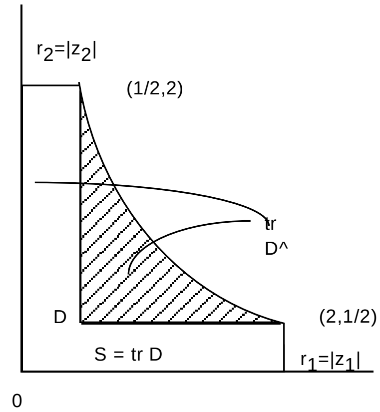


fig 1

The subject of analytic continuation will bring out a very remarkable difference between the case of $n \ge 2$ complex variables and the classical case of one variable. For a domain Ω in the complex plane $\mathbf{C}$ and any (finite) boundary point $b \in \partial\Omega$, there always exist analytic functions f on Ω which have no analytic continuation across the point b, think of $f(z) = 1/(z-b)$. By suitable distribution of singularities along $\partial\Omega$, one may even construct analytic functions on $\Omega \subset \mathbf{C}$ which can not be continued analytically across *any* boundary point; we say that Ω is their maximal domain of existence.



However, in $\mathbf{C}^n$ with $n \geq 2$ there are many domains Ω with the property that *all* functions in $\mathcal{O}(\Omega)$ can be continued analytically across a certain part of the boundary. Several examples of this phenomenon were discovered by Hartogs around 1905. We mention his striking *spherical shell theorem*: For $\Omega = B(a,R) - \overline{B}(a,\rho)$ where $0 < \rho < R$, every function in $\mathcal{O}(\Omega)$ has an analytic continuation to the whole ball $B(a,R)$ [cf. Sections 2.8, 3.4]. Another example is indicated in fig 1.6, where D stands for the union of two polydiscs in $\mathbf{C}^2$ with center 0. For every $f \in \mathcal{O}(D)$ the power series with center 0 converges throughout D, but any such power series will actually converge throughout the larger domain $\hat{D}$, thus providing an analytic continuation of f to $\hat{D}$ [cf. Section 2.4].

Many problems in complex analysis of several variables can only be solved on so-called DOMAINS OF HOLOMORPHY; for other problems, it is at least convenient to work with such domains. Domains of holomorphy Ω in $\mathbf{C}^n$ are characterized by the following property: For every boundary point b, there is a holomorphic function on Ω which has no analytic continuation to a neighbourhood of b. What this means precisely is explained in Section 2.1, cf. also the comprehensive definition in Section 6.1. The following sufficient condition is very useful in practice: Ω is a domain of holomorphy if for every sequence of points in Ω which converges to a boundary point, there is a function in $\mathcal{O}(\Omega)$ which is unbounded on that sequence [see Section 6.1]. Domains of holomorphy Ω will also turn out to be maximal domains of existence: there exist functions in $\mathcal{O}(\Omega)$ which can not be continued analytically across any part of the boundary [Section 6.4].

We will see in Section 6.1 that every convex domain in $\mathbf{C}^n \sim \mathbf{R}^{2n}$ is a domain of holomorphy. All domains of holomorphy have certain (weaker) convexity properties, going by names such as holomorphic convexity and *pseudoconvexity* [Chapter 6; fig 1.6 illustrates a pseudoconvex domain $\hat{D}$ in $\mathbf{C}^2$]. For many years it was a major question if all pseudoconvex domains are, in fact, domains of holomorphy (LEVI PROBLEM). The answer is yes [cf. Chapters 7, 11]. Work on the Levi problem has led to many notable developments in complex analysis.

We mention some problems where domains of holomorphy are important:

HOLOMORPHIC EXTENSION from affine subspaces. Let Ω be a given domain in $\mathbf{C}^n$ and let W denote an arbitrary affine subspace of $\mathbf{C}^n$. If f belongs to $\mathcal{O}(\Omega)$, the restriction of f

to the intersection $\Omega \cap W$ will be holomorphic for every choice of W. Conversely, suppose h is *some* holomorphic function on *some* intersection $\Omega \cap W$. Can h be extended to a function in $\mathcal{O}(\Omega)$? This problem turns out to be generally solvable for all affine subspaces W if and only if Ω is a domain of holomorphy [cf. Chapter 7].

SUBTRACTION of NONANALYTIC PARTS. Various problems fall into the following category. One seeks to determine a function h in $\mathcal{O}(\Omega)$ which satisfies a certain side-condition (S), and it turns out that it is easy to construct a smooth function g on Ω [$g \in C^2(\Omega)$, say] that satisfies condition (S). One then tries to obtain h by subtracting from g its "nonanalytic part" u without spoiling (S): $h = g - u$. What conditions does the correction term u have to satisfy? Since h must be holomorphic, it must satisfy the Cauchy-Riemann condition $\overline{\partial} h = 0$. It follows that u must solve an inhomogeneous problem of the form

$$\overline{\partial} u = \overline{\partial} g \text{ on } \Omega, \quad u : (S_0). \tag{9}$$

[Indeed, h must satisfy condition (S) the same as g, hence $u = g - h$ must satisfy an appropriate zero condition (S_0).] Solutions of the global problem (9) do not always exist, but the differential equation has solutions satisfying appropriate growth conditions if Ω is (pseudoconvex or) a domain of holomorphy [Chapter 11]. The spherical shell theorem of Hartogs may be proved by the method of subtracting the nonanalytic part, cf. Chapter 3.

GENERAL $\overline{\partial}$ EQUATIONS. The general first order $\overline{\partial}$ equation or *inhomogeneous Cauchy-Riemann equation* on $\Omega \subset \mathbf{C}^n$ has the form

$$\overline{\partial} u = \sum\nolimits_1^n \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j = v = \sum\nolimits_1^n v_j d\overline{z}_j$$

or, written as a system,

$$\partial u / \partial \overline{z}_j = v_j, \quad j = 1, \ldots, n.$$

The equation is locally solvable whenever the local integrability or compatibility conditions

$$\partial v_k / \partial \overline{z}_j \; [= \partial^2 u / \partial \overline{z}_k \partial \overline{z}_j = \partial^2 u / \partial \overline{z}_j \partial \overline{z}_k] = \partial v_j / \partial \overline{z}_k$$

are satisfied, as they are in the case of (9) [cf. Chapter 7]. There are also higher order $\overline{\partial}$ equations where the unknown is a differential form, not a function. Assuming that the natural local integrability conditions are satisfied, *all* $\overline{\partial}$ equations are globally solvable on Ω if and only if Ω is a domain of holomorphy, cf. Chapters 11, 12.

COUSIN PROBLEMS: see below.

1.10 Preview: zero sets, singularity sets and the Cousin problems. For holomorphic functions in $\mathbf{C}$, the best known singularities are the isolated ones: poles and essential singularities. However, holomorphic functions in $\mathbf{C}^n$ with $n \geq 2$ can not have isolated singularities. More accurately, it follows from Hartogs' spherical shell theorem that such singularities are removable, cf. Sections 1.9, 2.6.

From here on, let Ω be a connected domain in $\mathbf{C}^n$. We suppose first that f is holomorphic on Ω and not identically zero. In the case $n = 1$ it is well-known that the ZERO SET $Z(f) = Z_f$ of f is a discrete set without limit point in Ω, cf. exercises 1.15, 1.16. However, for $n \geq 2$ a zero set Z_f can not have isolated points [$1/f$ can not have isolated singularities]. Z_f will be a so-called analytic set of complex codimension 1 (complex dimension $n-1$). Example: a complex hyperplane ($2a$). The local behaviour of zero sets will be studied in Chapter 4.

Certain thin SINGULARITY SETS are also analytic sets of codimension 1 [Section 4.8].

We now describe some related global existence questions, the famous Cousin problems of 1895 which have had a great influence on the development of complex analysis in $\mathbf{C}^n$.

FIRST COUSIN PROBLEM. *Are there meromorphic functions on* $\Omega \subset \mathbf{C}^n$ *with arbitrarily prescribed local infinitary behaviour (of appropriate type)*?

A meromorphic function f is defined as a function which can locally be represented as a quotient of holomorphic functions. The local data may thus be supplied in the following way. One is given a covering $\{U_\lambda\}$ of Ω by (connected) open subsets and for each set U_λ, an associated quotient $f_\lambda = g_\lambda/h_\lambda$ of holomorphic functions with $h_\lambda \not\equiv 0$. One wants to determine a meromorphic function f on Ω which on each set U_λ becomes infinite just like f_λ, that is, $f - f_\lambda \in \mathcal{O}(U_\lambda)$. Naturally, the data U_λ, f_λ must be compatible in the sense that $f_\lambda - f_\mu \in \mathcal{O}(U_\lambda \cap U_\mu)$ for all $\lambda,\ \mu$.

For $n = 1$ *Mittag-Leffler* had shown that such a problem is always solvable. For example, if Ω is the right half-plane $\{\mathrm{Re}\, z > 0\}$ in $\mathbf{C}$, a meromorphic function f with pole set $\{\lambda = 1, 2, \ldots\}$ and such that $f(z) - 1/(z-\lambda)$ is holomorphic on a neighbourhood of λ is provided by the sum of the series

$$\sum_{\lambda=1}^{\infty} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} \right).$$

For $n \geq 2$ it turned out that the first Cousin problem is *not* generally solvable for every domain Ω in $\mathbf{C}^n$. However, the problem *is* generally solvable on domains of holomorphy Ω (Oka 1937). The global solution is constructed by patching together local pieces. There is a close connection between the solvability of the first Cousin problem and the global solvability of a related $\overline{\partial}$ equation [Chapters 7, 11]. Oka's original method has developed into the important technique of *sheaf cohomology* (Cartan-Serre 1951-1953, see Chapter 12 and cf. [Grauert-Remmert]).

SECOND COUSIN PROBLEM. *Are there holomorphic functions* f *on* $\Omega \subset \mathbf{C}^n$ *with arbitrarily prescribed local vanishing behaviour* (*of appropriate type*)?

The data will consist of a covering $\{U_\lambda\}$ of Ω by (connected) open subsets and for each set U_λ, an associated holomorphic function $f_\lambda \not\equiv 0$. One wants to determine a holomorphic function f on Ω which on each set U_λ vanishes just like f_λ. Here one must require that on the intersections $U_\lambda \cap U_\mu$, the functions f_λ and f_μ vanish in the same way, that is, f_λ/f_μ must be equal to a zero free holomorphic function. The family $\{U_\lambda, f_\lambda\}$ and equivalent Cousin-II data determine a so-called *divisor* D on Ω. The desired function $f \in \mathcal{O}(\Omega)$ must have the local vanishing behaviour given by D. One says that f must have D as a divisor. In the given situation this means that on every set U_λ, the quotient f/f_λ must be holomorphic and zero free.

For $n = 1$ *Weierstrass* had shown that such a problem is always solvable. For example, if Ω is the right half-plane $\{\operatorname{Re} z > 0\}$ in $\mathbf{C}$, a holomorphic function f with zero zet $\{\lambda = 1, 2, \ldots\}$ and corresponding multiplicities 1 is provided by the infinite product

$$\prod_{\lambda=1}^{\infty} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda}.$$

For $n \geq 2$ the second Cousin problem or divisor problem is not generally solvable, not *even* if Ω is a domain of holomorphy. General solvability on such a domain requires an additional condition of topological nature (Oka 1939) which may also be formulated in cohomological language (Serre 1953), see Chapter 12. The divisor problem is important for algebraic geometry.

From the preceding, the reader should not get the impression that all problems in the Cousin I, II area have now been solved. Actually, after the solution of the classical Cousin problems, the situation for $\mathbf{C}^n$ is much like the situation was for one complex variable after the work of Mittag-Leffler and Weierstrass. In the case of $\mathbf{C}$, one then turned to much more difficult problems such as the determination of holomorphic functions of prescribed growth with prescribed zero set, cf. [Boas]. The corresponding problems for $\mathbf{C}^n$ are largely open, although a start has been made, cf. [Ronkin] and [Lelong-Gruman].

Exercises

1.1. Use the definition of holomorphy (1.11) to prove that a holomorphic function on $\Omega \subset \mathbf{C}^n$ has a complex (partial) derivative with respect to each variable z_j throughout Ω.

1.2. Prove that $\mathcal{O}(\Omega)$ is a ring relative to ordinary addition and multiplication of functions. Which elements have a multiplicative inverse in $\mathcal{O}(\Omega)$? Cf. (1g) for the notation.

1.3. (i) Prove that there is exactly one complex line through any two distinct points a and b in $\mathbf{C}^n$.
(ii) Determine a parametric representation for the complex hyperplane $c \cdot (z - a) = 0$ in $\mathbf{C}^n$.

1.4. The real hyperplane V through $a = a' + ia''$ in $\mathbf{C}^n \sim \mathbf{R}^{2n}$ with normal direction $(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ is given by the equation

$$\alpha_1(x_1 - a_1') + \beta_1(y_1 - a_1'') + \ldots + \alpha_n(x_n - a_n') + \beta_n(y_n - a_n'') = 0.$$

Show that V can also be represented in the form

$$\operatorname{Re}\{c \cdot (z - a)\} = 0.$$

Verify that a real hyperplane through a in $\mathbf{C}^n$ contains precisely one complex hyperplane through a.

1.5. Prove that the composition of differentiable maps $\zeta = f(w) : D \subset \mathbf{C}^p \sim \mathbf{R}^{2p}$ to $\mathbf{C}$ and $w = g(z) : \Omega \subset \mathbf{C}^n \sim \mathbf{R}^{2n}$ to D is differentiable, and that

$$\frac{\partial(f \circ g)}{\partial \overline{z}_j} = \sum_{k=1}^{p} \left\{ \frac{\partial f}{\partial w_k}(g) \frac{\partial g_k}{\partial \overline{z}_j} + \frac{\partial f}{\partial \overline{w}_k}(g) \frac{\partial \overline{g}_k}{\partial \overline{z}_j} \right\}, \quad j = 1, \ldots, n.$$

Deduce that for holomorphic f and g (that is, f and $g_1, \ldots, g_p$ holomorphic), the composite function $f \circ g$ is also holomorphic.

1.6. Let f be holomorphic on $\Omega \subset \mathbf{C}^n$ and let V be a complex hyperplane intersecting Ω. Prove that the restriction of f to the intersection $\Omega \cap V$ may be considered as a holomorphic function on an open set in $\mathbf{C}^{n-1}$.

1.7. Analyze the boundary of the polydisc $\Delta_3(0, r)$. Then use the maximum principle for the case of one complex variable to prove that all holomorphic functions f on $\overline{\Delta}_3(0, r)$ assume their maximum absolute value on $T_3(0, r)$.

1.8. Let b be an arbitrary point of the torus $T(0, r) \subset \mathbf{C}^n$. Determine a holomorphic function f on the closed polydisc $\overline{\Delta}(0, r)$ for which $|f|$ assumes its maximum only at b. [First take $n = 1$, then $n \geq 2$.]

1.9. Let b be an arbitrary point of the sphere $S(0, r) \subset \mathbf{C}^n$. Prove that for $f(z) = \overline{b} \cdot z$, one has $|f(z)| \leq r^2$ on $\overline{B}(0, r)$ with equality if and only if $z = e^{i\theta} b$ for some $\theta \in \mathbf{R}$. Deduce that for $f(z) = \overline{b} \cdot z + 1$, one has $|f(z)| < |f(b)|$ throughout $\overline{B}(0, r) - \{b\}$.

1.10. Let f be holomorphic on $\overline{\Delta}(0,r)$. Apply Cauchy's integral formula to $g = f^p$ and let $p \to \infty$ in order to verify that

$$|f(z)| \leq \sup_{T(0,r)} |f(\zeta)|, \quad \forall z \in \Delta(0,r).$$

1.11. Extend the Cauchy integral formula for polydiscs to polydomains $D = D_1 \times \ldots \times D_n$, where $D_j \subset \mathbf{C}$ is the interior of a piecewise smooth simple closed curve Γ_j, $j = 1, \ldots, n$.

1.12. Represent the following functions by double power series with center $0 \in \mathbf{C}^2$ and determine the respective domains of convergence (without grouping the terms of the power series):

$$\frac{1}{(1-z_1)(1-z_2)}, \quad \frac{1}{1-z_1 z_2}, \quad \frac{1}{1-z_1-z_2}, \quad \frac{e^{z_1}}{1-z_2}.$$

1.13. Suppose that the power series $\sum c_\alpha (z-a)^\alpha$ converges throughout the open set $U \subset \mathbf{C}^n$. Prove that
   (i) the series is absolutely convergent on U;
   (ii) the convergence is locally uniform on U for any given order of the terms;
   (iii) the sum function is holomorphic on U.

1.14. Let f be analytic on a connected domain $\Omega \subset \mathbf{C}^n$ and such that $D^\alpha f(a) = 0$ for a certain point $a \in \Omega$ and all $\alpha \in \mathbf{N}_0$. Prove that $f \equiv 0$.

1.15. Let f be analytic on a connected domain $D \subset \mathbf{C}$ and $f \not\equiv 0$. Verify that for every point $a \in D$ there is an integer $m \geq 0$ such that $f(z) = (z-a)^m g(z)$, with g analytic on D and zero free on a neighbourhood of a. Show that in $\mathbf{C}^2$, there is no corresponding general factorization $f(z) = (z_1 - a_1)^{m_1}(z_2 - a_2)^{m_2} g(z)$, with g zero free around a.

1.16. Let D be a connected domain in $\mathbf{C}$ and $\{z_k\}$ a sequence of distinct points in D with limit $a \in D$. Verify that an analytic function f on D which vanishes at the points z_k must be identically zero. Devise possible extensions of this result to $\mathbf{C}^2$.

1.17. For the unit bidisc $\Delta(0,1) = \Delta_1(0,1) \times \Delta_1(0,1)$ in $\mathbf{C}^2$, a small planar domain around 0 may be a set of uniqueness, depending on what plane it lies in. Taking $0 < r < \frac{1}{2}$, show that the square

$$E_1 = \{x + iy \in \Delta : \ |x_1| < r, \ |x_2| < r, \ y_1 = y_2 = 0\}$$

is a set of uniqueness for the analytic functions f on Δ, whereas the square

$$E_2 = \{x + iy \in \Delta : \ |x_1| < r, \ |y_1| < r, \ x_2 = y_2 = 0\}$$

is not. [One may use a power series, or one may begin by considering $f(z_1, x_2)$ with fixed $x_2 \in (-r, r)$.]

1.18. Does the Cauchy transform ($6a$) define an analytic function on the exterior of the closed polydisc $\overline{\Delta}(a,r)$? Compare the cases $n=1$ and $n=2$.

1.19. Let $f(x_1+iy_1,\ldots,x_n+iy_n)$ be of class C^1 on $\Omega\subset\mathbf{C}^n\sim\mathbf{R}^{2n}$ as a function of $x_1,y_1,\ldots,x_n,y_n$ and such that $\overline{\partial}f\equiv 0$. Prove that $f(z)=f(z_1,\ldots,z_n)$ is analytic on Ω.

1.20. Let D be a connected domain in $\mathbf{C}^n$. Prove that the ring $\mathcal{O}(D)$ has no zero divisors: if $fg\equiv 0$ with $f,g\in\mathcal{O}(D)$ and $f(a)\neq 0$ at a point $a\in D$, then $g\equiv 0$.

1.21. (Extension of *Liouville's theorem*) Prove that a bounded holomorphic function on $\mathbf{C}^n$ must be constant.

1.22. Let f be holomorphic on a connected domain D of the form $\mathbf{C}^n-E$ where $n\geq 2$ and E is compact. Suppose that $f(z)$ remains bounded as $|z|\to\infty$. Prove that $f=$ constant (so that the "singularity set" E is removable). [Consider the restrictions of f to suitable complex lines.]

1.23. Let f be holomorphic on the closed polydisc $\overline{\Delta}(0,r)\subset\mathbf{C}^2$. Prove the following *mean value properties:*

$$f(0)=\frac{1}{m_2(T)}\int_{T(0,r)}f(\zeta)dm_2(\zeta)=\frac{1}{m_3(\partial\Delta)}\int_{\partial\Delta}f(\zeta)dm_3(\zeta).$$

Here dm_j denotes the appropriate area or volume element. [Since the circles $\zeta_1=r_1e^{it_1}$, $\zeta_2=$ constant and $\zeta_1=$ constant, $\zeta_2=r_2e^{it_2}$ on $T(0,r)$ intersect at right angles, the area element $dm_2(\zeta)$ is simply equal to the product of the elements of arc length, r_1dt_1 and r_2dt_2. Again by orthogonality, the volume element $dm_3(\zeta)$ of $C(0,r_1)\times\Delta_1(0,r_2)$ may be represented in the form $r_1dt_1\cdot\rho d\rho dt_2$, etc.]

1.24. Prove that holomorphic functions f on the closed unit ball $\overline{B}\subset\mathbf{C}^2$ have the following *mean value property:*

$$f(0)=\frac{1}{m_3(S)}\int_S f(\zeta)dm_3(\zeta),\quad S=\partial B.$$

[S is a union of tori $T(0,r)$ with $r_1=\rho,\ r_2=(1-\rho^2)^{\frac{1}{2}}$. The parametrization $\zeta_1=\rho e^{it_1}$, $\zeta_2=(1-\rho^2)^{\frac{1}{2}}e^{it_2}$ introduces orthogonal curvilinear coordinates on S and $dm_3(\zeta)=\rho dt_1(1-\rho^2)^{\frac{1}{2}}dt_2d\rho$.]
Used in conjunction with suitable holomorphic automorphisms of the ball, this mean value property gives a special integral representation for $f(z)$ on B in terms of the boundary values of f on S, cf. exercise 10.28.

1.25. Let $f(z_1,z_2)$ be continuous on the closed polydisc $\overline{\Delta}_2(a,r)$ and holomorphic on the interior. Take ζ_1 on $C(a_1,r_1)$. Now use Weierstrass's limit theorem to prove that $f(\zeta_1,w)$ is holomorphic on the disc $\Delta_1(a_2,r_2)$.

1.26. Prove the holomorphy of f in Corollary 1.72 by showing that $f(z)$ can be written as a Cauchy integral. [First write $K(z,t)$ as a Cauchy integral.]

1.27. Let $K(z, t)$ be defined and continuous on $\Omega \times I$ where $\Omega \subset \mathbf{C}^n$ is open and I is a compact rectangular block in $\mathbf{R}^m$. Suppose that $K(z, t)$ is holomorphic on Ω for each $t \in I$. Prove that $D_j K(z, t)$ is continuous on $\Omega \times I$ $(D_j = \partial/\partial z_j)$. Finally show that for $f(z) = \int_I K(z, t)dt$ one has $D_j f(z) = \int_I D_j K(z, t)dt$.

1.28. Prove that a locally bounded family $\mathcal{F}$ of functions in $\mathcal{O}(\Omega)$ is locally equicontinuous, that is, every point $a \in \Omega$ has a neighbourhood U with the following property. To any given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z') - f(z'')| < \epsilon$ for all z', $z'' \in U$ for which $|z' - z''| < \delta$ and for all $f \in \mathcal{F}$.

1.29. Give an example of a holomorphic map $f = (f_1, f_2)$ of $\mathbf{C}^2$ to $\mathbf{C}^2$, with nonconstant components f_1 and f_2, that fails to be open.

1.30. (Extension of *Schwarz's lemma*) Let f be holomorphic on the unit ball $B = B(0, 1)$ in $\mathbf{C}^n$ and in absolute value bounded by 1. Supposing that $f(0) = 0$, prove that $|f(z)| \le |z|$ on B. What can you say if f vanishes at 0 of order $\ge k$, that is, $D^\alpha f(0) = 0$ for all α's with $|\alpha| < k$? [One may work with complex lines.]

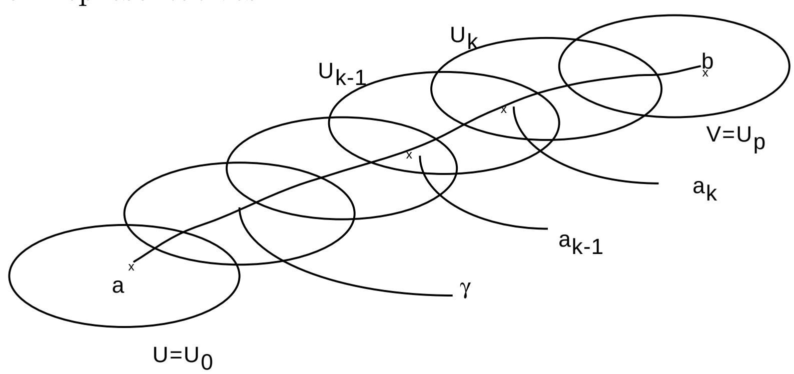
CHAPTER 2

Analytic continuation, part I

In the present chapter we discuss classical methods of analytic continuation – techniques based on power series, the Cauchy integral for a polydisc and Laurent series. More recent methods may be found in the next chapter.

After a general introduction on analytic continuation and a section on convexity, we make a thorough study of the domain of (absolute) convergence of a multiple power series with center 0. Such a domain is a special kind of connected multicircular domain: if $z = (z_1, \ldots, z_n)$ belongs to it, then so does every point $z' = (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$ with $\theta_j \in \mathbf{R}$. For $n = 1$ such connected domains are annuli or discs. Holomorphic functions on annuli are conveniently represented by Laurent series and the same is true for multicircular domains in $\mathbf{C}^n$.

2.1 General theory of analytic continuation. Consider tripels (a, U, f), where $a \in \mathbf{C}^n$, U is an open neighborhood of a and f is a function on U into some non specified, but fixed set X. Two tripels (a, U, f), (a', U', f') are called equivalent is $a = a'$ and $f = f'$ on a neighborhood U'' of a contained in $U \cap U'$. This is indeed an equivalence relation, as is easily seen. The equivalence class of (a, U, f) is called the *germ* of f at a. We will meet germs of continuous and of smooth functions, with values in R, C or worse, but the most prominent case will be that f is holomorphic. The tripel (a, U, f) is then called a *function element* (a, U, f) at a point $a \in \mathbf{C}^n$. Using theorem 1.54 one sees that elements (a, U, f) and $(a, \tilde{U}, \tilde{f})$ at the same point a are equivalent if f and $\tilde{f}$ have the same power series at $a : f_a = \tilde{f}_a$. Thus germs of holomorphic functions can be identified with convergent power series. If no confusion is possible we may occasionally identify germs of holomorphic functions with their representatives.



DEFINITION 2.11. A function element (b, V, f) is called a *direct analytic continuation* of the element (a, U, f) if $V \cap U$ is nonempty and $g = f$ on a component of $V \cap U$. [Some authors require that g be equal to f on every component of $V \cap U$.] More generally, an element (b, V, g) at b is called an *analytic continuation* of (a, U, f) if there is a finite *chain* of elements (a_k, U_k, f_k), $k = 0, 1, \ldots, p$ which links (a, U, f) to (b, V, g) by successive direct continuations:

$$(a_0, U_0, f_0) = (a, U, f), \;\; (a_p, U_p, f_p) = (b, V, g)$$

and

$$(a_k, U_k, f_k) \text{ is a direct analytic continuation of } (a_{k-1}, U_{k-1}, f_{k-1}) \text{ for } k = 1, \dots, p.$$

One loosely speaks of an analytic continuation of $f \in \mathcal{O}(U)$ to V. If $V \cap U$ is nonempty, the uniqueness theorem shows that (a, U, f) has at most one direct analytic continuation (b, V, g) for given $b \in V$ and a given component of $V \cap U$. [On a different component of $V \cap U$, g may be different from f.] In the case of a chain as above, one may insert additional elements to ensure that a_k belongs to $U_k \cap U_{k-1}$ for $k = 1, \dots, p$. Such a chain may be augmented further to obtain *analytic continuation along an arc* $\gamma : [0, 1] \to \mathbf{C}^n$ from a to b, namely, if γ is chosen as follows: $\gamma(0) = a$, $\gamma(1) = b$ and there is a partitioning $0 = t_0 < t_1 < \dots < t_p = 1$ such that $\gamma(t_k) = a_k$ and the subarc of γ corresponding to the interval $[t_{k-1}, t_k]$ belongs to U_{k-1}, $k = 1, \dots, p$. One can then define a continuous chain of elements (a^t, U^t, f^t), $0 \le t \le 1$ which links (a, U, f) to (b, V, g).

Given an element (a, U, f) at a and a point b, different chains starting with (a, U, f) may lead to *different* [more precisely, inequivalent] elements at b. For example, one may start with the function element

$$(1) \qquad (1, \{\operatorname{Re} z > 0\}, \;\; \text{p.v.} \log z)$$

at the point $z = 1$ of $\mathbf{C}$. Here the *p*rincipal *v*alue of

$$\log z = \log |z| + i \arg z, \quad z \neq 0$$

denotes the value with imaginary part $> -\pi$ but $\le +\pi$. Hence in our initial element, $\log z$ has imaginary part between $-\pi/2$ and $\pi/2$. One may continue this element analytically to the point $z = -1$ along the *upper half* of the unit circle. At any point e^{it}, $0 \le t \le \pi$ one may use the half-plane $\{t - \pi/2 < \arg z < t + \pi/2\}$ as basic domain and on it, one will by continuity obtain the holomorphic branch of $\log z$ with imaginary part between $t - \pi/2$ and $t + \pi/2$. On the half-plane $\{\operatorname{Re} z < 0\}$ as basic domain around $z = -1$, our analytic continuation will thus give the branch of $\log z$ with imaginary part between $\pi/2$ and $3\pi/2$. However, one may continue the original element (1) also along the *lower half* of the unit circle. The intermediate elements will be similar to those above, but this time $0 \ge t \ge -\pi$. Thus the new analytic continuation will give the branch of $\log z$ on the half-plane $\{\operatorname{Re} z < 0\}$ with imaginary part between $-\pi/2$ and $-3\pi/2$.

Definition 2.12 (WEIERSTRASS). *The totality of all equivalence classes of function elements (b, V, g) (or of all convergent power series g_b) at points $b \in \mathbf{C}^n$, which may be obtained from a given element (a, U, f) by unlimited analytic continuation, is called the* COMPLETE ANALYTIC FUNCTION *$\mathcal{F}$ generated by (a, U, f).*

RIEMANN DOMAIN for $\mathcal{F}$. As the example of $\log z$ shows, a complete analytic function $\mathcal{F}$ may be *multivalued* over $\mathbf{C}^n$. In order to get a better understanding of such a function, one introduces a multilayered Riemann domain $\mathcal{R}$ for $\mathcal{F}$ over $\mathbf{C}^n$ (a multisheeted Riemann surface when $n = 1$) on which $\mathcal{F}$ may be interpreted as a single-valued function. Most readers will have encountered concrete Riemann surfaces for $\log z$ and $\sqrt{z}$. We briefly describe the general case.

The *points* of the Riemann domain $\mathcal{R}$ for $\mathcal{F}$ in Definition 2.12 have the form $p = [(b, V, g)]$ or $p = (b, g_b)$ where $[(b, V, g)]$ stands for an equivalence class of elements at b. One says that the point p lies "above" b and the map $\pi : p = (b, g_b) \to b$ is called the *projection* of $\mathcal{R}$ to $\mathbf{C}^n$. The points $[(c, W, h)]$ or (c, h_c), corresponding to direct analytic continuations (c, W, h) of (b, V, g) for which c lies in V and $h_c = g_c$, will define a *basic neighbourhood* $\mathcal{N} = \mathcal{N}(p, V, g)$ of p in $\mathcal{R}$. Small basic neighbourhoods will separate the points of $\mathcal{R}$. The restriction $\pi \mid \mathcal{N}$ establishes a *homeomorphism* of $\mathcal{N}$ in $\mathcal{R}$ onto V in $\mathbf{C}^n$. Over each point b of $\mathbf{C}^n$, the Riemann domain $\mathcal{R}$ for $\mathcal{F}$ will have as many layers as there are different equivalence classes $[(b, V, g)]$ in $\mathcal{F}$ at b. If the element (b, V, g) is obtained by analytic continuation of (a, U, f) along an arc γ in $\mathbf{C}^n$, the Riemann domain will contain an arc σ above γ which connects the points of $\mathcal{R}$ corresponding to the two elements, cf. [Conway].

On the Riemann domain, the complete analytic function $\mathcal{F}$ is made into a *single-valued function* through the simple definition $\mathcal{F}(p) = \mathcal{F}((b, g_b)) = g(b)$. We now let $q = (z, h_z)$ run over the neighbourhood $\mathcal{N}(p, V, g)$ in $\mathcal{R}$. The result is

$$\mathcal{F}(q) = \mathcal{F}((z, h_z)) = h(z) = g(z), \quad \forall q = (z, h_z) \in \mathcal{N}(p, V, g).$$

Thus on the Riemann domain, $\mathcal{F}$ is locally given by an ordinary holomorphic function g on a domain $V \subset \mathbf{C}^n$ "under" $\mathcal{R}$. Taking this state of affairs as a natural definition of holomorphy on $\mathcal{R}$, the function $\mathcal{F}$ will be *holomorphic*. Setting $(a, U, f) = p_0$ and identifying $\mathcal{N}(p_0, U, f)$ with U, one will have $\mathcal{F} = f$ on U. In that way the Riemann domain $\mathcal{R}$ will provide a maximal continuation or existence domain for the function $f \in \mathcal{O}(U)$: every germ of every analytic continuation is represented by a point of $\mathcal{R}$. Cf. Section 5.6.

There are also more geometric theories of Riemann domains, not directly tied to functions $\mathcal{F}$. Riemann domains are examples of so-called *domains* $X = (X, \pi)$ *over* $\mathbf{C}^n$. The latter are Hausdorff spaces X with an associated projection π to $\mathbf{C}^n$. Every point of X must have a neighbourhood on which π establishes a homeomorphism onto a domain in $\mathbf{C}^n$. The $\mathbf{C}^n$ coordinates z_j can serve as local coordinates on X; different points of X over the same point $z \in \mathbf{C}^n$ may be distinguished by means of an additional coordinate. Cf. Section 5.6 and [Narasimhan].

Given a function element (a, U, f) and a boundary point b of U, there may or may not exist a direct analytic continuation (b, V, g) at b. In the case $n = 1$ there always exist functions $f \in \mathcal{O}(U)$ that can not be continued analytically across any boundary point of U. This is easily seen: using Weierstrass theorem mentioned at the end of 1.10 one constructs a holomorphic function f on U such that the boundary of U is in the closure of the zeroes of f, cf. Chapter 6. However, as mentioned already in Section 1.9, the situation is completely different in $\mathbf{C}^n$ with $n \geq 2$. There are connected domains $D \subset \mathbf{C}^n$ such that every function $f \in \mathcal{O}(D)$ can be continued analytically to a certain larger connected domain $D' \subset \mathbf{C}^n$ (*independent of* f). In many cases one can find a maximal continuation domain D^* in $\mathbf{C}^n$:

DEFINITION 2.13. A (connected) domain D^* in $\mathbf{C}^n$ is called a [or the] *envelope* or *hull of holomorphy* for $D \subset \mathbf{C}^n$ if

(i) $D \subset D^*$ and every $f \in \mathcal{O}(D)$ has an extension f^* in $\mathcal{O}(D^*)$;

(ii) For every boundary point b of D^*, there is a function $f \in \mathcal{O}(D)$ which has no analytic continuation to a neighbourhood of b. [The corresponding complete analytic function $\mathcal{F}$ has no element at b.]

It is perhaps surprising that there exist connected domains $D \subset \mathbf{C}^n$ which have no envelope of holomorphy in $\mathbf{C}^n$. However, for such a domain D, all functions in $\mathcal{O}(D)$ have an analytic continuation to a certain domain X_D over $\mathbf{C}^n$, see Section 2.9.

A maximal continuation domain D^* as in Definition 2.13 (which may coincide with D) will be a domain of holomorphy, cf. Chapter 6 where the latter domains are studied and characterized by special convexity properties. It will be useful to start here with a discussion of ordinary convexity.

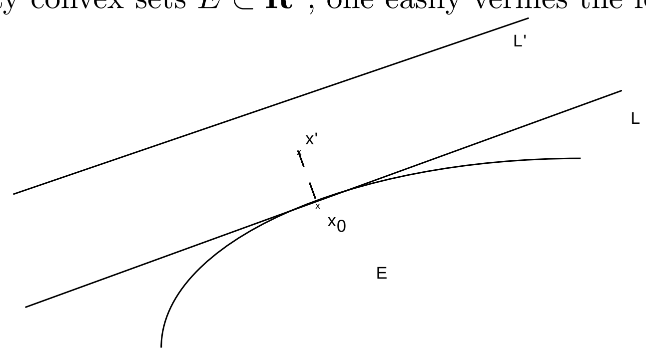
2.2 Auxiliary results on convexity. When we speak of convex sets we always think of them as lying in a real Euclidean space $\mathbf{R}^n$. Convex sets in $\mathbf{C}^n$ will be convex sets in the corresponding space $\mathbf{R}^{2n}$.

DEFINITION 2.21. A set $E \subset \mathbf{R}^n$ is called *convex* if for any pair of points x and y in E, the whole straight line segment with end points x and y belongs to E. In other words, $x \in E$, $y \in E$ must imply

$$(1-\lambda)x + \lambda y \in E, \quad \forall \lambda \in [0,1].$$

Every convex set is connected. The closure $\overline{E}$ and the interior E^0 of a convex set E are also convex. The intersection of any family of convex sets in $\mathbf{R}^n$ is convex.

For nonempty convex sets $E \subset \mathbf{R}^2$, one easily verifies the following properties:



(i) If there is a straight line $L' \subset \mathbf{R}^2$ which does not meet E, there is a *supporting line* L parallel to L', that is, a line L through a boundary point x_0 of E such that the interior E^0 lies entirely on one side of L.

(ii) If x' lies outside $\overline{E}$, there is a supporting line L separating x' from E^0 and passing through a point $x_0 \subset \overline{E}$ closest to x'. [Take L through x_0 perpendicular to $[x_0, x']$.]

(iii) If E is closed (or open), it is the intersection of the closed (or open, respectively) half-planes H containing E.

(iv) For every boundary point x_0 of E there are one or more supporting lines L passing through x_0. [The vectors $x - x_0$ for $x \in E$ belong to an angle $\leq \pi$.]

There are corresponding results for convex sets $E \subset \mathbf{R}^n$, $n \geq 3$. The supporting lines L then become *supporting hyperplanes* V, that is, affine subspaces of real dimension $n-1$. For a closed convex set $E \subset \mathbf{R}^n$, the intersection of E with a supporting hyperplane V is a closed convex set of lower dimension. More precisely, $E \cap V$ will be a closed convex set, congruent to a closed convex set in $\mathbf{R}^{n-1}$.

DEFINITION 2.22. For an arbitrary (nonempty) set S in $\mathbf{R}^n$, the smallest convex set containing S is called its *convex hull*, notation $E = \mathrm{CH}(S)$.

It is easy to verify that the convex hull $\mathrm{CH}(S)$ consists of all finite sums of the form

$$x = \sum_{j=1}^{m} \lambda_j s_j \text{ with } s_j \in S,\, \lambda_j \geq 0,\, \sum \lambda_j = 1. \tag{2}$$

Indeed, induction on m and the definition of convexity will show that $\mathrm{CH}(S)$ must contain all points of the form (2). On the other hand, the set of all those points is convex and contains S, hence it contains $\mathrm{CH}(S)$.

In the case of a compact set S in the plane, one readily shows that m can always be taken ≤ 3. [If x belongs to $\mathrm{CH}(S)$ but not to S, one may choose an arbitrary point $s_1 \in S$ and join it to x; the half-line from s_1 through x must meet the boundary of $\mathrm{CH}(S)$ at or beyond x.] For any set S in $\mathbf{R}^n$, every point x in $\mathrm{CH}(S)$ has a representation (2) with $m \leq n+1$ (Carathéodory's theorem, cf. [Cheney]). For our application to power series we need the notion of logarithmic convexity. Let $\mathbf{R}^n_+$ denote the set of points $x \in \mathbf{R}^n$ with $x_j \geq 0,\ \forall j$. We would like to say that $F \subset \mathbf{R}^n_+$ is logarithmically convex if the set

$$\log F \stackrel{\text{def}}{=} \{(\log r_1, \ldots, \log r_n) : (r_1, \ldots, r_n) \in F\}$$

is convex. However, in order to avoid difficulties when $r_j = 0$ for some j so that $\log r_j = -\infty$ [cf. exercise 2.7], we will use the following

DEFINITION 2.23. A set F in $\mathbf{R}^n_+$ is called *logarithmically convex* if $r' \in F$ and $r'' \in F$ always implies that F contains every point r of the symbolic form

$$r = (r')^{1-\lambda}(r'')^{\lambda},\ \ 0 \leq \lambda \leq 1,$$

that is,

$$r_j = (r'_j)^{1-\lambda}(r''_j)^{\lambda},\ \ \forall j.$$

The *logarithmically convex hull* of a set $S \subset \mathbf{R}^n_+$ is the smallest logarithmically convex set containing S.

EXAMPLE 2.24. Let S be the union of the rectangles

$$S_1 = \{(r_1, r_2) \in \mathbf{R}^n_+ : r_1 < 2,\, r_2 < \tfrac{1}{2}\},\ \ S_2 = \{(r_1, r_2) \in \mathbf{R}^n_+ : r_1 < \tfrac{1}{2},\, r_2 < 2\}.$$

Then $\log S$ is the union of the quadrants

$$\log S_1 = \{(\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 < \log 2, \;\; \rho_2 < \log \tfrac{1}{2}\},$$

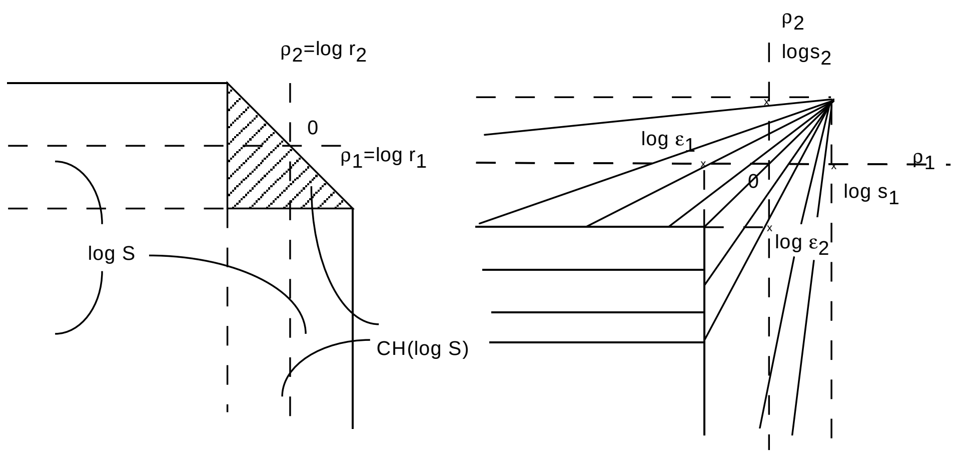
$$\log S_2 = \{(\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 < \log \tfrac{1}{2}, \;\; \rho_2 < \log 2\},$$

including some points with a coordinate $-\infty$. The convex hull of $\log S$ consists of the points (ρ_1, ρ_2) such that

$$\rho_1 < \log 2, \;\; \rho_2 < \log 2, \;\; \rho_1 + \rho_2 < 0$$

(fig 2.3). The logarithmically convex hull of S consists of the points $(r_1, r_2) = (e^{\rho_1}, e^{\rho_2})$ with $(\rho_1, \rho_2) \in \mathrm{CH}(\log S)$, or more precisely, of the points $(r_1, r_2) \geq 0$ such that (cf. fig 1.6):

$$r_1 < 2, \;\; r_2 < 2, \;\; r_1 r_2 = e^{\rho_1 + \rho_2} < 1.$$



EXAMPLE 2.25. Let S consist of a single point $s = (s_1, \dots, s_n) > 0$ and a neighbourhood of 0 in $\mathbf{R}^n_+$ given by $0 \leq r_j < \epsilon_j \,(< s_j), \;\; j = 1, \dots, n$. Then the logarithmically convex hull of S contains the set given by $0 \leq r_j < s_j$, $j = 1, \dots, n$, cf. fig 2.4.

2.3 Multiple power series and multicircular domains. In the following we will study sets of convergence of power series and of more general Laurent series

$$(3a) \qquad \sum_{\alpha \in \mathbf{Z}^n} c_\alpha z^\alpha = \sum_{\alpha_1 \in \mathbf{Z}, \dots, \alpha_n \in \mathbf{Z}} c_{\alpha_1 \dots \alpha_n} z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

In order to avoid problems with the order of the terms, we only consider absolute convergence here.

DEFINITION 2.31. Let A be the set of those points $z \in \mathbf{C}^n$ where the Laurent series $(3a)$ [or power series $(3b)$] is absolutely convergent. The interior A^0 of A will be called the *domain of* (absolute) *convergence* of the series.

In the case $n = 1$ the domain of convergence is an open annulus or disc (or empty). For general n, our first observation is that the absolute convergence of a Laurent series ($3a$) at a point z implies its absolute convergence at every point z' with $|z'_j| = |z_j|,\ \forall j$. Indeed, one will have $|c_\alpha (z')^\alpha| = |c_\alpha z^\alpha|,\ \forall \alpha$. It is convenient to give a name to the corresponding sets of points:

DEFINITION 2.32. $E \subset \mathbf{C}^n$ is called a *multicircular set* (or Reinhardt set) if

$$a = (a_1, \ldots, a_n) \in E \text{ implies } a' = (e^{i\theta_1} a_1, \ldots, e^{i\theta_n} a_n) \in E$$

for all real $\theta_1, \ldots, \theta_n$. A *multicircular domain* is an open multicircular set.

Multicircular sets are conveniently represented by their "trace" in the space $\mathbf{R}^n_+$ "of absolute values", in which all coordinates are nonnegative. Cf. fig 1.6, where the multicircular domain $D = \Delta(0, 0; 2, \frac{1}{2}) \cup \Delta(0, 0; \frac{1}{2}, 2)$ in $\mathbf{C}^2$ is represented by its trace.

DEFINITION 2.33. The *trace* of a multicircular set $E \subset \mathbf{C}^n$ is given by

$$\operatorname{tr} E = \{(|a_1|, \ldots, |a_n|) \in \mathbf{R}^n_+ : (a_1, \ldots, a_n) \in E\}.$$

A multicircular set E is determined by its trace. If E is *connected*, then so is $\operatorname{tr} E$ (and conversely). If E is open, $\operatorname{tr} E$ is open in $\mathbf{R}^n_+$.

Our primary interest is in multiple power series

$$\sum_{\alpha \in \mathbf{N}^n_0} c_\alpha z^\alpha = \sum_{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0} c_{\alpha_1 \ldots \alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}. \tag{3b}$$

The absolute convergence of a power series ($3b$) at a point z implies its absolute convergence at every point z' with $|z'_j| \leq |z_j|,\ \forall j$. The corresponding sets are called complete multicircular sets:

DEFINITION 2.34. $E \subset \mathbf{C}^n$ is called a *complete multicircular set* (or complete Reinhardt set) if

$$(a_1, \ldots, a_n) \in E \text{ implies } (a'_1, \ldots, a'_n) \in E$$

whenever $|a'_j| \leq |a_j|,\ \forall j$.

Observe that a complete multicircular set E is *connected*: $a \in E$ is joined to the origin by the segment $z = \lambda a$, $0 \leq \lambda \leq 1$ in E. A *complete multicircular domain* (= open set) will be a union of (open) polydiscs centered at the origin, and conversely. Cf. D and $\hat{D}$ illustrated in fig 1.6.

Proposition 2.35. *The domain of* (*absolute*) *convergence* A^0 *of a multiple power series* ($3b$) *with center* 0 *is a complete multicircular domain* [but may be empty].

PROOF. Let A^0 be nonempty and choose any point a in A^0. Then A^0 contains a ball $B(a, \delta)$, and this ball will contain a point b such that $|b_j| > |a_j|,\ \forall j$. The absolute convergence of the series ($3b$) at $z = b$ implies its absolute convergence throughout the polydisc $\Delta(0, \ldots, 0; |b_1|, \ldots, |b_n|)$. This polydisc in A^0 contains all points a' with $|a'_j| \leq |a_j|,\ \forall j$. Thus A^0 is a complete multicircular set. A^0 will be a connected domain.

2.4 Convergence domains of power series and analytic continuation. Let B denote the set of those points $z \in \mathbf{C}^n$ at which the terms $c_\alpha z^\alpha,\ \alpha \in \mathbf{N}_0^n$ of the power series (3b) form a bounded sequence:

$$B = \{z \in \mathbf{C}^n : |c_\alpha z^\alpha| \le M = M(z) < +\infty,\ \ \forall \alpha \in \mathbf{N}_0^n\}. \tag{4}$$

The set B is clearly multicircular and it also has a certain convexity property:

Lemma 2.41. *The trace of B is logarithmically convex.*

PROOF. Let $r' \ge 0$ and $r'' \ge 0$ be any two points in $\operatorname{tr} B$. Then there is a constant M [for example, $M = \max\{M(r'), M(r'')\}$] such that

$$|c_\alpha|(r_1')^{\alpha_1} \dots (r_n')^{\alpha_n} \le M,\ \ |c_\alpha|(r_1'')^{\alpha_1} \dots (r_n'')^{\alpha_n} \le M, \quad \forall \alpha \in \mathbf{N}_0^n.$$

It follows that for any $r = (r_1, \dots, r_n)$ with components of the form $r_j = (r_j')^{1-\lambda}(r_j'')^\lambda$ [with $\lambda \in [0,1]$ independent of j] and for all α's,

$$|c_\alpha| r_1^{\alpha_1} \dots r_n^{\alpha_n} = \{|c_\alpha|(r_1')^{\alpha_1} \dots (r_n')^{\alpha_n}\}^{1-\lambda}\{|c_\alpha|(r_1'')^{\alpha_1} \dots (r_n'')^{\alpha_n}\}^\lambda \le M.$$

Thus $r \in B$ and hence $\operatorname{tr} B$ is logarithmically convex [Definition 2.23].

Abusing the language, a multicircular domain is called logarithmically convex when its trace is. We can now prove

Theorem 2.42. *The domain of* (*absolute*) *convergene A^0 of a multiple power series* (3b) *with center* 0 *is a logarithmically convex complete multicircular domain* [but may be empty]. *The power series will converge uniformly on every compact subset of A^0.*

PROOF. Consider a power series (3b) for which A^0 is nonempty. We know that A^0 is a complete multicircular domain [Proposition 2.35], hence it suffices to verify its logarithmic convexity. Clearly, $A \subset B$, cf. (4), hence $A^0 \subset B^0$. We will show that also $B^0 \subset A^0$. Choose $b \in B^0$. Then B must contain a point c with $|c_j| = r_j > |b_j|,\ \ \forall j$. The boundedness of the sequence of terms $\{c_\alpha z^\alpha\}$ at $z = c$ or $z = r$ implies the absolute convergence of the power series (3b) throughout the polydisc $\Delta(0, r)$ [Lemma 1.41]. Hence $b \in A^0$ so that $B^0 \subset A^0$; as a result, $A^0 = B^0$. Since $\operatorname{tr} B$ is logarithmically convex [Lemma 2.41], $\operatorname{tr} A^0 = \operatorname{tr} B^0$ will also be logarithmically convex [basically because the interior of a convex set is convex].

We know that A^0 is a union of polydiscs $\Delta(0, s)$. The convergence of our power series is uniform on every smaller polydisc $\Delta(0, \lambda s)$ with $\lambda \in (0, 1)$ [Lemma 1.41], hence it is uniform on every compact subset of A^0. Indeed, such a set may be covered by finitely many polydiscs $\Delta(0, s)$ in A^0 and hence by finitely many polydiscs $\Delta(0, \lambda s)$.

COROLLARY 2.43 (*Analytic continuation by power series*). Suppose f is holomorphic on a complete multicircular domain D in $\mathbf{C}^n$. Then f has an analytic continuation to the logarithmically convex hull $\hat{D}$ of D. The continuation is furnished by the sum of the power series for f with center 0.

Indeed, D is a union of polydiscs $\Delta(0, r)$. On each of those polydiscs, the power series $\sum c_\alpha z^\alpha$ for f with center 0 converges absolutely, and it converges to $f(z)$, cf. Corollary

1.63. The domain of convergence A^0 of the power series thus contains D. Being logarithmically convex, A^0 must contain $\hat{D}$, the smallest logarithmically convex multicircular domain containing D. The power series is uniformly convergent on every compact subset of $\hat{D} \subset A^0$. Its sum is therefore holomorphic on $\hat{D}$; it extends f analytically throughout $\hat{D}$.

Fig 1.6 illustrates the case

$$D = \Delta(0,0;2,\tfrac{1}{2}) \cup \Delta(0,0;\tfrac{1}{2},2)$$

in $\mathbf{C}^2$, cf. Example 2.24. Here the logarithmically convex hull $\hat{D}$ is the exact domain of convergence of the power series with center 0 for the function

$$f(z) = \frac{1}{2-z_1} + \frac{1}{2-z_2} + \frac{1}{1-z_1z_2}.$$

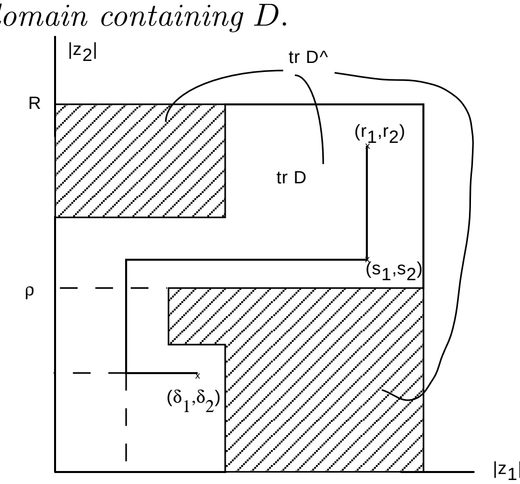
The logarithmically convex hull $\hat{D}$ of a complete multicircular domain D in $\mathbf{C}^n$ is at the same time its envelope of holomorphy [Definition 2.13]. Indeed, $\hat{D}$ will be a domain of holomorphy and (hence) also the maximal domain of existence for a certain holomorphic function [see Sections 6.3, 6.4.] The latter property implies that every logarithmically convex complete multicircular domain is the exact domain of convergence for some power series with center 0.

2.5 Analytic continuation by Cauchy integrals. We will show how the Cauchy integral or CAUCHY TRANSFORM

$$(5a) \qquad \hat{f}_r(z) \stackrel{\text{def}}{=} \frac{1}{(2\pi i)^n} \int_{T(0,r)} \frac{f(\zeta)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n, \quad \forall z \in \Delta(0,r),$$

can be used for analytic continuation. Here $T(0,r) = C(0,r_1) \times \ldots \times C(0,r_n)$, with positive orientation of the circles $C(0,r_j)$.

Theorem 2.51. *Let $D \subset \mathbf{C}^n$ be a connected multicircular domain containing the origin and let f be holomorphic on D. Then the Cauchy transforms $\hat{f}_r$, where $r > 0$ runs over the interior of* trace D *jointly furnish an analytic continuation of f to D', the smallest complete multicircular domain containing D.*



PROOF. We take $n = 2$ and choose $\delta = (\delta_1, \delta_2) > 0$ such that the closed polydisc $\overline{\Delta}(0, \delta)$ belongs to D. To each point $r = (r_1, r_2) > 0$ in $\operatorname{tr} D$ we associate the Cauchy transform $\hat{f}_r$ (5a).

(i) Since f is holomorphic on $\overline{\Delta}(0, \delta)$ we have

$$\hat{f}_\delta(z) = f(z), \quad \forall z \in \Delta(0, \delta),$$

cf. the Cauchy integral formula, Theorem 1.31.

(ii) We next show that for arbitrary $r > 0$ and $s > 0$ in $\operatorname{tr} D$:

$$\hat{f}_r(z) = \hat{f}_s(z) \ \text{ on some polydisc } \ \Delta(0, \delta'), \ \ \delta' = \delta'_{rs}. \tag{5b}$$

To this end we connect r to s in the interior of $\operatorname{tr} D$ by a polygonal line S, whose straight segments are parallel to the coordinate axes (fig 2.5). In order to prove (5b) it is sufficient to consider the special case where S is a segment S_1 parallel to one of the axes, for example

$$S_1 = \{(t_1, t_2) \in \mathbf{R}^2_+ : \ s_1 = t_1 = r_1, \ s_2 \le t_2 \le r_2\}.$$

For fixed ζ_1 with $|\zeta_1| = r_1 = s_1$ and fixed z_2 with $|z_2| < s_2$, the function

$$g(w) \stackrel{\text{def}}{=} \frac{f(\zeta_1, w)}{w - z_2}$$

will be holomorphic on some annulus $\{\rho < |w| < R\}$ in $\mathbf{C}$ such that $\rho < s_2 < r_2 < R$, cf. fig 2.5, 2.6. Hence by Cauchy's theorem for an annulus,

$$\int_{C(0,r_2)} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = \int_{C(0,r_2)} g(w)dw = \int_{C(0,s_2)} g(w)dw = \int_{C(0,s_2)} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2. \tag{5c}$$

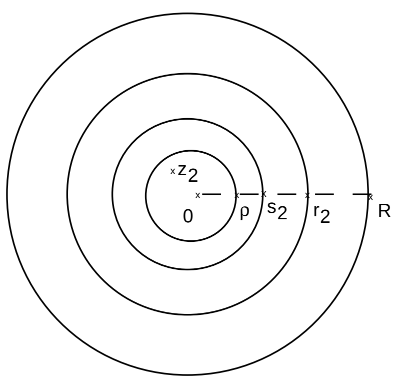


fig 2

We now multiply the first and the last member of (5c) by $1/(\zeta_1 - z_1)$, taking $|z_1| < s_1 = r_1$, and integrate the result with respect to ζ_1 along $C(0, r_1) = C(0, s_1)$. Replacing the repeated integrals by double integrals, we obtain (5b) for the end points r and s of S_1 [and, in our example, for all $z \in \Delta(0, s)$]. The general case (5b) follows by a finite number of steps of this kind.

Combining (i) and (ii) we conclude that for every $r > 0$ in $\operatorname{tr} D$

$$\hat{f}_r(z) = f(z) \text{ on some polydisc } \Delta(0, \delta''), \ \ \delta'' = \delta''_r. \tag{5d}$$

Now the Cauchy transform $\hat{f}_r$ is holomorphic on $\Delta(0, r)$ [Theorem 1.61]. It follows that $\hat{f}_r$ is equal to f throughout $\Delta(0, r) \cap D$ [uniqueness theorem 1.54]. Jointly, the Cauchy transforms $\hat{f}_r$ furnish an analytic continuation F of f to the domain

$$D' = \cup_r \Delta(0, r), \ \ \text{union over all } \ r > 0 \ \text{ in } \ \operatorname{tr} \mathrm{D}. \tag{5e}$$

D' is the smallest complete multicircular domain containing D, cf. fig 2.5.

COROLLARY 2.52 (*Once again, analytic continuation by power series*). Suppose f is holomorphic on a connected multicircular domain $D \subset \mathbf{C}^n$ that contains the origin. Then the power series for f with center 0 converges to f throughout D and it furnishes an analytic continuation of f to the logarithmically convex hull $\hat{D}$ of D.

Indeed, for any $r > 0$ in $\operatorname{tr} D$, the power series for $\hat{f}_r$ with center 0 converges to $\hat{f}_r$ on $\Delta(0, r)$ [Corollary 1.63], but this series is none other than the power series for f with center 0, cf. (5d). Hence the latter converges to the analytic continuation F of f throughout D' (5e), and in particular to f throughout D. By Corollary 2.43 applied to F and the complete multicircular domain D', the power series for F or f around 0 actually provides an analytic continuation of F or f to the logarithmically convex hull $(D')^\wedge$ of D'.

Observe finally that $(D')^\wedge = \hat{D}$. Indeed, $\hat{D}$ must contain D' [and hence $(D')^\wedge$], because $\hat{D}$ will contain every polydisc $\Delta(0, s)$ with $s > 0$ in $\operatorname{tr} D$, cf. Example 2.25.

2.6 Laurent series in one variable with variable coefficients; removability of isolated singularities. Let $A = A(0; \rho, R)$ denote the *annulus* $0 \le \rho < |w| < R \le +\infty$ in $\mathbf{C}$ (cf. fig 2.6) and let $g(w)$ be holomorphic on A. Then there is a unique *Laurent series* with center 0 that converges to g for some total ordering of its terms at each point of A. It is the series

$$\sum_{-\infty}^{\infty} c_k w^k \ \text{ with } \ c_k = \frac{1}{2\pi i} \int_{C(0,r)} g(w) w^{-k-1} dw, \tag{6a}$$

where one may integrate over any (positively oriented) circle $C(0, r)$ with $\rho < r < R$. The series actually converges absolutely, and uniformly on every compact subset of $A(0; \rho, R)$. To prove the existence of the series representation one uses the Cauchy integral formula for an annulus: for $\rho < r_1 < |w| < r_2 < R$,

$$g(w) = \frac{1}{2\pi i} \int_{C(0,r_2)} \frac{g(v)}{v - w} dv - \frac{1}{2\pi i} \int_{C(0,r_1)} \frac{g(v)}{v - w} dv. \tag{6b}$$

The first integral gives a power series $\sum_0^\infty c_k w^k$ on the disc $\Delta(0, r_2)$ [which will in fact converge throughout the disc $\Delta(0, R)$]. The second integral gives a power series in $1/w$, which may be written as $-\sum_{-\infty}^{-1} c_k w^k$ and which converges for $|w| > r_1$ [and in fact, for

$|w| > \rho]$. Combining the series one obtains $(6a)$. As to the other assertions above, cf. Section 2.7.

A holomorphic function $g(w)$ on $A(0;\rho,R)$ will have an *analytic continuation* to the disc $\Delta(0,R)$ if and only if all Laurent coefficients c_k with negative index are equal to 0. Indeed, if there is such a continuation [which we also call g], then by Cauchy's theorem, the second integral in $(6b)$ is identically zero for $|w| > r_1$.

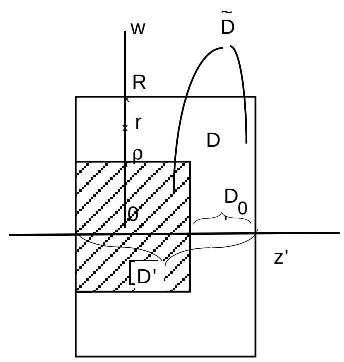
We now move on to $\mathbf{C}^n$ with $n \geq 2$. Treating our complex variables $z_1,\ldots,z_n$ asymmetrically for the time being, we will write z' for $(z_1,\ldots,z_{n-1})$ and w for z_n. Using Laurent series in w with coefficients depending on z', we will prove:

Theorem 2.61 (HARTOGS' CONTINUITY THEOREM). *Let $f(z',w) = f(z_1,\ldots,z_{n-1},w)$ be holomorphic on a domain $D \subset \mathbf{C}^n$ $(n \geq 2)$ of the form*

$$D = D' \times A(0;\rho,R) \cup D_0' \times \Delta(0,R),$$

where D' is a connected domain in $\mathbf{C}^{n-1}$ and D_0' a nonempty subdomain of D' (fig 2.7). *Then f has an analytic continuation to the domain*

$$\tilde{D} = D' \times \Delta(0,R).$$



PROOF. For fixed $z' \in D'$ the function $g(w) = f(z',w)$ is holomorphic in w on the annulus $A(0;\rho,R)$, hence it may be represented by a Laurent series in w,

$$(6c) \qquad f(z',w) = g(w) = \sum_{-\infty}^{\infty} c_k(z')w^k.$$

By $(6a)$ the coefficients are given by

$$(6d) \quad c_k(z') = \frac{1}{2\pi i}\int_{C(0,r)} f(z',w)w^{-k-1}dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z',re^{it})r^{-k}e^{-ikt}dt, \quad \rho < r < R.$$

We may now apply the holomorphy theorem for integrals [Corollary 1.72] to conclude that each of the coefficients $c_k(z')$ is *holomorphic* in z' on D'. Indeed, $f(z',w)$ is holomorphic and hence continuous in (z',w) on D. It follows that the final integrand

$$K(z',t) = f(z',re^{it})r^{-k}e^{-ikt}$$

is continuous on $D' \times [-\pi, \pi]$. Furthermore, for fixed t this integrand will be holomorphic in z' on D', because $f(z', w)$ is holomorphic in z' for fixed $w = re^{it}$ in $A(0; \rho, R)$.

We next observe that for fixed z' in D'_0, the function $g(w) = f(z', w)$ is holomorphic on the whole disc $\Delta(0, R)$. Hence for such z', the Laurent series (6*c*) must reduce to a power series. In other words, for every $k < 0$,

$$c_k(z') = 0 \ \text{ throughout } \ D'_0.$$

Thus by the uniqueness theorem for holomorphic functions, $c_k(z') = 0$ on all of D' for each $k < 0$. Conclusion:

$$f(z', w) = \sum_0^\infty c_k(z') w^k \ \text{ throughout } \ D' \times A. \tag{6e}$$

This power series with holomorphic coefficients actually defines a holomorphic function $\tilde{f}(z', w)$ throughout $\tilde{D} = D' \times \Delta(0, R)$. Indeed, we will show that the series is absolutely and uniformly convergent on every compact subset of $\tilde{D}$; Weierstrass's theorem 1.71 on the holomorphy of uniform limits will do the rest. Let E' be any compact subset of D' and set $E = E' \times \overline{\Delta}(0, s)$ where $s < R$. Choosing $r \in (\rho, R)$ such that $r > s$, the coefficient formula (6*d*) furnishes a uniform estimate

$$|c_k(z')| \leq M r^{-k} \ \text{ for } \ z' \in E', \ \text{ with } \ M = \sup|f| \ \text{ on } \ E' \times C(0, r) \subset D.$$

This estimate implies the uniform convergence of the series in (6*e*) on E, where $|w| \leq s < r$.

The holomorphic sum function $\tilde{f}$ on $\tilde{D}$ is equal to f on $D' \times A$ and hence on D. Thus it provides the desired analytic continuation of f to $\tilde{D}$.

APPLICATION 2.62 (*Removability of isolated singularities when* $n \geq 2$). Let f be holomorphic on a "punctured polydisc" $D = \Delta_n(a, r) - \{a\}$. Then f has an analytic extension to $\tilde{D} = \Delta_n(a, r)$.

[By translation, it may be assumed that $a = 0$. Now apply Theorem 2.61, taking $D' = \Delta_{n-1}(0, r')$ with $r' = (r_1, \ldots, r_{n-1})$, $\rho = 0$, $R = r_n$ and $D'_0 = D' - \{0\}$. An alternative proof may be based on the one-dimensional Cauchy integral formula, cf. exercise 2.14.]

APPLICATION 2.63. Holomorphic functions on open sets $\Omega \subset \mathbf{C}^n$, $n \geq 2$ can *not* have *isolated zeros.*

[An isolated zero of f would be a nonremovable isolated singularity for $1/f$.]

2.7 Multiple Laurent series on general multicircular domains. For the time being, we assume that our connected multicircular domain D in $\mathbf{C}^n$ does not contain any point z with a vanishing coordinate; the exceptional case will be considered in Section 2.8. When $n = 1$, D is an annulus $A(0; \rho, R)$ on which holomorphic functions are uniquely representable by Laurent series with center 0. The analog for general n is a Laurent series in n variables:

THEOREM 2.71. Let f be holomorphic on a connected multicircular domain $D \subset \mathbf{C}^n$, $n \geq 1$ that does not meet any hyperplane $\{z_j = 0\}$. Then there is a unique n-variable

Laurent series with center 0 (and constant coefficients) which converges to f at every point of D for some total ordering of its terms. It is the series

$$(7a)\qquad \sum_{\alpha_1\in\mathbf{Z},\ldots,\alpha_n\in\mathbf{Z}} c_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1}\ldots z_n^{\alpha_n}$$

whose coefficients are given by the formula

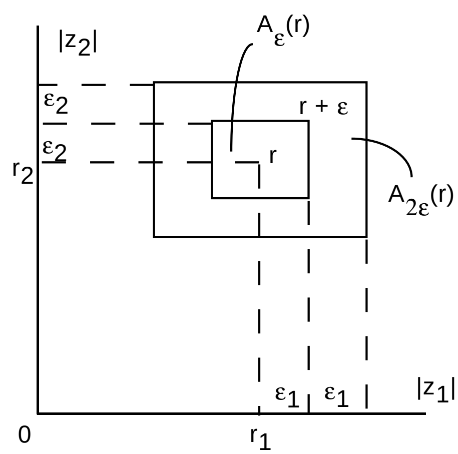
$$(7b)\qquad c_{\alpha_1\ldots\alpha_n} = \frac{1}{(2\pi i)^n}\int_{T(0,r)} f(z) z_1^{-\alpha_1-1}\ldots z_n^{-\alpha_n-1} dz_1\ldots dz_n$$

for any $r=(r_1,\ldots,r_n)>0$ in the trace of D. The series will actually be absolutely convergent on D and it will converge uniformly to f on any compact subset of D.

PROOF. We treat the typical case $n=2$. For $r=(r_1,r_2)>0$ and $0<\delta=(\delta_1,\delta_2)<r$ we introduce the "annular domains"

$$A_\delta(r)=\{(z_1,z_2)\in\mathbf{C}^2 : r_j-\delta_j<|z_j|<r_j+\delta_j,\ \ j=1,2\}.$$

(i) *Uniqueness of the Laurent series and coefficient formula.* For given $r>0$ in $\operatorname{tr} D$ we choose $\epsilon<\frac{1}{2}r$ so small that $\overline{A}_{2\epsilon}(r)$ belongs to D. Suppose now that we have a series $(7a)$ which converges pointwise to some function $f(z)$ on $\overline{A}_{2\epsilon}(r)$, either for some total ordering of the terms or when the series is written as a repeated series. In the former case we know and in the latter case we explicitly postulate that the terms form a *bounded sequence* at each point of $\overline{A}_{2\epsilon}(r)$.



From the boundedness of the sequence $\{c_\alpha z^\alpha\}$ at the point $z=r+2\epsilon$, it follows that the power series

$$\sum_{\alpha_1\geq 0,\alpha_2\geq 0} c_\alpha z_1^{\alpha_1} z_2^{\alpha_2}$$

is absolutely and uniformly convergent on the polydisc $\Delta(0,r+\epsilon)$ and in particular on $A_\epsilon(r)$, cf. Lemma 1.41. We next use the boundedness of the sequence $\{c_\alpha z^\alpha\}$ at the point $z = (r_1 - 2\epsilon_1, r_2 + 2\epsilon_2)$ or $1/z_1 = 1/(r_1 - 2\epsilon_1)$, $z_2 = r_2 + 2\epsilon_2$. It implies that the power series

$$\sum_{-\alpha_1>0,\alpha_2\geq 0} c_\alpha \left(\frac{1}{z_1}\right)^{-\alpha_1} z_2^{\alpha_2}$$

in $1/z_1$ and z_2 is absolutely and uniformly convergent for $|1/z_1| < 1/(r_1-\epsilon_1)$ or $|z_1| > r_1-\epsilon_1$ and $|z_2| < r_2+\epsilon_2$, hence in particular on $A_\epsilon(r)$. Also using the boundedness of the sequence $\{c_\alpha z^\alpha\}$ at $z = r - 2\epsilon$ and at $z = (r_1 + 2\epsilon_1,\, r_2 - 2\epsilon_2)$, we conclude that the whole series (7a) is *absolutely and uniformly convergent on* $A_\epsilon(r)$. The sum will be equal to $f(z)$ for any arrangement of the terms.

Termwise integration of the absolutely and uniformly convergent series

$$f(z)z^{-\beta-1} = \sum_{\alpha\in\mathbf{Z}^2} c_\alpha z^{\alpha-\beta-1} \quad [\beta + 1 = (\beta_1 + 1,\, \beta_2 + 1)]$$

over the torus $T(0, r)$ in $A_\epsilon(r)$ gives

$$\text{(7}c\text{)} \qquad \frac{1}{(2\pi i)^2}\int_{T(0,r)} f(z)z^{-\beta-1}dz = \sum_\alpha c_\alpha \frac{1}{(2\pi i)^2}\int_{T(0,r)} z^{\alpha-\beta-1}dz,$$

where dz stands for dz_1dz_2. Since

$$\frac{1}{2\pi i}\int_{C(0,r_j)} z_j^{\alpha_j-\beta_j-1}dz_j = \begin{cases} 1 \text{ for } \alpha_j = \beta_j \\ 0 \text{ for } \alpha_j \neq \beta_j, \end{cases}$$

the sum in (7c) reduces to c_β. We have thus proved formula (7b) at least for $n = 2$ and with β instead of α.

If f is represented by a series (7a) at each point of D [in the sense indicated at the beginning of (i)], the coefficients are given by (7b) for each $r > 0$ in $\operatorname{tr} D$, hence such a representation is surely *unique*. We will then have *absolute and uniform convergence* of the Laurent series on any compact subset $E \subset D$, since such an E can be covered by finitely many annular domains $A_\epsilon(r)$ for which $\overline{A}_{2\epsilon}(r)$ belongs to D.

(ii) *Existence of the Laurent series.* Let f be holomorphic on D. For $r > 0$ in $\operatorname{tr} D$, so that $T(0, r) \subset D$, the right-hand side of (7b) defines coefficients $c_\alpha(r)$ which might depend on r. Do they really? No, using the method of polygonal lines as in part (ii) of the proof of Theorem 2.51, one readily shows that $c_\alpha(r)$ is *independent* of r. Indeed, referring to fig 2.5,

$$\int_{C(0,r_2)} f(z_1, z_2)z_2^{-\alpha_2-1}dz_2 = \int_{C(0,s_2)} f(z_1, z_2)z_2^{-\alpha_2-1}dz_2$$

whenever $z_1 \in C(0, r_1) = C(0, s_1)$. Multiplying by $z_1^{-\alpha_1-1}$ and integrating with respect to z_1, one concludes that $c_\alpha(r) = c_\alpha(s)$.

Thus we may use (7b) to associate constant coefficients c_α to f. With these coefficients, the terms in the series (7a) will form a *bounded sequence* at each point of D. Indeed, choose w in D and take $r_j = |w_j|$, $j = 1, 2$ [as we may]. Then by (7b),

$$
(7d) \qquad |c_{\alpha_1\alpha_2} w_1^{\alpha_1} w_2^{\alpha_2}| = \left| \frac{1}{(2\pi i)^2} \int_{T(0,r)} f(z) \left(\frac{w_1}{z_1}\right)^{\alpha_1} \left(\frac{w_2}{z_2}\right)^{\alpha_2} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \right|
$$

$$
\leq \sup{}_{T(0,r)} |f(z)|, \quad \forall \alpha.
$$

We now fix $r > 0$ in $\operatorname{tr} D$ and take $\epsilon < r$ so small that $A_\epsilon(r)$ belongs to D. For fixed z_2 in the annulus $r_2 - \epsilon_2 < |z_2| < r_2 + \epsilon_2$, the function $f(z_1, z_2)$ is holomorphic in z_1 on the annulus $r_1 - \epsilon_1 < |z_1| < r_1 + \epsilon_1$, hence f has the absolutely convergent one-variable Laurent representation

$$
(7e) \qquad f(z_1, z_2) = \sum_{\alpha_1 \in \mathbf{Z}} d_{\alpha_1}(z_2) z_1^{\alpha_1}, \quad z \in A_\epsilon(r)
$$

with

$$
(7f) \qquad \begin{aligned} d_{\alpha_1}(z_2) &= \frac{1}{2\pi i} \int_{C(0,r_1)} f(z_1, z_2) z_1^{-\alpha_1 - 1} dz_1 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r_1 e^{it}, z_2) r_1^{-\alpha_1} e^{-i\alpha_1 t} dt\,. \end{aligned}
$$

The coefficients $d_{\alpha_1}(z_2)$ will be holomorphic on the annulus $r_2 - \epsilon_2 < |z_2| < r_2 + \epsilon_2$, cf. the holomorphy theorem for integrals 1.72. Hence the coefficients have the absolutely convergent Laurent representations

$$
(7g) \qquad d_{\alpha_1}(z_2) = \sum_{\alpha_2 \in \mathbf{Z}} d_{\alpha_1\alpha_2} z_2^{\alpha_2}
$$

with

$$
(7h) \qquad d_{\alpha_1\alpha_2} = d_{\alpha_1\alpha_2}(r) = \frac{1}{2\pi i} \int_{C(0,r_2)} d_{\alpha_1}(z_2) z_2^{-\alpha_2 - 1} dz_2.
$$

Substituting $(7g)$ into $(7e)$ we finally obtain the representation

$$
f(z_1, z_2) = \sum_{\alpha_1} \left\{ \sum_{\alpha_2} d_{\alpha_1\alpha_2}(r) z_2^{\alpha_2} \right\} z_1^{\alpha_1}, \quad \forall z \in A_\epsilon(r).
$$

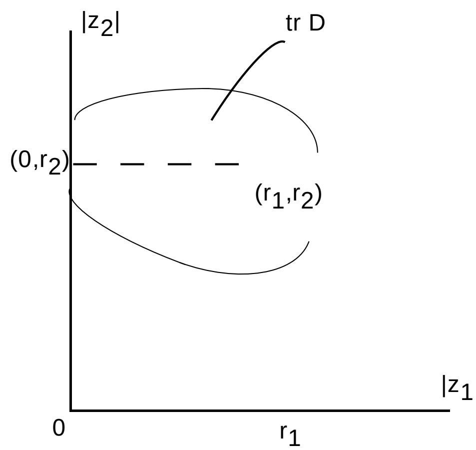
Here by $(7h)$ and $(7f)$, also making use of the continuity of f on $T(0,r)$ to rewrite a repeated integral as a double integral,

$$d_{\alpha_1\alpha_2}(r) = \frac{1}{(2\pi i)^2}\int_{C(0,r_2)}\left\{\int_{C(0,r_1)} f(z_1,z_2)z_1^{-\alpha_1-1}dz_1\right\}z_2^{-\alpha_2-1}dz_2$$
$$= \frac{1}{(2\pi i)^2}\int_{T(0,r)} f(z_1,z_2)z_1^{-\alpha_1-1}z_2^{-\alpha_2-1}dz_1dz_2 = c_{\alpha_1\alpha_2}(r) = c_{\alpha_1\alpha_2}.$$

Conclusion: f has a representation as a series $(7a,b)$, locally on D and hence globally, when the terms are arranged in a repeated series. The terms form a bounded sequence at each point $w \in D$ $(7d)$. Thus by part (i), the corresponding double series is absolutely convergent and hence converges to f on D for any arrangement of the terms.

2.8 Spherical shell theorem. In $\mathbf{C}$, a connected multicircular domain D containing the point $w = 0$ is just a disc around 0. In that case the Laurent series $(6a)$ for $f \in \mathcal{O}(D)$ on $D - \{0\}$ reduces to a power series: $c_k = 0$ for all $k < 0$. Something similar happens in $\mathbf{C}^n$:

LEMMA 2.81. Let $D \subset \mathbf{C}^n$ be a connected multicircular domain containing a point b with *jth* coordinate $b_j = 0$. Let f be holomorphic on D and let $\sum c_\alpha z^\alpha$ be its Laurent series $(7a,b)$ on $\tilde{D} = D - \{z \in \mathbf{C}^n : z_1 \dots z_n = 0\}$. [All points with a vanishing coordinate have here been removed from D so as to make Theorem 2.71 applicable as it stands.] Then $c_{\alpha_1\dots\alpha_n} = 0$ for all α's with $\alpha_j < 0$.



PROOF. We may take $n = 2$ and $j = 1$. Shifting $b = (0, b_2)$ a little if necessary, we may assume $r_2 = |b_2| \neq 0$ [D is open]. Since $\operatorname{tr} D$ is open in $\mathbf{R}^2_+$, it will contain a short closed horizontal segment from the point $(0, r_2)$ to a point $(r_1, r_2) > 0$. D will then contain all points (z_1, z_2) with $|z_1| \leq r_1,\ |z_2| = r_2$ (fig 2.9). Thus for each fixed $z_2 \in C(0, r_2),\ f(z_1, z_2)$ will be analytic in z_1 on the closed disc $|z_1| \leq r_1$. Hence by Cauchy's theorem,

$$\int_{C(0,r_1)} f(z_1,z_2)z_1^{-\alpha_1-1}dz_1 = 0 \text{ for } z_2 \in C(0,r_2)$$

whenever $-\alpha_1 - 1 \geq 0$. Conclusion:

$$c_{\alpha_1\alpha_2} = \frac{1}{(2\pi i)^2}\int_{C(0,r_2)}\left(\int_{C(0,r_1)} f(z_1,z_2)z_1^{-\alpha_1-1}dz_1\right)z_2^{-\alpha_2-1}dz_2 = 0,\ \ \forall\alpha_1 < 0.$$

We single out the important case where D meets *each* of the hyperplanes $\{z_j = 0\}$:

THEOREM 2.82 (*Analytic continuation based on multiple Laurent series*). Let $D \subset \mathbf{C}^n$ be a connected multicircular domain which for each $j,\ 1 \le j \le n$ contains a point z with $z_j = 0$, and let f be any holomorphic function on D. Then the Laurent series for f on $\tilde{D} = D - \{z_1 \dots z_n = 0\}$ with center 0 is a power series. Its sum function furnishes an analytic continuation of f to the logarithmically convex hull $\hat{D}$ of D.

PROOF. By Lemma 2.81 the Laurent series $(7a, b)$ for f [on $\tilde{D}$] has $c_\alpha = c_{\alpha_1\dots\alpha_n} = 0$ whenever [at least] one of the numbers α_j is negative, hence the Laurent series is a power series. This power series converges to f throughout $\tilde{D}$ [Theorem 2.71], hence it converges on every polydisc $\Delta(0, r)$ with $r > 0$ in $\operatorname{tr}\tilde{D}$ or $\operatorname{tr} D$ [Proposition 1.42]. Since the sum is equal to f on $\Delta(0, r) \cap \tilde{D}$ it is equal to f on $\Delta(0, r) \cap D$ [uniqueness theorem] and hence throughout D. Naturally, the power series furnishes an analytic continuation of f to the smallest complete multicircular domain D' containing D and to its logarithmically convex hull $(D')^\wedge$, cf. the discussion following Corollary 2.52. As in that case, $(D')^\wedge$ will coincide with $\hat{D}$. [Indeed, $\hat{D}$ will contain a neighbourhood of the origin, cf. exercise 2.7, hence it contains every polydisc $\Delta(0, s)$ with $s > 0$ in $\operatorname{tr} D$ (fig 2.4), and thus $\hat{D}$ contains D'.]

Application 2.83 (HARTOGS' SPHERICAL SHELL THEOREM). *Let f be holomorphic on the spherical shell given by $\rho < |z| < R$ in $\mathbf{C}^n$ with $n \ge 2$, $\rho \ge 0$. Then f has an analytic continuation to the ball $B(0, R)$.*

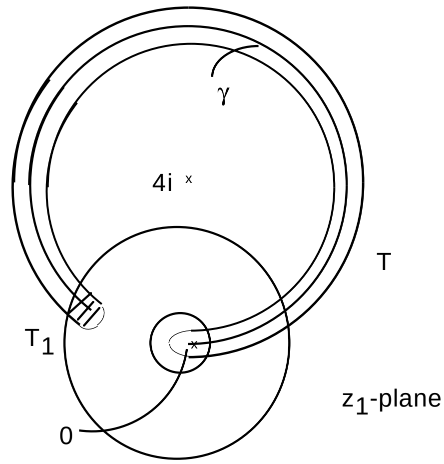
[A more general theorem of this kind will be proved in Chapter 3.]

2.9 Envelopes of holomorphy may extend outside $\mathbf{C}^n$! We will first construct a domain D in $\mathbf{C}^2$ and a Riemann domain X over $\mathbf{C}^2$ with the following property: every holomorphic function on D can be continued analytically to X, and the Riemann domain X is really necessary to accomodate single-valued analytic continuations of all functions in $\mathcal{O}(D)$.

We start with the multicircular domain D_0 in $\mathbf{C}^2$ given by

$$D_0 = \{|z_1| < 1,\ |z_2| < 2\} \cup \{|z_1| < 2,\ 1 < |z_2| < 2\}.$$

[Make a picture of trace D_0! Fig 2.11 shows, among other things, the 3-dimensional intersection of D_0 with the real hyperplane $y_2 = 0$.] As we know, every function in $\mathcal{O}(D_0)$ extends analytically to the bidisc $\Delta(0, 2) = \Delta(0, 0; 2, 2)$, cf. Section 2.5.



We next choose an arc γ of the circle $C(4i, 4)$ in the z_1-plane $(z_2 = 0)$ as follows: γ starts at the origin and, running counterclockwise, it terminates in the half-plane $\{\operatorname{Re} z_1 < 0\}$ between the circles $C(0,1)$ and $C(0,2)$ (fig 2.10). For example,

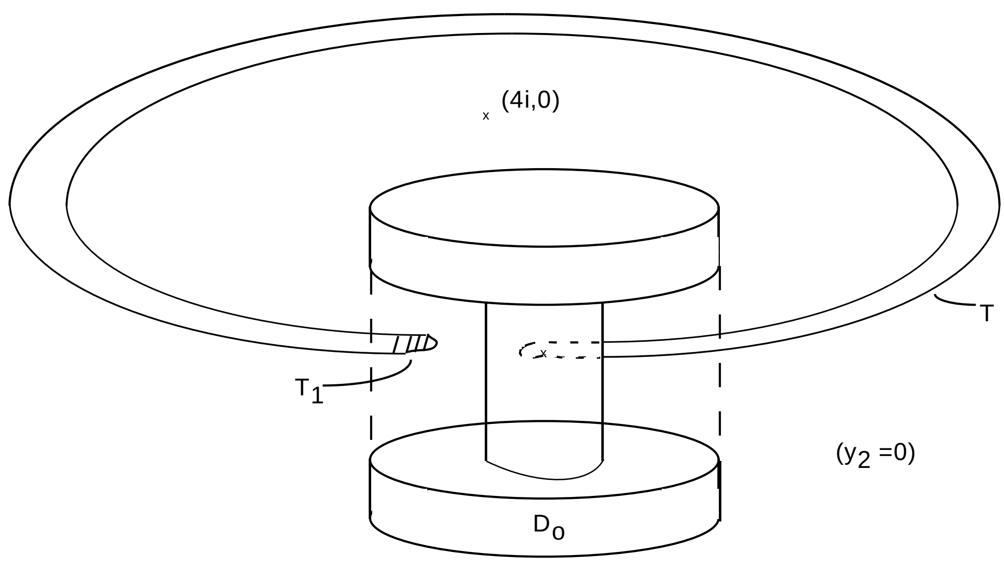
$$\gamma : z_1 = 4i + 4e^{it}, \;\; z_2 = 0, \;\; -\pi/2 \leq t \leq 3\pi/2 - \pi/8.$$

Around this arc we construct a *thin tube* T in $\mathbf{C}^2$, say an ϵ-neighbourhood of γ. Here ϵ is chosen so small that the part

$$T_1 \stackrel{\text{def}}{=} T \cap \Delta(0,2) \cap \{\operatorname{Re} z_1 < 0\}$$

does not meet D_0. Our domain $D \subset \mathbf{C}^2$ will be (cf. fig 2.11)

$$D \stackrel{\text{def}}{=} D_0 \cup T.$$



Now let f be any function in $\mathcal{O}(D)$. Then the restriction $f \mid D_0$ has an analytic continuation to $\Delta(0,2)$. However, on the part T_1 of T that continuation may very well be different from the original function f. For example, one may take for $f(z)$ that holomorphic branch of

$$\log(z_1 - 4i) \quad \text{on } D$$

for which $\operatorname{Im} f$ runs from $-\pi/2$ to $3\pi/2 - \pi/8$ on γ. On T_1 the values of $\operatorname{Im} f$ will be approximately $3\pi/2 - \pi/8$, while on the part of T close to the origin, $\operatorname{Im} f$ will be approximately $-\pi/2$. Hence the analytic continuation f^* of the restriction $f \mid D_0$ to $\Delta(0,2)$ will have its imaginary part on T_1 in the vicinity of $\pi/2 - \pi/8$!

All functions in $\mathcal{O}(D)$ have an analytic continuation to a Riemann domain X over $\mathbf{C}^2$ which contains $\Delta(0,2)$ and a copy of the tube T. The two are connected where $\operatorname{Re} z_1 > 0$, but where T (going "counterclockwise") again reaches $\Delta(0,2)$, now in the half-space $\{\operatorname{Re} z_1 < 0\}$, the end T_1 must remain separate from $\Delta(0,2)$: it may be taken "over $\Delta(0,2)$".

SIMULTANEOUS ANALYTIC CONTINUATION: *general theory.* The construction of the maximal Riemann continuation domain $\mathcal{R}$ for a holomorphic function f in Section 2.1 can be extended to the case of simultaneous analytic continuation for the members of a family of holomorphic functions. We deal with *indexed families*; in the following discussion, the index set Λ is kept fixed. Mimicking the procedure for a single function, we now define Λ-*elements* $(a, U, \{f_\lambda\})$ at points $a \in \mathbf{C}^n$. Such elements consist of a connected domain $U \subset \mathbf{C}^n$ containing a and a family of functions $\{f_\lambda\} \subset \mathcal{O}(U)$ with index set Λ. Two Λ-elements $(a, U, \{f_\lambda\})$ and $(a, \tilde{U}, \{\tilde{f}_\lambda\})$ at the same point a are called *equivalent* if the power series $(f_\lambda)_a$ and $(\tilde{f}_\lambda)_a$ agree for every $\lambda \in \Lambda$. A Λ-element $(b, V, \{g_\lambda\})$ is called a direct Λ-continuation of $(a, U, \{f_\lambda\})$ if $V \cap U$ is nonempty and $g_\lambda = f_\lambda$, $\forall \lambda$ on a fixed component of $V \cap U$. General Λ-*continuations* are introduced by means of both finite chains and continuous chains of direct Λ-continuations.

Starting with a given Λ-element $(a, U, \{f_\lambda\})$ and carrying out unlimited Λ-continuation, one arrives at a *Riemann domain* $X = (X, \pi)$ over $\mathbf{C}^n$ whose points p are equivalence classes of Λ-continuations at points $b \in \mathbf{C}^n$: let us write $p = [(b, V, \{g_\lambda\})]$. Basic neighbourhoods $\mathcal{N} = \mathcal{N}(p, V, \{g_\lambda\})$ in X shall consist of the points q corresponding to the direct Λ-continuations $(c, W, \{h_\lambda\})$ of $(b, V, \{g_\lambda\})$ for which $c \in V$ and $(h_\lambda)_c = (g_\lambda)_c$, $\forall \lambda$. The projection $\pi : \pi(p) = b$, when restricted to $\mathcal{N}$, establishes a homeomorphism of $\mathcal{N}$ in X onto V in $\mathbf{C}^n$. Every function f_λ of the original Λ-element has an *analytic continuation* F_λ to X given by $F_\lambda(p) = g_\lambda(b)$. Indeed,

$$F_\lambda(q) = h_\lambda(z) = g_\lambda(z), \ \ \forall q = [(z, W, \{h_\lambda\})] \in \mathcal{N}(p, V, \{g_\lambda\}),$$

so that F_λ is holomorphic on X in the accepted sense: on $\mathcal{N} \subset X$ it is given by an ordinary holomorphic function on the domain $V = \pi(\mathcal{N}) \subset \mathbf{C}^n$. Finally, setting $p_0 = [(a, U, \{f_\lambda\})]$ and identifying $\mathcal{N}(p_0, U, \{f_\lambda\})$ with U, one has $F_\lambda = f_\lambda$ on U.

We have thus obtained a common continuation domain X for the family of holomorphic functions $\{f_\lambda\}$ on U. It is plausible and one can show that this "Λ-continuation domain" X is maximal; one speaks of a Λ-envelope for U, cf. [Narasimhan].

APPLICATION 2.91 (*Envelope of holomorphy*). Let D be a connected domain in $\mathbf{C}^n$. Applying the preceding construction to $U = D$ or $U \subset D$ and $\Lambda = \mathcal{O}(D)$, one obtains an $\mathcal{O}(D)$-continuation domain $X_D = (X_D, \pi)$ for D. Being maximal, X_D is called an envelope of holomorphy for D. Every function $f \in \mathcal{O}(D)$ has a (unique) analytic continuation F_f to X_D. On a suitable neighbourhood $\mathcal{N}$ of p in X_D, the functions F_f are given by $F_f(q) = g_f(z)$, $z = \pi \circ q$, where the functions g_f on $V = \pi(\mathcal{N})$ may be obtained from the functions f on U by analytic continuation along a *common path*. Observe in particular that for $f \equiv c$, also $F_f \equiv c$. More generally, if $F_f = c$ on some neighbourhood $\mathcal{N}$ in X, then $F_f = c$ everywhere.

What was said in the last four sentences is also true for arbitrary (connected) $\mathcal{O}(D)$-continuation domains X for D, in or over $\mathbf{C}^n$. It is perhaps surprising that on such a domain X, the analytic continuations f^* of the functions $f \in \mathcal{O}(D)$ can not take on new values:

PROPOSITION 2.92. Suppose that the equation $f(z) = c$, $c \in \mathbf{C}$ has no solution $z \in D$. Then the equation $f^*(q) = c$ can not have a solution q in any $\mathcal{O}(D)$-continuation domain X.

Indeed, by the hypothesis there is a function $g \in \mathcal{O}(D)$ such that

$$\{f(z) - c\}g(z) \equiv 1 \text{ on } D.$$

Introducing the simultaneous analytic continuations to X, one obtains

$$\{f^*(q) - c\}\, g^*(q) = (f - c)^*(q)\, g^*(q) = 1^* = 1 \text{ on } X.$$

It is not hard to deduce the following corollary, cf. exercise 2.24:

COROLLARY 2.93. Let $D \subset D'$ be a connected domain in $\mathbf{C}^n$ to which all functions in $\mathcal{O}(D)$ can be continued analytically. Then D' belongs to the convex hull $\mathrm{CH}(D)$.

Exercises

2.1. Give an example of two function elements (a, U, f), (b, V, g) such that $g = f$ on one component of $V \cap U$, while $g \neq f$ on another component.

2.2. Let b be an arbitrary boundary point of the polydisc $\Delta(0, r)$ in $\mathbf{C}^2$. Show that there is a holomorphic function on $\Delta(0, r)$ that tends to infinity as $z \to b$. [One may conclude that $\Delta(0, r)$ is a domain of holomorphy, cf. Section 1.9.]

2.3. Prove that hulls of holomorphy in $\mathbf{C}^n$ are unique when they exist [Definition 2.13].

2.4. (i) Let E be a compact convex set in $\mathbf{R}^n$ and let V be a supporting hyperplane. Prove that the intersection $E \cap V$ is also a compact convex set.
(ii) Let S be a compact subset of $\mathbf{R}^n$. Prove the Carathéodory representation (2) for the points of the convex hull $\mathrm{CH}(S)$ with $m \leq n + 1$.

2.5. Let S be a compact set in $\mathbf{R}^n$. Show that

(i) For every direction (or unit vector) c there is a point $b \in S$ such that $\max_{x\in S}\, c \cdot x = c \cdot b$;

(ii) The convex hull $\mathrm{CH}(S)$ is the set of all points $x \in \mathbf{R}^n$ such that $c \cdot x \le \max_{s\in S}\, c \cdot s$ for every vector $c \in \mathbf{R}^n$.

2.6. What sort of equation "$y = f(x)$" does the logarithmically convex hull of the set of two points $r' > 0$ and $r'' > 0$ in $\mathbf{R}^2_+ = \{(r_1, r_2) \ge 0\}$ have?

2.7. Determine the logarithmically convex hull in $\mathbf{R}^2_+$ of :

(i) the set $\{s, t\}$ of two points $s = (s_1, s_2) > 0$ and $t = (t_1, 0)$;

(ii) the set $\{s, t\}$ when $s = (s_1, 0)$ and $t = (0, t_2)$;

(iii) the set consisting of the neighbourhood $\{s_1 - \epsilon < r_1 < s_1 + \epsilon,\ 0 \le r_2 < \epsilon\}$ of $(s_1, 0)$ and the neighbourhood $\{0 \le r_1 < \epsilon,\ t_2 - \epsilon < r_2 < t_2 + \epsilon\}$ of $(0, t_2)$.

2.8. Prove that a closed or open set F in $\mathbf{R}^n_+$ is logarithmically convex if and only if $r' \in F$ and $r'' \in F$ always implies that $r = (r'r'')^{\frac{1}{2}}$ is in F. Deduce that the unit ball $B = B(0, 1)$ in $\mathbf{C}^2$ is logarithmically convex [that is, $\mathrm{tr}\, B$ is logarithmically convex].

2.9. Let $S = S(0, 1)$ denote the unit sphere in $\mathbf{C}^n \sim \mathbf{R}^{2n}$:

$$S = \{z \in \mathbf{C}^n : z_1\overline{z}_1 + \ldots + z_n\overline{z}_n = 1\}$$

and let b be any point of S. Show that

(i) The $(2n-1)$-dimensional (real) tangent hyperplane to S at b may be represented by the equation $\mathrm{Re}\,(\overline{b} \cdot z) = 1$;

(ii) $\mathrm{Re}\,(\overline{b} \cdot z) < 1$ throughout the unit ball B;

(iii) The complex tangent hyperplane at b, of complex dimension $n-1$ (real dimension $2n-2$) may be represented by the equation $c \cdot (z - b) = 0$ with $c = \ldots$;

(iv) There is a holomorphic function f on the unit ball B that tends to infinity as $z \to b$. [Thus B is a domain of holomorphy, cf. Section 1.9.]

2.10. Let D be a connected (multicircular) domain in $\mathbf{C}^2$. Prove that $D - \{z_1 = 0\}$ and $D - \{z_1 z_2 = 0\}$ are also connected (multicircular) domains.

2.11. Prove that the multicircular domain in $\mathbf{C}^2$ given by $|z_1| < 2,\ |z_2| < 2,\ |z_1 z_2| < 1$ is a domain of holomorphy. [Cf. exercise 2.2.]

2.12. (Relation between the sets A and A^0 of Section 2.3) Let $\sum c_\alpha z^\alpha$ be a multiple power series with center 0. Prove that every point $a \in A$ (point of absolute convergence) with $|a_j| > 0$, $\forall j$ belongs to $\mathrm{clos}\, A^0$. Then give an example to show that a point $b \in A$ for which one coordinate is zero may be very far from A^0.

2.13. Let $\sum c_\alpha z^\alpha$ be a power series in $\mathbf{C}^2$ for which $\Delta(0, 0; r_1, r_2)$ is a polydisc of convergence that is maximal for the given r_1 as far as r_2 is concerned, that is, the power series does not converge throughout any polydisc $\Delta(0, 0; r_1, s_2)$ with $s_2 > r_2$. Prove that the sum function $f(z)$ of the power series must become singular somewhere on the torus $T(0, r) = C(0, r_1) \times C(0, r_2)$. [Hence if $f \in \mathcal{O}(\Delta(0, r))$ becomes singular at a point b in $\Delta_1(0, r_1) \times C(0, r_2)$, it must have a singularity on every torus $C(0, \rho) \times C(0, r_2)$ with $|b_1| < \rho \le r_1$.]

2.14. (*Analytic continuation across a compact subset*) Let $D \subset \mathbf{C}^2$ be the domain given by

$$\{|z_1| < 1+\epsilon,\ 1-\epsilon < |z_2| < 1+\epsilon\} \cup \{1-\epsilon < |z_1| < 1+\epsilon,\ |z_2| < 1+\epsilon\}, \quad (0 < \epsilon < 1),$$

or more generally, let D be any connected domain in $\mathbf{C}^n$ $(n \geq 2)$ that contains the boundary $\partial\Delta(0,1)$ of the unit polydisc, but not all of $\Delta(0,1)$ itself. Let f be any function in $\mathcal{O}(D)$. Prove that the formula

$$F(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(z',w)}{w-z_n}dw, \quad z \in \Delta(0,1), \quad z' = (z_1,\ldots,z_{n-1})$$

furnishes an analytic continuation of f to $\Delta(0,1)$.

2.15. Let B denote the set of those points $z \in \mathbf{C}^n$ at which the terms $c_\alpha z^\alpha$, $\alpha \in \mathbf{Z}^n$ of the Laurent series $(3a)$ form a bounded sequence. Prove that (the trace of) B is logarithmically convex.

2.16. Prove that the domain of (absolute) convergence A^0 of a Laurent series with center 0 in $\mathbf{C}^n$ is logarithmically convex.

2.17. Let D be a multicircular domain in $\mathbf{C}^n$ which contains the origin, or which at least for each j contains a point z with $z_j = 0$. Prove that the logarithmically convex hull $\hat{D}$ contains the whole polydisc $\Delta(0,s)$ whenever $s > 0$ belongs to $\operatorname{tr} D$.

2.18. Use appropriate results to determine the envelopes of holomorphy for the following multicircular domains in $\mathbf{C}^2$:

(i) $\{|z_1| < 1,\ |z_2| < 2\} \cup \{|z_1| < 2,\ 1 < |z_2| < 2\}$;
(ii) $\{1 < |z_1| < 2,\ |z_2| < 2\} \cup \{|z_1| < 2,\ 1 < |z_2| < 2\}$.

2.19. Let f be holomorphic on $\mathbf{C}^n$, $n \geq 2$ and $f(0) = 0$. Prove that the zero set Z_f of f is closed but unbounded.

2.20. Let f be holomorphic on $D = B(0,1) - \{z_1 = 0\}$ in $\mathbf{C}^2$. Suppose f has an analytic continuation to a neighbourhood of the point $(0, \frac{1}{2})$. Prove that f has an analytic continuation to $B(0,1)$.

2.21. Let D be a multicircular domain in $\mathbf{C}^2$ that contains the point $(0, r_2)$ with $r_2 > 0$ and let $\epsilon = (\epsilon_1, \epsilon_2) > 0$ be so small that D contains the closure of the domain $U_{2\epsilon}$ given by $|z_1| < 2\epsilon_1,\ r_2 - 2\epsilon_2 < |z_2| < r_2 + 2\epsilon_2$. Prove that the Laurent series for $f \in \mathcal{O}(D)$ is absolutely and uniformly convergent on U_ϵ. [Cf. Lemma 2.81 and part (i) of the proof of Theorem 2.71.]

2.22. (*Isolated singularities in* $\mathbf{C}^n$, $n \geq 2$ *are removable*) Give two alternative proofs for Application 2.62.

2.23. Derive the spherical shell theorem from Hartogs' continuity theorem. [Let f be holomorphic for $\rho < |z| < R$ and suppose that the boundary point $(\rho, 0, \ldots, 0)$ would be singular for f.]

2.24. Let $D \subset \mathbf{C}^n$ be connected and bounded and let $D' \subset \mathbf{C}^n$ be a connected domain containing D to which all functions in $\mathcal{O}(D)$ can be continued analytically. Determine the analytic continuations of the functions $f(z) = c \cdot z$, where $c = \alpha - i\beta \in \mathbf{C}^n$ and prove that

$$\operatorname{Re} c \cdot z \leq \max_{\zeta \in \overline{D}} \operatorname{Re} c \cdot \zeta, \quad \forall z \in D'.$$

Deduce that D' belongs to the convex hull of $\overline{D}$, hence of D. [Cf. exercise 2.5.]

2.25. Let $D \subset \mathbf{C}^2$ be a bounded connected multicircular domain containing the origin. Use the monomials $p(z) = z_1^{\alpha_1} z_2^{\alpha_2}$ to show that a point $z \in \mathbf{C}^2$ outside the closure of the logarithmically convex hull $\hat{D}$ of D can not belong to an $\mathcal{O}(D)$-continuation domain $D' \supset D$ in $\mathbf{C}^2$.

2.26. Try to find an example of a domain D in $\mathbf{C}^n$, for which the envelope of holomorphy X_D over $\mathbf{C}^n$ has infinitely many layers.

CHAPTER 3

Analytic continuation, part II

This chapter deals with more recent methods of analytic continuation, based on the $\overline{\partial}$ equation and the so-called partial derivatives lemma.

We have already discussed Hartogs' spherical shell theorem 2.83, but there is a much more general result on the removal of compact singularity sets, the Hartogs-Osgood-Brown continuation theorem [Section 3.4]. We will present a modern proof of that result (due to Ehrenpreis) in which one starts with a C^∞ continuation g across an appropriate compact set and then subtracts off the "nonanalytic part" u, cf. Section 1.9. In the present instance the correction term u has to satisfy a $\overline{\partial}$ equation

$$\overline{\partial} u = v = \sum_1^n v_j d\overline{z}_j \quad \text{or} \quad \frac{\partial u}{\partial \overline{z}_j} = v_j, \;\; j = 1, \ldots, n$$

on $\mathbf{C}^n$ with C^∞ coefficients v_j of *compact support*. The analytic continuation problem requires a smooth solution u on $\mathbf{C}^n$ which likewise has compact support. The local integrability conditions

$$\partial v_k / \partial \overline{z}_j = \partial v_j / \partial \overline{z}_k, \quad \forall\, j, k$$

being satisfied, it turns out that there is a compactly supported solution u whenever $n \geq 2$ [Section 3.2]. It will be obtained with the aid of a useful one-variable device, Pompeiu's integral formula for smooth functions.

There are various situations in real and complex analysis where one has good bounds on a family of directional derivatives

$$\left(\frac{d}{dt}\right)^m f(a + t\xi)\Big|_{t=0}, \quad \xi \in E \subset S(0,1), \quad m = 1, 2, \ldots \;\; (a \text{ fixed})$$

of a C^∞ function f. If the set of directions E is substantial enough, a partial derivatives lemma of the author and Wiegerinck provides related bounds for all derivatives $D^\alpha f(a)$. Under appropriate conditions, the power series for f with center a can then be used for analytic extension.

To illustrate the method we give a simple proof of the Behnke–Kneser "recessed-edge theorem" [Section 3.5]. Another application leads to a form of Bogolyubov's famous edge-of-the-wedge theorem. This result which came from a problem in quantum field theory provides a remarkable $\mathbf{C}^n$ extension of Schwarz's classical reflection principle.

3.1 Inhomogeneous Cauchy-Riemann equation for $n = 1$. As preparation for the case of $\mathbf{C}^n$ we consider the case of one variable,

$$\text{(1a)} \qquad \frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y}\right) = v \quad \text{on} \quad \mathbf{C},$$

where v is a function with *compact support*. The SUPPORT, abbreviation supp, of a function [or distribution, or differential form] is the smallest closed set outside of which it is equal to zero. Our functions $v(z) = v(x+iy)$ will be *smooth*, that is, at least of class C^1 on $\mathbf{C} = \mathbf{R}^2$ as functions of x and y.

For the solution of equation ($1a$) we start with POMPEIU'S FORMULA [also called the Cauchy-Green formula]:

Proposition 3.11. *Let D be a bounded domain in $\mathbf{C}$ whose boundary Γ consists of finitely many piecewise smooth Jordan curves, oriented in such a way that D lies to the left of Γ. Let $f(z) = f(x+iy)$ be of class C^1 on $\overline{D}$ as a function of x and y. Then*

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz - \frac{1}{\pi}\int_D \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a}dxdy, \quad \forall a \in D. \tag{1b}$$

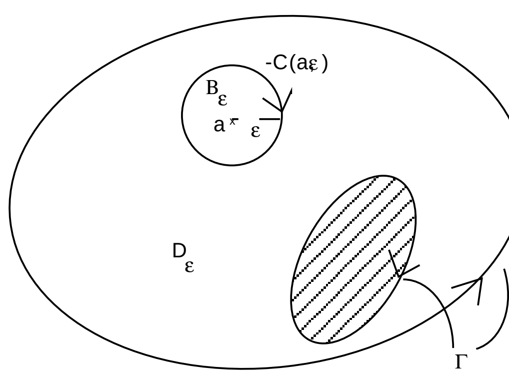
Observe that the area integral over D is well-defined because $1/(z-a)$ is absolutely integrable over a neighbourhood of a, cf. the proof below. Formula ($1b$) reduces to Cauchy's integral formula if f is holomorphic on $\overline{D}$, so that $\partial f/\partial\overline{z} = 0$. The formula occurred in work of Pompeiu around 1910, but its usefulness for complex analysis only became apparent around 1950.

The proof will be based on GREEN'S FORMULA for integration by parts in the plane:

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

where $P(x,y)$ and $Q(x,y)$ are functions of class $C^1(\overline{D})$ and ∂D denotes the oriented boundary of D. Applied to $P = F$ and $Q = iF$ with $F(z) = F(x+iy)$ in $C^1(\overline{D})$, we obtain a *complex form* of Green's formula:

$$\begin{aligned}\int_{\partial D} F(z)dz &= \int_{\partial D} Fdx + iFdy = \int_D \left(i\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right) dxdy \\ &= 2i\int_D \frac{\partial F}{\partial \overline{z}}dxdy.\end{aligned} \tag{1c}$$



PROOF of Proposition 3.11. One would like to apply Green's formula ($1c$) to the function

$$F(z) = \frac{f(z)}{z-a}, \;\; a \in D.$$

However, this F is in general not smooth at $z = a$. We therefore exclude a small closed disc $\overline{B}_\epsilon = \overline{B}(a,\epsilon)$ from D, of radius $\epsilon < d(a,\Gamma)$. Below, we will apply Green's formula to F on

$$D_\epsilon \stackrel{\text{def}}{=} D - \overline{B}_\epsilon.$$

The correctly oriented boundary ∂D_ϵ will consist of Γ and $-C(a,\epsilon)$: the circle $C(a,\epsilon)$ traversed clockwise.

Since $1/(z-a)$ is holomorphic throughout $\overline{D}_\epsilon$, the product rule of differentiation gives

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a} + f(z)\frac{\partial}{\partial \overline{z}}\frac{1}{z-a} = \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a}, \quad z \in \overline{D}_\epsilon.$$

Thus by $(1c)$,

$$(1d) \qquad \int_\Gamma \frac{f(z)}{z-a}dz + \int_{-C(a,\epsilon)} \frac{f(z)}{z-a}dz = 2i\int_{D_\epsilon} \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a}dxdy.$$

Passage to the limit as $\epsilon \downarrow 0$ will give $(1b)$. Indeed, by the continuity of f at a,

$$\int_{-C(a,\epsilon)} \frac{f(z)}{z-a}dz = -i\int_{-\pi}^{\pi} f(a+\epsilon e^{it})dt \to -2\pi i f(a) \text{ as } \epsilon \downarrow 0.$$

Furthermore, since $\partial f/\partial \overline{z}$ is continuous on $\overline{D}$ while $1/(z-a)$ is (absolutely) integrable over discs $B(a,R)$, the product is integrable over D, hence the last integral in $(1d)$ will tend to the corresponding integral over D. In fact, if M denotes a bound for $|\partial f/\partial \overline{z}|$ on $\overline{D}$, then

$$\begin{aligned} \Big| \int_D - \int_{D_\epsilon} \Big| &= \Big| \int_{\overline{B}_\epsilon} \frac{\partial f}{\partial \overline{z}}\frac{1}{z-a}dxdy \Big| \le M\int_{\overline{B}_\epsilon} \frac{1}{|z-a|}dxdy \\ &= M\int_0^\epsilon \int_{-\pi}^{\pi} \frac{1}{r}\, r\, dr dt = M\; 2\pi\epsilon \to 0 \quad \text{as} \quad \epsilon \downarrow 0. \end{aligned}$$

Corollary 3.12. *Any C^1 function $f(z) = f(x+iy)$ on $\mathbf{C}$ of* COMPACT SUPPORT *has the representation*

$$(1e) \qquad f(z) = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial f}{\partial \overline{z}}(\zeta)\frac{1}{\zeta - z}d\xi d\eta \quad (\zeta = \xi + i\eta), \quad \forall z \in \mathbf{C}.$$

Indeed, fixing $a \in \mathbf{C}$, one may apply Pompeiu's formula $(1b)$ to a disc $D = B(0,R)$ which contains both a and the support of f. Then the integral over $\Gamma = C(0,R)$ will vanish; the integral over $D = B(0,R)$ will be equal to the corresponding integral over $\mathbf{C}$ or over $\operatorname{supp} f$. One may finally replace the variable z under the integral sign in $(1b)$ by $\zeta = \xi + i\eta$ and then replace a by z.

Formula $(1e)$ can also be verified directly and the condition that f have compact support may be relaxed to a smallness condition on f and $\partial f/\partial \bar z$ at infinity, cf. exercises 3.1, 3.2. Thus *if* our equation $\partial u/\partial \bar z = v$ has a solution which is small at infinity, it will be given by the CAUCHY-GREEN TRANSFORM u of v :

$$(1f)\qquad u(z) \stackrel{\text{def}}{=} -\frac{1}{\pi}\int_{\mathbf{C}\text{ or supp}\,v} \frac{v(\zeta)}{\zeta - z}\,d\xi d\eta \quad (\zeta = \xi + i\eta),\ \ \forall z \in \mathbf{C}.$$

We will show that this candidate is indeed a solution:

Theorem 3.13. *Let v be a C^p function $(1 \le p \le \infty)$ on $\mathbf{C}$ of compact support [briefly, $v \in C_0^p(\mathbf{C})$]. Then the Cauchy-Green transform u of v $(1f)$ provides a C^p solution of the equation $\partial u/\partial \bar z = v$ on $\mathbf{C}$. It is the unique smooth solution which tends to 0 as $|z| \to \infty$.*

PROOF. Replacing ζ by $\zeta' + z$ in $(1f)$ and dropping the prime $'$ afterwards, we may write the formula for u as

$$(1g)\qquad u(z) = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{v(z+\zeta')}{\zeta'}\,d\xi' d\eta' = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{v(z+\zeta)}{\zeta}\,d\xi d\eta.$$

We will show that u has first order partial derivatives and that they may be obtained by differentiation under the integral sign. Fixing a and varying $z = a + h$ over a small disc $B(a, r)$, the function $v(z+\zeta)$ will vanish for all ζ outside a fixed large disc $B = B(0, R)$. Focusing on $\partial u/\partial x$ we take h *real* and $\neq 0$, so that

$$(1h)\qquad \frac{v(a+h+\zeta) - v(a+\zeta)}{h} - \frac{\partial v}{\partial x}(a+\zeta) = \frac{1}{h}\int_0^h \left\{\frac{\partial v}{\partial x}(a+t+\zeta) - \frac{\partial v}{\partial x}(a+\zeta)\right\} dt.$$

Since $\partial v/\partial x$ is continuous and of compact support it is uniformly continuous on $\mathbf{C}$, hence the right-hand side $\rho(\zeta, h)$ of $(1h)$ tends to 0 as $h \to 0$ uniformly in ζ. Multiplying $(1h)$ by the absolutely integrable function $1/\zeta$ on B and integrating over B, we conclude that

$$-\pi\frac{u(a+h) - u(a)}{h} - \int_B \frac{\partial v}{\partial x}(a+\zeta)\cdot\frac{1}{\zeta}\,d\xi d\eta = \int_B \rho(\zeta, h)\frac{1}{\zeta}\,d\xi d\eta \to 0$$

as $h \to 0$. Thus the partial derivative $\partial u/\partial x$ exists at a and

$$(1i)\qquad \frac{\partial u}{\partial x}(a) = -\frac{1}{\pi}\int_{B\text{ or }\mathbf{C}} \frac{\partial v}{\partial x}(a+\zeta)\cdot\frac{1}{\zeta}\,d\xi d\eta.$$

The uniform continuity of $\partial v/\partial x$ also ensures continuity of $\partial u/\partial x$.

Differentiation with respect to y goes in much the same way, hence u is of class C^1. Combining the partial derivatives we find that

$$(1j)\qquad \frac{\partial u}{\partial \bar z}(a) = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial v}{\partial \bar z}(a+\zeta)\cdot\frac{1}{\zeta}\,d\xi d\eta = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial v}{\partial \bar z}(\zeta)\frac{1}{\zeta - a}\,d\xi d\eta.$$

Now v is a C^p function of bounded support, hence by Corollary 3.12, the final member of $(1j)$ is equal to $v(a)$. Thus u satisfies the differential equation $(1a)$.

If $p \geq 2$, one may also form higher order partial derivatives by differentiation under the integral sign in $(1g)$ to show that all partial derivatives of u of order $\leq p$ exist and are continuous on $\mathbf{C}$.

The Cauchy-Green transform $u(z)$ tends to 0 as $|z| \to \infty$ and it is the only smooth solution of $(1a)$ with that property. Indeed, the other smooth solutions have the form $u + f$, where f is smooth and satisfies the Cauchy-Riemann condition $\partial f/\partial \overline{z} = 0$, hence f must be an entire function. However, by Liouville's theorem, $f(z) \to 0$ as $|z| \to \infty$ only if $f \equiv 0$.

REMARKS 3.14. For $v \in C_0^p$, $p \in \mathbf{N}_0$, the Cauchy-Green transform u $(1f)$ will actually be of class $C^{p+\alpha}$, $\forall \alpha \in (0,1)$: $u \in C^p$ and its partial derivatives of order p will satisfy a Lipschitz condition of order α, cf. exercise 3.6. In general, the transform u will not have compact support, in fact, as $|z| \to \infty$,

$$zu(z) \to (1/\pi) \int_{\mathbf{C}} v(\zeta) d\xi d\eta$$

and this limit need not vanish. [Cf. also exercise 3.5.] Formula $(1f)$ defines a function u under much weaker conditions than we have imposed in the Theorem: continuity of v and integrability of $|v(\zeta)/\zeta|$ over $\mathbf{C}$ will suffice. The corresponding transform u will be a weak or distributional solution of equation $(1a)$, cf. exercise 3.8.

3.2 Inhomogeneous $C - R$ equation for $n \geq 2$, compact support case. Saying that a differential form

$$f = \sum_{j=1}^{n} (u_j dz_j + v_j d\overline{z}_j) \tag{2a}$$

is defined and *of class* C^p on $\Omega \subset \mathbf{C}^n$ means that the coefficients u_j, v_j are defined and of class C^p on Ω as functions of the real variables $x_1, y_1, \ldots, x_n, y_n$. By definition, such a form vanishes on an open subset of Ω only if all coefficients vanish there. There will be a maximal open subset of Ω on which $f = 0$; its complement in Ω is the *support* of f. The differential form f in $(2a)$ is called a first order form or a 1-form; if it contains no terms $u_j dz_j$, one speaks of a $(0,1)$-*form.*

Theorem 3.21. *Let E be a compact subset of $\mathbf{C}^n$, $n \geq 2$ with* CONNECTED *complement $E^c = \mathbf{C}^n - E$. Let*

$$v = \sum_{1}^{n} v_j d\overline{z}_j$$

be a $(0,1)$-form of class C^p $(1 \leq p \leq \infty)$ on $\mathbf{C}^n$ whose support belongs to E and which satisfies the integrability conditions $\partial v_k/\partial \overline{z}_j = \partial v_j/\partial \overline{z}_k$, $\forall\, j,k$. Then the equation $\overline{\partial} u = v$, or equivalently, the system

$$\frac{\partial u}{\partial \overline{z}_j} = v_j, \quad j = 1, \ldots, n \tag{2b}$$

has a unique solution u of class C^p on $\mathbf{C}^n$ with support in E.

A result of this kind is sometimes called a Grothendieck-Dolbeault lemma, cf. [Grauert-Remmert]. The solution u will actually be of class $C^{p+\alpha}$, $\forall\alpha \in (0,1)$, see exercise 3.9. For arbitrary compact E, supp u need not be contained in supp v [cf. the proof below].

PROOF. We will solve the first equation ($2b$) by means of the Cauchy-Green transform relative to z_1, cf. Theorem 3.13. It will then miraculously follow from the integrability conditions that the other equations are also satisfied!

It is convenient to set $(z_2, \ldots, z_n) = z'$, so that $z = (z_1, z')$. For fixed z', the smooth function $v_1(z_1, z')$ of z_1 has bounded support in $\mathbf{C}$, hence Theorem 3.13 gives us a solution of the equation $\partial u/\partial \overline{z}_1 = v_1$ in the form of the Cauchy-Green transform of v_1 relative to z_1 :

$$(2c) \qquad u(z) = u(z_1, z') = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{v_1(\zeta, z')}{\zeta - z_1} d\xi d\eta = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{v_1(z_1 + \zeta, z')}{\zeta} d\xi d\eta, \quad z \in \mathbf{C}^n.$$

Here the integration variable $\zeta = \xi + i\eta$ runs just over the complex plane. The method of differentiation under the integral sign of Section 3.1, applied to the last integral, shows that u is of class C^p on $\mathbf{C}^n$ as a function of $x_1, y_1, \ldots, x_n, y_n$.

We now go back to the first integral in ($2c$) to obtain an expression for $\partial u/\partial \overline{z}_j$ when $j \geq 2$. In the second step below we will use the integrability condition $\partial v_1/\partial \overline{z}_j = \partial v_j/\partial \overline{z}_1$:

$$(2d) \qquad \frac{\partial u}{\partial \overline{z}_j}(z) = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial v_1}{\partial \overline{z}_j}(\zeta, z')\frac{1}{\zeta - z_1} d\xi d\eta = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial v_j}{\partial \overline{z}_1}(\zeta, z')\frac{1}{\zeta - z_1} d\xi d\eta.$$

Observe that for fixed z', the smooth function $v_j(z_1, z')$ of z_1 also has bounded support in $\mathbf{C}$. Hence by the representation for such functions in Corollary 3.12, the last integral ($2d$) is just equal to $v_j(z_1, z') = v_j(z)$. Since we knew already that $\partial u/\partial \overline{z}_1 = v_1$, we conclude that $\overline{\partial} u = v$.

It follows in particular that $\overline{\partial} u = 0$ throughout E^c, hence u is holomorphic on the domain E^c. We will show that $u = 0$ on E^c. For suitable $R > 0$, the set E and hence supp v will be contained in the ball $B(0, R)$. Thus $v_1(\zeta, z') = 0$ for $|z'| > R$ and arbitrary ζ. Hence by ($2c$), $u(z_1, z') = 0$ for $|z'| > R$ and all z_1, so that $u = 0$ on an open subset of E^c. The uniqueness theorem for holomorphic functions 1.54 now shows that $u = 0$ throughout the connected domain E^c, in other words, supp $u \subset E$.

Naturally, the equation $\overline{\partial} u = v$ can not have another smooth solution on $\mathbf{C}^n$ with support in E. [What could one say about the difference of two such solutions?]

3.3. Smooth approximate identities and cutoff functions. In various problems, the first step towards a holomorphic solution is the construction of smooth approximate solutions. For that step we need smooth cutoff functions and they are constructed with the aid of suitable C^∞ *functions of compact support.* The latter play an important role in analysis, for example, as *test functions* in the theory of distributions, cf. Chapter 11.

The precursor is the C^∞ function on $\mathbf{R}$ defined by

$$\sigma(x) = \begin{cases} e^{-1/x} & \text{for } \ x > 0 \\ 0 & \text{for } \ x \leq 0; \end{cases}$$

its first and higher derivatives at 0 are all equal to 0. One next defines a C^∞ function τ on $\mathbf{R}$ with support $[-1,1]$ by setting

$$\tau(x)=\sigma\{2(1+x)\}\sigma\{2(1-x)\}=\begin{cases}\exp\left(-\frac{1}{1-x^2}\right) & \text{for } \ |x|<1\\ \\ 0 & \text{for } \ |x|\geq 1.\end{cases}$$

Moving on to $\mathbf{R}^n$, the function $\tau(|x|)$ will provide a C^∞ function whose support is the closed unit ball $\overline{B}(0,1)$; here $|x|$ stands for the length of $x : |x|^2 = x_1^2+\ldots+x_n^2$. We like to make the integral over $\mathbf{R}^n$ equal to 1, hence we introduce

$$\text{(3}a\text{)}\qquad \rho(x)=c_n\tau(|x|)=\begin{cases}c_n\ \exp\left(-\frac{1}{1-|x|^2}\right) & \text{for } \ |x|<1\\ \\ 0 & \text{for } \ |x|\geq 1,\ \ x\in\mathbf{R}^n,\end{cases}$$

where the constant c_n is chosen such that $\int_{\mathbf{R}^n}\rho(x)dx=1$. [Here dx denotes the volume element of $\mathbf{R}^n$.]

From the function ρ we derive the important family of C^∞ functions

$$\text{(3}b\text{)}\qquad \rho_\epsilon(x)\overset{\text{def}}{=}\frac{1}{\epsilon^n}\rho\left(\frac{x}{\epsilon}\right),\ \ x\in\mathbf{R}^n,\ \ \epsilon>0$$

with supp $\rho_\epsilon=\overline{B}(0,\epsilon)$. Observe that by change of scale,

$$\int_{\mathbf{R}^n}\rho_\epsilon(x)dx=\int_{B(0,\epsilon)}\frac{1}{\epsilon^n}\rho\left(\frac{x}{\epsilon}\right)dx=\int_{B(0,1)}\rho(x)dx=1.$$

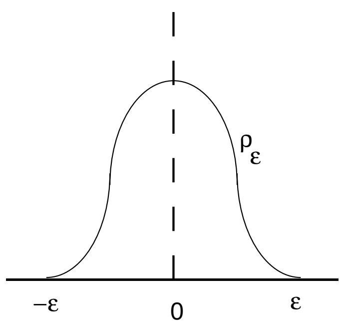


fig 3.2

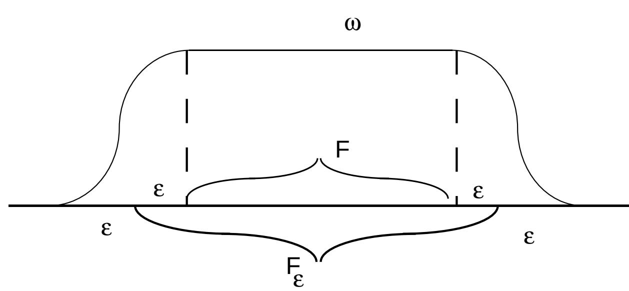


fig 3.3

APPROXIMATE IDENTITIES 3.31. The directed family of functions $\{\rho_\epsilon\}$, $\epsilon\downarrow 0$ of $(3a,b)$ is the standard example of a C^∞ (nonnegative) approximate identity on $\mathbf{R}^n$ relative to convolution. The usual requirements on an approximate identity $\{\rho_\epsilon\}$ are:

(i) $\rho_\epsilon(x)\to 0$ as $\epsilon\downarrow 0$, uniformly outside every neighbourhood of 0;
(ii) ρ_ϵ is integrable over $\mathbf{R}^n$ and $\int_{\mathbf{R}^n}\rho_\epsilon(x)dx=1$;

(iii) $\rho_\epsilon(x) \geq 0$ throughout $\mathbf{R}^n$.

Properties (i)–(iii) readily imply that for any continuous function f on $\mathbf{R}^n$ of compact support, the CONVOLUTION $f \star \rho_\epsilon$ converges to f as $\epsilon \downarrow 0$:

$$(f \star \rho_\epsilon)(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}^n} f(x-y)\rho_\epsilon(y)dy \to f(x) = \int_{\mathbf{R}^n} f(x)\rho_\epsilon(y)dy$$

uniformly on $\mathbf{R}^n$.

[An approximation ρ_ϵ to the identity may be considered as an approximation to the so-called delta function or delta distribution δ. The latter acts as the identity relative to convolution: $\delta \star f = f \star \delta = f$, cf. exercise 11.5]

Proposition 3.32. *To any set S in $\mathbf{R}^n$ and any $\epsilon > 0$ there is a C^∞ " CUTOFF FUNCTION" ω on $\mathbf{R}^n$ which is equal to 1 on S and equal to 0 at all points of $\mathbf{R}^n$ at a distance $\geq 2\epsilon$ from S. One may require that $0 \leq \omega \leq 1$.*

PROOF. We will obtain ω as the convolution of the characteristic function of a neighbourhood of S with the C^∞ approximation ρ_ϵ to the identity of $(3a, b)$ [taking $\epsilon > 0$ fixed]. Let S_ϵ denote the ϵ-neighbourhood of S, that is, the set of all points $x \in \mathbf{R}^n$ at a distance $< \epsilon$ from S [S_ϵ is an open set containing S]. Let χ_ϵ be the characteristic function of S_ϵ, that is, χ_ϵ equals 1 on S_ϵ and 0 elsewhere. We define ω as the convolution of χ_ϵ and ρ_ϵ :

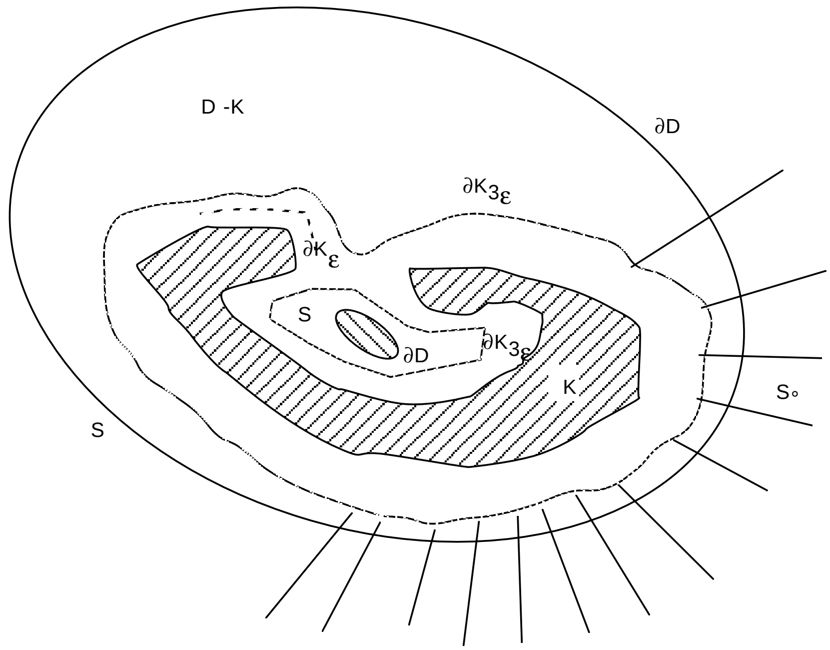
$$\begin{aligned}\omega(x) = (\chi_\epsilon \star \rho_\epsilon)(x) &= \int_{\mathbf{R}^n} \chi_\epsilon(x-y)\rho_\epsilon(y)dy = \int_{B(0,\epsilon)} \chi_\epsilon(x-y)\rho_\epsilon(y)dy \\ &= \int_{\mathbf{R}^n} \chi_\epsilon(y)\rho_\epsilon(x-y)dy = \int_{S_\epsilon} \rho_\epsilon(x-y)dy.\end{aligned} \tag{3c}$$

First taking $x \in S$, the second integral shows that $\omega(x) = 1$: the points $x - y$ will belong to S_ϵ for all $y \in B(0, \epsilon)$, so that $\chi_\epsilon(x - y) = 1$ throughout $B(0, \epsilon)$ and $\omega(x) = \int_{B(0,\epsilon)} \rho_\epsilon(y)dy = 1$. Next taking x outside $S_{2\epsilon}$, the same integral shows that now $\omega(x) = 0$: this time, all points $x - y$ with $|y| < \epsilon$ lie outside S_ϵ. Furthermore, since $\rho_\epsilon \geq 0$ we have $0 \leq \omega(x) \leq \int \rho_\epsilon = 1$ throughout $\mathbf{R}^n$.

In order to prove that ω is of class C^∞ one may use the last integral in $(3c)$. For x in the vicinity of a point a, one need only integrate over the intersection of S_ϵ with some fixed ball $B(a, r)$, hence over a bounded set independent of x. The existence and continuity of the partial derivatives $\partial\omega/\partial x_1$, etc. may now be established by the method of formula $(1h)$ [cf. exercise 3.12; the partial derivatives of ρ_ϵ are uniformly continuous on $\mathbf{R}^n$]. Repeated differentiation under the integral sign will show that ω has continuous partial derivatives of all orders.

3.4 Use of the $\overline{\partial}$ equation for analytic continuation. We can now prove the Hartogs-Osgood-Brown continuation theorem:

Theorem 3.41. *Let D be a connected domain in $\mathbf{C}^n$ with $n \geq 2$ and let K be a compact subset of D such that $D - K$ is connected. Then every holomorphic function f on $D - K$ has an analytic continuation to D.*



PROOF. Let f be any function in $\mathcal{O}(D-K)$.

(i) We first construct a C^∞ *approximate solution* g to the continuation problem. Since we do not know how f behaves near K, we will start with the values of f at some distance from K. For any $\rho > 0$, let K_ρ denote the ρ-neighbourhood of K. Choosing $0 < \epsilon < d(K, \partial D)/3$, we set $S = \mathbf{C}^n - \overline{K}_{3\epsilon}$, so that the open set S contains the whole boundary ∂D. For later use, the unbounded component of S is called S_∞.

Now select a C^∞ cutoff function ω on $\mathbf{C}^n \sim \mathbf{R}^{2n}$ which is equal to 1 on S and equal to 0 on K_ϵ. [Use Proposition 3.32 with $2n$ instead of n.] We then define g on D by setting

$$g = \begin{cases} \omega f & \text{on } \; D-K, \\ 0 & \text{on } \; K. \end{cases}$$

This g is of class C^∞ [because ω vanishes near ∂K] and

$$g = f \text{ on } D \cap S$$

[where $\omega = 1$]. Thus g furnishes a C^∞ *continuation* to D of the restriction of f to $D \cap S$.

(ii) We will modify g so as to obtain an *analytic continuation*

$$h = g - u$$

of f. By the uniqueness theorem 1.54, it will be enough to require that h be holomorphic on D and equal to f on a subdomain of $D-K$; here $D \cap S_\infty$ will work best. The correction term u then has to vanish on $D \cap S_\infty$ [where $g = f$] and it must make $\overline{\partial} h = 0$. Hence u must solve the $\overline{\partial}$ problem

$$\overline{\partial} u = \overline{\partial} g \text{ on } D, \;\; u = 0 \text{ on } D \cap S_\infty. \tag{4a}$$

One may extend g to a C^∞ form v on $\mathbf{C}^n$ by setting

$$v = \begin{cases} \overline{\partial} g & \text{on } D \\ 0 & \text{on } \mathbf{C}^n - D; \end{cases} \tag{4b}$$

indeed, $\overline{\partial} g = \overline{\partial} f = 0$ on $D \cap S$ and hence near ∂D. The $(0,1)$-form v of course satisfies the integrability conditions $\partial v_k / \partial \overline{z}_j = \partial v_j / \partial \overline{z}_k$. Its support belongs to $\mathbf{C}^n - S = \overline{K}_{3\epsilon}$ which is part of the compact set $E = \mathbf{C}^n - S_\infty$.

We now take for u the C^∞ solution of the extended $\overline{\partial}$ problem

$$\overline{\partial} u = v \text{ on } \mathbf{C}^n, \;\; u = 0 \text{ on } S_\infty = E^c.$$

[Existence and uniqueness of u are assured by Theorem 3.21.] Then the function $h = g - u$ will be holomorphic on D by $(4a, b)$. Being equal to $g = f$ on $D \cap S_\infty$, h will be equal to f throughout the connected domain $D - K$. Thus h provides the desired analytic continuation of f to D.

REMARK. Another proof of Theorem 3.41 may be obtained by means of the integral formula of Martinelli and Bochner, see Section 10.7.

3.5 Partial derivatives lemma and recessed-edge theorem. The following special case of the partial derivatives lemma suffices for most applications. For the general case and for a proof, see Section 8.7.

Lemma 3.51. *For any nonempty open subset E of the unit sphere S^{n-1} in $\mathbf{R}^n$, there exists a constant $\beta = \beta(E) > 0$ such that for every C^∞ function f in a neighbourhood of a point $a \in \mathbf{R}^n$ and every integer $m \geq 0$,*

$$\max_{|\alpha|=m} \frac{1}{\alpha!} |D_x^\alpha f(a)| \leq \sup_{\xi \in E} \frac{1}{m!} \left| \left(\frac{d}{dt} \right)^m f(a + t\xi) \big|_{t=0} \right| \Big/ \beta^m.$$

We will use the Lemma to prove an interesting result on analytic continuation which goes back to Behnke and Kneser, cf. [Kneser 1932]. Let Ω be a connected domain in $\mathbf{C}^n \sim \mathbf{R}^{2n}$ with $n \geq 2$ and let $X \subset \Omega$ be the intersection of two real hypersurfaces $V : \varphi = 0$ and $W : \psi = 0$, with φ and ψ of class $C^1(\Omega)$, $\operatorname{grad} \varphi \neq 0$ on V, $\operatorname{grad} \psi \neq 0$ on W. The hypersurface V will divide Ω into two parts, one where $\varphi > 0$ and one where $\varphi < 0$; similarly for W. We suppose that $\operatorname{grad} \varphi$ and $\operatorname{grad} \psi$ are linearly independent at each point of X, so that the real tangent hyperplanes to V and W are different along X. We finally set

$$\Omega_0 = \{ z = x + iy \in \Omega : \min[\varphi(x,y), \psi(x,y)] < 0 \}$$

(fig 3.5). For Ω_0, X is a *"recessed edge"*.

Theorem 3.52. (i) *Suppose that the vectors*

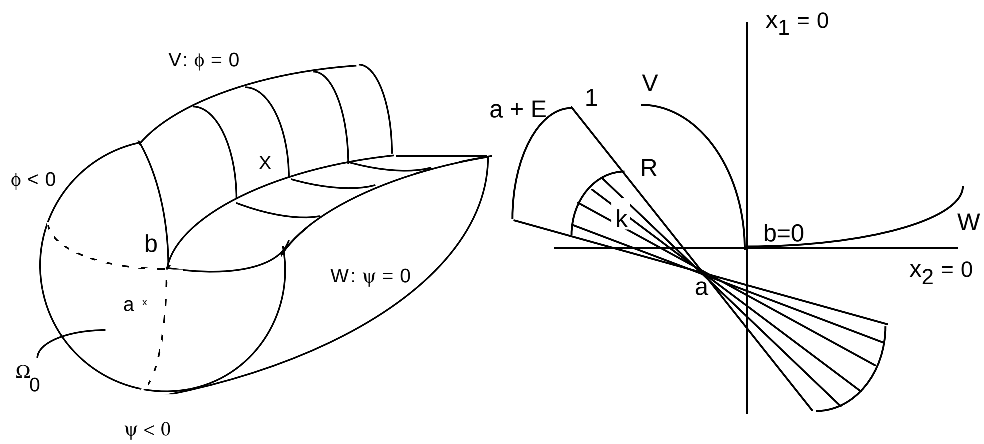
$$p = \left(\frac{\partial \varphi}{\partial z_1}, \ldots, \frac{\partial \varphi}{\partial z_n} \right) \text{ and } q = \left(\frac{\partial \psi}{\partial z_1}, \ldots, \frac{\partial \psi}{\partial z_n} \right)$$

are linearly independent over $\mathbf{C}$ *at the point* $b \in X$*, so that the hypersurfaces* V *and* W *even have different* COMPLEX *tangent hyperplanes at* b [*cf. Example 1.21*]. *Then there is a neighbourhood of* b *to which all holomorphic functions* f *on* Ω_0 *can be continued analytically.*

(ii) *If for every point* $b \in X$ *there is a holomorphic function on* Ω_0 *which can not be continued analytically to a neighbourhood of* b*, then* X *is a* COMPLEX ANALYTIC HYPERSURFACE. *More precisely, after appropriate complex linear coordinate transformation,* X *has local representation* $z_n = g(z_1, \ldots, z_{n-1})$ *with holomorphic* g.

PROOF of part (i). Since $p_j = \partial\varphi/\partial z_j = \frac{1}{2}\partial\varphi/\partial x_j - \frac{1}{2}i\partial\varphi/\partial y_j$, etc., the real tangent hyperplanes to V and W at 0 have the respective representations

$$\mathrm{Re}\,(p_1 z_1 + \ldots + p_n z_n) = 0, \;\; \mathrm{Re}\,(q_1 z_1 + \ldots + q_n z_n) = 0$$



[cf. 1.21]. The vectors p and q being linearly independent, there is a $1-1$ complex linear coordinate transformation of the form

$$z_1' = p_1 z_1 + \ldots + p_n z_n, \;\; z_2' = q_1 z_1 + \ldots + q_n z_n, \;\; z_3' = \ldots, \ldots, \;\; z_n' = \ldots .$$

Carrying out such a transformation, it may be assumed that the real tangent hyperplanes to V and W are given by the equations

$$x_1 = 0, \;\; x_2 = 0$$

and that the point

$$a = (a_1, a_2, a_3, \ldots, a_n) = (-\epsilon, -\epsilon, 0, \ldots, 0)$$

of $\mathbf{C}^n$ belongs to Ω_0 for all small $\epsilon > 0$.

Geometric considerations (cf. fig 3.6) next show that there is a constant $R > 0$ such that for all small $\epsilon > 0$, Ω_0 contains the compact set

$$K = \overline{B}(a, R) \cap \{z \in \mathbf{C}^n : x_2 + \epsilon = (x_1 + \epsilon)\tan\theta, \; 5\pi/8 \leq \theta \leq 7\pi/8\}.$$

Observe that the corresponding real "directions" or unit vectors $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with $\xi_2 = \xi_1 \tan\theta,\ 5\pi/8 < \theta < 7\pi/8$ form a nonempty open subset E of the unit sphere S^{n-1} in $\mathbf{R}^n$.

Now let f be in $\mathcal{O}(\Omega_0)$, $\sup_K |f| = C = C(f, K)$. Any complex line L through a of the form $z = a + w\xi,\ \xi \in E$ intersects K in a disc $\overline{\Delta}$ of radius R. The restriction of f to $\overline{\Delta}$ is represented by the function

$$h(w) = f(a + w\xi),\ \ |w| \le R.$$

Applying the Cauchy inequalities to h on $\overline{\Delta}_1(0, R)$ we find that

$$\frac{1}{m!}\left|\left(\frac{d}{dw}\right)^m f(a + w\xi)\big|_{w=0}\right| \le \frac{C}{R^m},\ \ \forall \xi \in E,\ \ \forall m \in \mathbf{N}_0. \tag{5a}$$

Hence by the partial derivatives lemma, considering the restriction of f to $\Omega_0 \cap \mathbf{R}^n$ and taking $w = t \in \mathbf{R}$,

$$\max_{|\alpha|=m} \frac{1}{\alpha!}\big|D_x^\alpha f(a)\big| \le C/(\beta R)^m,\ \ \forall m;\ \ \beta = \beta(E) > 0. \tag{5b}$$

Since f is analytic on Ω_0, the derivatives $D_z^\alpha f(a)$ are equal to the derivatives $D_x^\alpha f(a)$. Thus around a,

$$f(z) = \sum c_\alpha (z - a)^\alpha = \sum_{\alpha \ge 0} \frac{1}{\alpha!} D_x^\alpha f(a)\,(z_1 - a_1)^{\alpha_1} \dots (z_n - a_n)^{\alpha_n}.$$

By $(5b)$, the power series will converge at every point z with $|z_j - a_j| < \beta R,\ \forall j$:

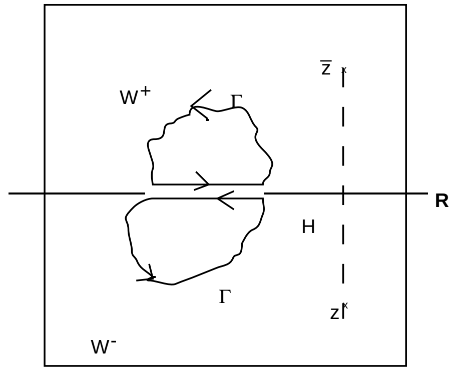
$$\big|c_\alpha (z - a)^\alpha\big| \le C \left|\frac{z_1 - a_1}{\beta R}\right|^{\alpha_1} \dots \left|\frac{z_n - a_n}{\beta R}\right|^{\alpha_n},\ \ \forall \alpha.$$

Conclusion: f has an analytic continuation to the polydisc $\Delta(a, \beta R)$ and in fact, letting $\epsilon \downarrow 0$ so that $a \to 0$, to the polydisc $\Delta(0, \beta R)$.

REMARK on part (ii). The crucial observation is that under the hypothesis of part (ii), the complex tangent hyperplanes to V and W must coincide along X. As a consequence, the $(2n-2)$-dimensional real tangent spaces to X are complex hyperplanes. This being the case, one may conclude that X is complex analytic ("Levi-Civita lemma"). For more detailed indications of the proof, see exercise 3.19.

3.6 The edge-of-the-wedge theorem. We will discuss a simple version for $\mathbf{C}^n$ and begin with the special case $n = 1$ in order to bring out more clearly why the theorem is so remarkable for $n \ge 2$. Let W^+ be a(n open) rectangular domain in the upper half-plane in $\mathbf{C}$, of which one side falls along the real axis. The reflected rectangle in the lower half-plane is called W^- and the (open) common boundary segment is called H (fig 3.7). We finally set

$$W = W^+ \cup H \cup W^-.$$



For $n = 1$ our simple edge-of-the-wedge theorem reduces to the following well-known facts:

(i) (*A segment as removable singularity set*). Any continuous function f on W which is holomorphic on W^+ and on W^- is actually holomorphic on W.

[The integral of fdz along any piecewise smooth simple closed curve Γ in W will be zero, cf. fig 3.7, hence f is analytic on W. One may appeal to Morera's theorem here, or observe directly that f will have a well-defined primitive $F(z) = \int_a^z f(\zeta)d\zeta$ on W. Since F is differentiable in the complex sense, it is analytic, hence so is $f = F'$.]

(ii) (*Analytic continuation by Schwarz reflection*). Any continuous function g on $W^+ \cup H$ which is holomorphic on W^+ and real-valued on H has an analytic continuation to W. For $z \in W^-$ the continuation is given by reflection: $g(z) = \overline{g(\overline{z})}$.

[Apply part (i) to the extended function g. The condition that g (or f in part (i)) be continuous at the points of H can be weakened, cf. [Carleman] and Remarks 3.62.

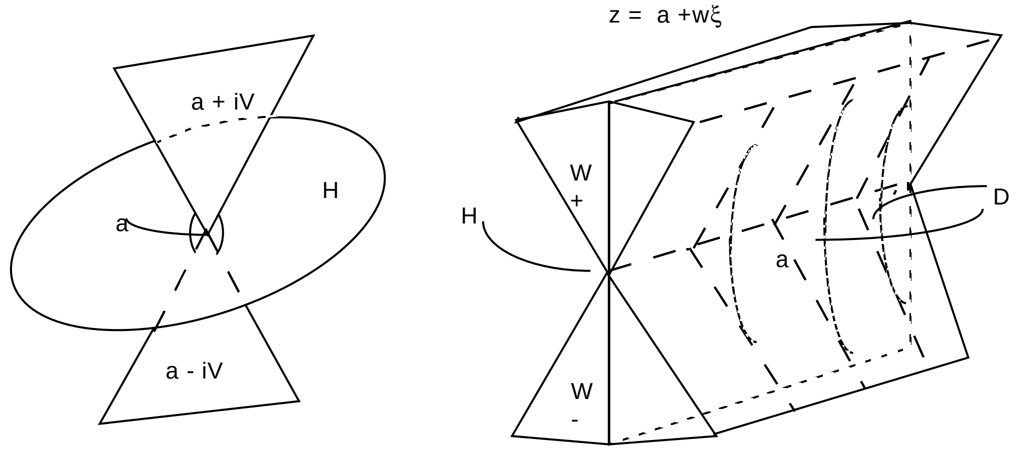
THE CASE OF $\mathbf{C}^n$ ($n \geq 1$). Let H (for "horizontal") be a connected domain in the real space $\mathbf{R}^n = \mathbf{R}^n + i0$ in $\mathbf{C}^n$ and let V (for "vertical") be a (usually truncated) connected open cone with vertex at the origin in (another) $\mathbf{R}^n$. To get a simple picture, we assume that $\overline{V}$ and $-\overline{V}$ meet only at the origin. To H and V we associate two (connected) domains in $\mathbf{C}^n$ as follows:

$$W^+ = H + iV = \{z = x + iy \in \mathbf{C}^n : x \in H,\ y \in V\},\ \ W^- = H - iV$$

("wedges" with common "edge" H). We again define

$$W = W^+ \cup H \cup W^-.$$

Observe that W is not an open set when $n \geq 2$: W does *not* contain a $\mathbf{C}^n$ *neighbourhood of any point* $a \in H$ (fig 3.8). For $n \geq 2$, the set H is a peculiarly small part of the boundary of W^+: it only has real dimension n instead of $2n - 1$, as one would expect of a "normal" piece of the boundary of a $\mathbf{C}^n$ domain. For the purpose of illustration when $n = 2$, only one line segment of H has been drawn in fig 3.9. In that way one clearly sees two wedges with a common edge.



Theorem 3.61. *Let H, V, W^+, W^- and W be as described above. Then there exists a connected domain D in $\mathbf{C}^n$ containing W such that the following is true:*

(i) *Any continuous function f on W which is holomorphic on W^+ and on W^- has an analytic continuation to D;*

(ii) *Any continuous function g on $W^+ \cup H$ which is holomorphic on W^+ and real-valued on H has an analytic continuation to D; for $z \in W^-$, the continuation is given by reflection: $g(z) = \overline{g(\overline{z})}$.*

REMARKS 3.62. For every point $a \in H$ there will be a fixed $\mathbf{C}^n$ neighbourhood to which all functions f and g as in the Theorem can be analytically continued. Actually, the hypothesis that $f \in \mathcal{O}(W^+ \cup W^-)$ has a continuous extension to W can be weakened considerably. It is sufficient if for $y \to 0$ in V, the function

$$F_y(x) = f(x+iy) - f(x-iy), \;\; x \in H \subset \mathbf{R}^n$$

tends to 0 in *weak* or *distributional* sense ([Bogolyubov-Vladimirov], cf. [Rudin 4], [Korevaar 1991]; weak convergence is defined in Chapter 11). It even suffices to have convergence here in the sense of hyperfunctions [De Roever]. There are several forms of the edge-of-the-wedge theorem and many different proofs have been given, cf. [Rudin 4] and [Shabat].

PROOF of the Theorem. One need only consider part (i) since part (ii) will follow as in the case $n = 1$. It is sufficient to show that there is a polydisc $\Delta(a, \rho)$, $\rho = \rho_a$ around each point $a \in H$ to which all functions f as in part (i) can be continued analytically. We focus on one such function f. The key observation will be that W contains closed squares $Q_\xi(a)$ of constant size with center a in a substantial family of complex lines (cf. fig 3.9):

$$z = a + w\xi, \;\; \xi \in E \subset S^{n-1} \subset \mathbf{R}^n, \;\; w = u + iv \in \mathbf{C}.$$

By the one-variable result, the (continuous) restrictions $f|Q_\xi(a)$ are analytic. The Cauchy inequalities now imply bounds on certain directional derivatives of $f|H$. Such bounds (at and around a) and the partial derivatives lemma will ensure that $f|H$ is locally represented by a power series $\sum c_\alpha (x-a)^\alpha$; replacing x by z one obtains the desired analytic continuation.

Let us first look at V. The directions from 0 that fall within the cone V determine a nonempty open subset E' of the unit sphere S^{n-1}. We choose some open subset E with compact closure in E'. There will then be a number $R > 0$ such that $V \cup 0$ contains the closed truncated cone

$$V_0 = \{y \in \mathbf{R}^n : y = v\xi, \ \xi \in \overline{E}, \ 0 \le v \le R\}.$$

For $a \in H$, the domain $H \subset \mathbf{R}^n$ contains the real ball $U(a,d) : |x-a| < d = d(a, \partial H)$. Choosing $R < d$, our set W will contain the squares

$$Q_\xi(a) = \{z \in \mathbf{C}^n : z = a + w\xi, \ \xi \in \overline{E}, \ -R \le u, v \le R\}.$$

The union of these squares for ξ running over $\overline{E}$ is a compact subset of W [contained in $\overline{U}(a,R) \pm iV_0$], on which $|f|$ will be bounded, say by $C = C_f$.

The restriction of f to $Q_\xi(a)$ is represented by

$$h(w) = f(a + w\xi), \ -R \le u, v \le R.$$

The function h is continuous on its square and analytic for $v = \operatorname{Im} w \neq 0$, hence it is analytic on the whole square. Thus as in Section 3.5, the Cauchy inequalities give a family of inequalities ($5a$). Here C and R (may) depend on a, but if we fix b in H and restrict a to a small neighbourhood H_0 of b in H, we may take C_f and R constant.

The inequalities ($5a$), with $w = t \in \mathbf{R}$ and a running over H_0, will imply that $f_0 = f|H_0$ is of class C^∞ around b and that it has a holomorphic extension $\tilde{f}$ to a $\mathbf{C}^n$ neighbourhood of b. We sketch a proof; another proof may be derived from exercise 3.23. Choose a convex neighbourhood H_1 of b in H_0 such that $d(H_1, \partial H_0) = \delta > 0$. Now "regularize" f_0 through convolution with the members of the approximate identity $\{\rho_\epsilon\}$ of $(3a,b)$, $0 < \epsilon < \delta$. One thus obtains C^∞ functions $f_\epsilon = f_0 \star \rho_\epsilon$ on H_1 whose derivatives in the directions $\xi \in E$ satisfy the inequalities ($5a$) (with $w = t \in \mathbf{R}$) for all $a \in H_1$ and all $\epsilon \in (0,\delta)$. By the partial derivatives lemma, the derivatives D_x^α of the functions f_ϵ will then satisfy the inequalities ($5b$) for every $a \in H_1$. Taylor's formula with remainder next shows that the real Taylor series for f_ϵ with center $a \in H_1$ converges to f_ϵ on $\Delta(a, \beta R) \cap H_1$. Complexifying such Taylor series for f_ϵ, one obtains holomorphic extensions $\tilde{f}_\epsilon$ of the functions f_ϵ to the union D_1 of the polydiscs $\Delta(a, \beta R)$ with $a \in H_1$. [H_1 is a set of uniqueness for D_1, cf. exercise 1.17.] The family $\{\tilde{f}_\epsilon\}$, $0 < \epsilon < \delta$ will be locally bounded on D_1 and $\tilde{f}_\epsilon \to f_0$ on H_1 as $\epsilon \downarrow 0$. Thus by Vitali's theorem 1.74, the functions $\tilde{f}_\epsilon$ converge to a holomorphic extension $\tilde{f}$ of f_0 on D_1.

Since $\tilde{f} = f_0 = f$ on H_1, $\tilde{f}$ will also agree with f on the intersection of D_1 with any complex line $z = a + w\xi$ for which $a \in H_1$ and $\xi \in E$. It follows that $\tilde{f} = f$ throughout open subsets of W^+ and W^-. Conclusion: $\tilde{f}$ provides an analytic continuation of f to D_1 and in particular, to the polydisc $\Delta(a, \beta R)$ for every $a \in H_1$.

Exercises

3.1. (Direct verification of the representation $(1e)$) Let a be fixed and z variable in $\mathbf{C}$, $z = a + re^{i\theta}$. Let f be a C^1 function on $\mathbf{C}$ of compact support. Prove that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}e^{i\theta}\left(\frac{\partial f}{\partial r} - \frac{1}{ir}\frac{\partial f}{\partial \theta}\right)$$

and deduce that for $\epsilon \downarrow 0$:

$$\int_{|z-a|>\epsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} dxdy = -\frac{1}{2}\int_{-\pi}^{\pi} f(a+\epsilon e^{i\theta})d\theta \to -\pi f(a).$$

3.2. Extend formula $(1e)$ to arbitrary functions f in $C^1(\mathbf{C})$ which tend to 0 as $|z| \to \infty$ while $|\partial f/\partial \overline{z}|/|z|$ is integrable over $\mathbf{C}$.

3.3. Why can not one calculate $\partial u/\partial \overline{z}$ by differentiation under the integral sign in formula $(1f)$ as it stands?

3.4. Prove a formula for $(\partial u/\partial y)(a)$ analogous to $(1i)$, starting with an appropriate analog to $(1h)$.

3.5. Let $v \in C(\mathbf{C})$ be of compact support and let u be its Cauchy-Green transform $(1f)$. Prove that u is holomorphic outside $\operatorname{supp} v$. Expand u in a Laurent series around ∞ to obtain conditions on the "moments" $\int_{\mathbf{C}} v(\zeta)\zeta^k d\xi d\eta$ which are necessary and sufficient in order that u vanish on a neighbourhood of ∞.

3.6. Let v be a continuous function on $\mathbf{C}$ of compact support and let u be its Cauchy-Green transform $(1f)$. Prove that u is of class $\operatorname{Lip}\alpha$ for each $\alpha \in (0,1)$ or even better, that

$$|u(z+h) - u(z)| \le M|h| \log (1/|h|)$$

for some constant M and all $z \in \mathbf{C}$, all $|h| \le \frac{1}{2}$. [Take $0 < |h| \le \frac{1}{2}$, $\zeta \in \operatorname{supp} v \subset B(0,R)$ for some $R \ge 1$, $|z| \le 2R$. Substituting $\zeta - z = h\zeta'$, the variable ζ' may be restricted to the disc $B(0, 3R/|h|)$.]

3.7. Let $D \subset \mathbf{C}$ be a domain as in Proposition 3.11 and let v be of class $C^1(\overline{D})$. Suppose one knows that the equation $\partial u/\partial \overline{z} = v$ has a solution f on D which extends to a C^1 function on $\overline{D}$. Prove that

$$u(z) \stackrel{\text{def}}{=} -\frac{1}{\pi}\int_D \frac{v(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta$$

is also a C^1 solution on D.

3.8. (Continuation) Let D be as in Proposition 3.11 and let v be continuous on $\overline{D}$. Prove that the Cauchy-Green transform u of v on D (exercise 3.7) is also continuous and that it provides a *weak solution* of the equation $\partial u/\partial \overline{z} = v$ on D. That is,

$$\langle \frac{\partial u}{\partial \overline{z}}, \varphi\rangle \stackrel{\text{def}}{=} -\langle u, \frac{\partial \varphi}{\partial \overline{z}}\rangle \stackrel{\text{def}}{=} -\int_D u \frac{\partial \varphi}{\partial \overline{z}} dxdy$$

is equal to

$$\langle v, \varphi \rangle \stackrel{\text{def}}{=} \int_D v\varphi \, dxdy$$

for all test functions φ on D (all C^∞ functions φ of compact support in D). [If v belongs to $C^1(\overline{D})$ the function u will be an ordinary solution, cf. Section 11.2.]

3.9. Verify that the function $u(z)$ in formula (2c), with v as in Proposition 3.21, is of class C^p as a function of $x_1, y_1, \ldots, x_n, y_n$. Next use the method of exercise 3.6 to show that the partial derivatives of u of order p are of class $\mathrm{Lip}\,\alpha$, $\forall \alpha \in (0,1)$.

3.10. Verify that the functions σ and τ introduced in Section 3.3 are of class C^∞ on $\mathbf{R}$.

3.11. Verify that the functions ρ_ϵ of (3a, b) constitute a C^∞ approximate identity on $\mathbf{R}^n$ as $\epsilon \downarrow 0$.

3.12. Let ω be the cutoff function of (3c) and let e_1 denote the unit vector in the x_1 direction. Prove that

$$\frac{\partial \omega}{\partial x_1}(a) = \lim_{h \to 0} \frac{\omega(a + he_1) - \omega(a)}{h} \text{ exists and } = \int_{S_\epsilon} \frac{\partial \rho_\epsilon}{\partial x_1}(a - y)dy.$$

3.13. Let u be continuous on $\mathbf{R}^n$ and let $\{\rho_\epsilon\}$ be a C^∞ approximate identity with $\mathrm{supp}\,\rho_\epsilon \subset \overline{B}(0,\epsilon)$. Prove that the *"regularization"* $u_\epsilon = u \star \rho_\epsilon$ is of class C^∞ and that $u_\epsilon \to u$ as $\epsilon \downarrow 0$, uniformly on every compact subset of $\mathbf{R}^n$.

3.14. (*Weak solutions of* $\overline{D}u = 0$ *are holomorphic*) Let u be continuous on $\mathbf{C}$ and such that $\partial u/\partial \overline{z} = 0$ in the weak sense, cf. exercise 3.8. Prove that u is holomorphic. [Show first that $\partial u_\epsilon/\partial \overline{z} = 0$, where u_ϵ is as in exercise 3.13.]

3.15. Show by an example that there is no Hartogs-Osgood-Brown continuation theorem for $n = 1$. Where does the proof of Theorem 3.41 break down when $n = 1$?

3.16. Let D be a simply connected domain in $\mathbf{C}$, $K \subset D$ a compact subset such that $D - K$ is connected. Prove that a holomorphic function f on $D - K$ can be continued analytically to D if and only if $\int_\Gamma \{f(\zeta)/(\zeta - z)\}d\zeta = 0$ for some [and then for every] piecewise smooth simple closed curve Γ around K in $D - K$ and for all z outside Γ.

3.17. (Continuation) Prove that the following moment conditions are also necessary and sufficient for the possibility of analytic continuation of f across K : $\int_\Gamma f(\zeta)\zeta^k d\zeta = 0$, $\forall k \geq 0$ for some curve Γ as above.

3.18. Prove that every holomorphic function on the domain $D = \{z = x + iy \in \mathbf{C}^2 : \min(x_1, x_2) < 0\}$ has an analytic continuation to all of $\mathbf{C}^2$.

3.19. (*Proof of Theorem 3.52 part* (ii)) Let V, W, Ω_0 and X satisfy the hypotheses of Theorem 3.52 part (ii). Verify the following assertions:
   (i) At every point $b \in X$, the hypersurfaces V and W have the same complex tangent hyperplane.
   (ii) The real tangent spaces to X are complex hyperplanes.

(iii) The real tangent hyperplanes to V and W at $b \in X$ being different, one has $\partial\psi/\partial z_j = \lambda\partial\varphi/\partial z_j$ at b, $j = 1, \ldots, n$, with $\lambda = \lambda(b)$ nonreal.

(iv) Supposing from here on that $\partial\varphi/\partial z_n \neq 0$ at the point $b \in X$, the vectors $(\partial\varphi/\partial x_n,\ \partial\varphi/\partial y_n)$ and $(\partial\psi/\partial x_n,\ \partial\psi/\partial y_n)$ are linearly independent at b.

(v) By real analysis, X has a local representation $z_n = x_n + iy_n = g(z') = g(z_1, \ldots, z_{n-1)}$ around b. [Cf. Remarks 5.13.]

(vi) The function g satisfies the Cauchy-Riemann equations around b'. [Cf. assertion (ii) above.]

3.20. Let $H = \mathbf{R}^2$, let V be the positive "octant" $\{y_1 > 0,\ y_2 > 0\}$ of (another) $\mathbf{R}^2$ and set $W = (H + iV) \cup H \cup (H - iV)$ in $\mathbf{C}^2$. Which points $z = a + iy$ near $a \in H$ are outside W? [Cf. fig 3.8.]

3.21. (Continuation) Prove that any function f which is continuous on W and analytic on W^+ and W^- has an analytic continuation to all of $\mathbf{C}^2$.

3.22. Let f_0 be a continuous function on the domain $H_0 \subset \mathbf{R}^n$ which possesses derivatives of all orders in the directions $\xi \in E$ throughout H_0. Suppose that these derivatives satisfy the inequalities $(5a)$ with $w = t \in \mathbf{R}$ at all points $a \in H_0$. Let H_1 be a subdomain of H_0 such that $d(H_1, \partial H_0) = \delta > 0$ and let $\{\rho_\epsilon\}$ be the approximate identity of $(3a, b)$, with $0 < \epsilon < \delta$. Prove that the regularizations $f_\epsilon = f_0 \star \rho_\epsilon$ satisfy the inequalities $(5a)$ (with $w = t$) at every point $a \in H_1$.

3.23. Let f be a continuous function on a domain H in $\mathbf{R}^n$ such that for n linearly independent unit vectors ξ and every $m \geq 1$, the directional derivatives $(d/dt)^m f(x + t\xi)|_{t=0}$ exist and are bounded functions on a neighbourhood H_0 of each point $b \in H$. Prove that f is of class C^∞. [By a linear coordinate transformation it may be assumed that the unit vectors ξ are equal to $e_1, \ldots, e_n$. Multiplying f by a C^∞ cutoff function with support in H_0 which is equal to 1 around b, one may assume that f has its support in the hypercube $-\pi < x_1, \ldots, x_n < \pi$. Taking $n = 2$ for a start, one knows that $D_1^m f$ and $D_2^m f$ exist and are bounded for $m = 1, 2, \ldots$. Introducing the Fourier series $\sum c_{pq} \exp\{i(px_1 + qx_2)\}$ for f on the square $-\pi < x_1,\ x_2 < \pi$, one may conclude that the multiple sequence $\{(|p|^m + |q|^m)c_{pq}\}$, $(p, q) \in \mathbf{Z}^2$ is bounded for each m. Deduce that the (formal) series for $D^\alpha f$ is uniformly convergent for every α, hence ... .]

3.24. (Alternative proof of 3.61) We adopt the notation of 3.6..

i. Show that we may assume that H contains the cube $|x_i| < 6$ and that V contains the truncated cone $0 < v_i < 6$ and that assuming this, it suffices to show that f extends to the unit polydisc $\Delta = \Delta(0, 1)$.

ii. Let $c = \sqrt{2} - 1$ and let $\phi(w, \lambda) = \frac{w + \lambda/c}{1 + c\lambda w}$. Check the following:

a. If $|\lambda| = 1$ or w is real, then $\mathrm{Im}\phi . \mathrm{Im}\lambda \geq 0$.

b. $|\phi| < 6$ for $|\lambda|, |w| < 1$.

c. $\phi(w, 0) = w$.

iii. Form

$$\Phi(z, \lambda) = (\phi(z_1, \lambda), \ldots, \phi(z_n, \lambda)$$

and consider $g_z(\lambda) = f(\Phi(z, \lambda))$. Show that g_z is well defined for $z \in H \cap \Delta$, $|\lambda| \leq 1$ and for $|\lambda| = 1$ and $z \in \Delta$. Show that for $z \in H \cap \Delta$, $g_z(\lambda)$ is analytic on

$|\lambda| < 1$. Use Corr.1.72 to see that

$$F(z) := \int_{-\pi}^{\pi} g_z(e^{i\theta}) \frac{d\theta}{2\pi}$$

is analytic on Δ.

iv. Show that for $z \in H \cap \Delta$ one has $F(z) = g_z(0) = f(z)$.

v. Show that F is an analytic extension of f to Δ.

CHAPTER 4

Local structure of holomorphic functions Zero sets and singularity sets

We will study germs of holomorphic functions at a point a. These germs form a ring $\mathcal{O}_a$, addition and multiplication being defined by the like operations on representatives. One loosely speaks of the ring of holomorphic functions at a.

For the study of $\mathcal{O}_0$ in $\mathbf{C}^n$, it is customary to single out one of the variables. In the following this will be z_n; we denote $(z_1, \ldots, z_{n-1})$ by z', so that

$$z = (z_1, \ldots, z_{n-1}, z_n) = (z', z_n), \;\; z' \in \mathbf{C}^{n-1}, \;\; z_n \in \mathbf{C}.$$

We similarly split the radii of polydiscs $\Delta(0, r) \subset \mathbf{C}^n$:

$$r = (r_1, \ldots, r_{n-1}, r_n) = (r', r_n), \;\; r_j > 0.$$

In this context the origin of $\mathbf{C}^{n-1}$ will usually be called O'.

Suppose now that f is holomorphic in some unspecified neigbourhood of 0 in other words: $[f] \in \mathcal{O}_0$ and that $f(0) = 0$, $f \not\equiv 0$. In the case $n = 1$ the local structure of f and the local *zero set* Z_f are very simple: in a suitably small neighbourhood of 0, the function $f(z)$ can be written as $E(z)z^k$, where $k \geq 1$ and E is zero free in a neighbourhood of 0. In the case $n \geq 2$ the origin can not be an isolated zero of f, but the fundamental WEIERSTRASS PREPARATION THEOREM (Section 4.4) will furnish a related factorization. After an initial linear transformation which favors the variable z_n, one obtains a local representation

$$f(z) = E(z)W(z)$$

on some small neighborhood of 0, that is $[f] = [E][W]$ in $\mathcal{O}_0$. Here W is a so-called Weierstrass polynomial in z_n and E is zero free and holomorphic in some neighbourhood of the origin. This means: W is a polynomial in z_n with leading coefficient 1; the other coefficients are analytic in $z' = (z_1, \ldots, z_{n-1})$ near $0'$ and they vanish at $0'$. Around 0, W will have the same zero set as f and this fact prepares the way for further study of Z_f.

The detailed investigation of Z_W will be based on a study of the polynomial ring $\mathcal{O}'_0[z_n]$, where $\mathcal{O}'_0$ stands for the ring $\mathcal{O}_0$ in $\mathbf{C}^{n-1}$ (Sections 4.5, 4.6).

After we have obtained a good description of the zero set, it becomes possible to prove some results on removable singularities. We will also see that certain "thin" singularity sets are at the same time zero sets.

Normalization relative to z_n and a basic auxiliary result. Let f be holomorphic in a neighbourhood of the origin in $\mathbf{C}^n$. In the (absolutely convergent) power series $\sum c_\alpha z^\alpha$ for $f(z)$ around 0, we may collect terms of the same degree:

$$\begin{aligned} & f(z) = P_0(z) + P_1(z) + P_2(z) + \ldots, \\ (1a) \qquad & P_j(z) = \sum_{|\alpha|=j} c_\alpha z^\alpha \text{ homogeneous in } z_1, \ldots, z_n \text{ of degree } j. \end{aligned}$$

DEFINITION 4.11. The function f is said to *vanish* (exactly) *of order* $k \geq 1$ at the origin if

$$P_j \equiv 0, \ \ j = 0, \ldots, k-1; \ \ P_k \not\equiv 0.$$

An equivalent statement would be [cf. Section 1.5]:

$$\begin{cases} D^\alpha f(0) & = D_1^{\alpha_1} \ldots D_n^{\alpha_n} f(0) = 0, \quad \forall \alpha \ \text{ with } \ |\alpha| = \alpha_1 + \ldots + \alpha_n < k, \\ \\ D^\beta f(0) & \neq 0 \ \text{ for some } \ \beta \ \text{ with } \ |\beta| = k. \end{cases}$$

DEFINITION 4.12. The function f is said to be *normalized relative to* z_n at the origin if $f(0', z_n)$ does not vanish identically in a neighbourhood of $z_n = 0$. Such an f is said to *vanish* (exactly) *of order* k relative to z_n at the origin if $f(0', z_n)$ has a zero of order (exactly) k for $z_n = 0$. Equivalently:

$$D_n^j f(0) = 0, \ j = 0, \ldots, k-1; \ \ D_n^k f(0) \neq 0.$$

Corresponding definitions apply at a point $a \in \mathbf{C}^n$: one may consider $f(a+z)$ instead of $f(z)$. If f is normalized relative to z_n at 0 [so that $D_n^k f(0) \neq 0$, say], then f is also normalized relative to z_n at all points a close to 0.

EXAMPLE 4.13. The function $f(z) = z_1 z_2$ on $\mathbf{C}^2$ vanishes of order 2 at the origin. It is not normalized relative to z_2, but one can normalize it relative to the final variable by the substitution $z_1 = \zeta_1 + \zeta_2, \ z_2 = \zeta_2$. Then f goes over into the analytic function $g(\zeta) = \zeta_1 \zeta_2 + \zeta_2^2$ which vanishes of order 2 relative to ζ_2 at the origin.

Lemma 4.14. *Suppose f vanishes (exactly) of order k at the origin. Then one can always carry out a $1-1$ linear coordinate transformation $z = A\zeta$ in $\mathbf{C}^n$ to ensure that f vanishes of order (exactly) k at 0 relative to the (new) n-th coordinate.*

PROOF. Write f as a sum of homogeneous polynomials of different degree as in $(1a)$, so that $P_k \not\equiv 0$. Choose $b \neq 0$ such that $P_k(b) \neq 0$ and then construct an invertible $n \times n$ matrix A with n-th column b. Now put $z = A\zeta$ and set

$$g(\zeta) \stackrel{\text{def}}{=} f(A\zeta) = P_k(A\zeta) + P_{k+1}(A\zeta) + \ldots.$$

Observe that A times the (column) vector $(0, \ldots, 0, \zeta_n)$ equals $(b_1\zeta_n, \ldots, b_n\zeta_n)$, so that

$$\begin{aligned} g(0, \ldots, 0, \zeta_n) = f(b_1\zeta_n, \ldots, b_n\zeta_n) = P_k(b_1\zeta_n, \ldots, b_n\zeta_n) + \\ + P_{k+1}(\ldots) + \ldots = P_k(b)\zeta_n^k + P_{k+1}(b)\zeta_n^{k+1} + \ldots. \end{aligned}$$

Clearly $D_n^k g(0) \neq 0$.

Auxiliary Theorem 4.15. *Let f be holomorphic on the polydisc $\Delta(0,r) \subset \mathbf{C}^n$, $n \geq 2$ and suppose that f vanishes (exactly) of order k relative to z_n at the origin. Then there exist a smaller polydisc $\Delta(0,\rho)$:*

$$0 < \rho = (\rho_1, \ldots, \rho_{n-1}, \rho_n) = (\rho', \rho_n) < r = (r', r_n)$$

and a number $\epsilon > 0$ such that

$$(1b) \qquad f(0', z_n) \neq 0 \text{ for } 0 < |z_n| \leq \rho_n,$$

$$(1c) \qquad f(z', z_n) \neq 0 \text{ for } z' \in \Delta_{n-1}(0', \rho'), \;\; \rho_n - \epsilon < |z_n| < \rho_n + \epsilon (\leq r_n).$$

For any $z' \in \Delta(0', \rho')$, the function $g(z_n) = f(z', z_n)$ will have precisely k zeros in the disc $\Delta_1(0, \rho_n)$ (counting multiplicities).

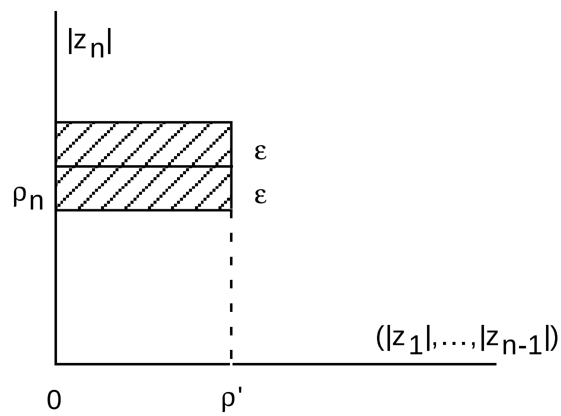
PROOF. The function $f(0', z_n)$ is holomorphic on the disc $\Delta_1(0, r_n)$ and it has a zero of order k at $z_n = 0$. Since the zeros of $f(0', z_n)$ are isolated, there exists a number $\rho_n \in (0, r_n)$ such that $f(0', z_n) \neq 0$ for $0 < |z_n| \leq \rho_n$.

The function $f(z', z_n)$ is holomorphic and hence continuous on a $\mathbf{C}^n$ neighbourhood of the circle

$$\gamma : \{z' = 0', \; |z_n| = \rho_n\}.$$

It is different from 0 on γ, hence $\neq 0$ on some $\mathbf{C}^n$ polydisc around each point $(0', w) \in \gamma$. Covering γ by a finite number of such polydiscs $\Delta_{n-1}(0', s') \times \Delta_1(w, s_n)$, we conclude that $f(z', z_n) \neq 0$ on a $\mathbf{C}^n$ neigbourhood of γ in $\Delta(0, r)$ of the form

$$\Delta_{n-1}(0', \rho') \times \{\rho_n - \epsilon < |z_n| < \rho_n + \epsilon\}.$$



We now fix $z' \in \Delta(0', \rho')$ for a moment. The function $g(w) = f(z', w)$ is holomorphic on the closed disc $\overline{\Delta}_1(0, \rho_n)$ and zero free on the circumference $C(0, \rho_n)$. The number of zeros $N_g = N(z')$ of g in $\Delta_1(0, \rho_n)$ (counting multiplicities) may be calculated with the aid of the residue theorem:

$$(1d) \qquad N(z') = N_g = \frac{1}{2\pi i} \int_{C(0,\rho_n)} \frac{g'(w)}{g(w)} dw = \frac{1}{2\pi i} \int_{C(0,\rho_n)} \frac{\partial f(z', w)/\partial w}{f(z', w)} dw.$$

[Cf. Section 1.8. For any holomorphic $h(w)$, the residue of hg'/g at a μ-fold zero w_0 of g will be $\mu h(w_0)$.]

With formula (1d) in hand, we let z' vary over $\Delta(0', \rho')$. The final integrand is continuous in (z, w) on $\Delta(0', \rho') \times C(0, \rho_n)$ [which is a subset of $\Delta(0, r)$], since the denominator $f(z', w)$ does not vanish there [see (1c)]. Furthermore, the integrand is holomorphic in z' for each w on $C(0, \rho_n)$. Applying the holomorphy theorem for integrals 1.72 [cf. also Section 2.6], it follows that $N(z')$ is holomorphic on $\Delta(0', \rho')$. Since $N(z')$ is integer-valued, it must be constant, hence

$$N(z') = N(0') = k,$$

the number of zeros of $f(0', w)$ or $f(0', z_n)$ in $\Delta_1(0, \rho_n)$ [always counting multiplicities].

4.2 An implicit function theorem. Let f, r and ρ be as in Auxiliary Theorem 4.15 so that in particular f is holomorphic on $\overline{\Delta}(0, \rho)$. Supposing that $k = 1$, the equation

$$g(w) = f(z', w) = 0 \quad [\text{with arbitrary fixed } z' \text{ in } \Delta(0', \rho')]$$

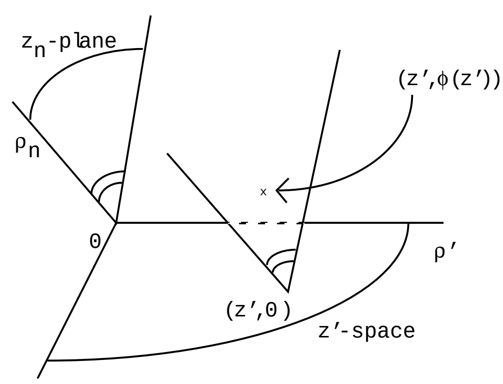
has precisely one root $w = w_0 = \varphi(z')$ inside the disc $\Delta_1(0, \rho_n)$ [and no root on the boundary $C(0, \rho_n)$]. With the aid of the residue theorem we can represent this root by an integral similar to (1d) :

$$\text{(2)} \qquad \varphi(z') = w_0 = \frac{1}{2\pi i} \int_{C(0,\rho_n)} w \frac{g'(w)}{g(w)} dw = \frac{1}{2\pi i} \int_{C(0,\rho_n)} w \frac{\partial f(z', w)/\partial w}{f(z', w)} dw.$$

Letting z' vary over $\Delta(0', \rho')$, this integral shows that $\varphi(z')$ is *holomorphic*, cf. the preceding proof. The result is important enough to be listed as a theorem:

Theorem 4.21 (IMPLICIT FUNCTION THEOREM)**.** *Let f be holomorphic on the polydisc $\Delta(0, r) \subset \mathbf{C}^n$ and suppose that f vanishes (exactly) of order* 1 *relative to z_n at the origin:*

$$f(0) = 0, \quad D_n f(0) \neq 0.$$



Then there exists $\rho = (\rho', \rho_n)$ with $0 < \rho < r$ such that on the polydisc $\Delta(0', \rho') \subset \mathbf{C}^{n-1}$, there is a unique holomorphic function $\varphi(z')$ with the following properties:

(i) $\varphi(0') = 0$,

(ii) $\varphi(z') \subset \Delta_1(0, \rho_n), \forall z' \in \Delta(0', \rho')$,

(iii) $f(z', z_n) = 0$ at a point $z \in \Delta(0, \rho)$ if and only if $z_n = \varphi(z')$ with $z' \in \Delta(0', \rho')$.

COROLLARY 4.22. Let f be holomorphic on $D \subset \mathbf{C}^n$ and vanish (exactly) of order 1 at the point $a \in D$. Then there is a neighbourhood of a in which the zero set Z_f is *homeomorphic to a domain in* $\mathbf{C}^{n-1}$. [In this case a is called a *regular point* of Z_f. Since homeomorphisms preserve dimension, the zero set has complex dimension $n-1$ or real dimension $2n-2$.]

Indeed, taking $a = 0$ and normalizing relative to z_n as in Lemma 4.14, we will have $f(0) = 0,\ D_n f(0) \neq 0$. By Theorem 4.21 there is then a polydisc $\Delta(0, \rho) \subset D$ in which Z_f has the form

$$Z_f \cap \Delta(0, \rho) = \{(z', \phi(z')) \in \mathbf{C}^n : z' \in \Delta(0', \rho')\}$$

with $\varphi \in \mathcal{O}(\Delta(0', \rho'))$. The correspondence $z' \leftrightarrow (z', \varphi(z'))$ between $\Delta(0', \rho')$ and Z_f (the graph of φ) in $\Delta(0, \rho)$ is $1-1$ and bicontinuous.

In the following sections we will investigate the zero set in the vicinity of a point where f vanishes of order > 1.

4.3 Weierstrass polynomials. Let f, r and ρ again be as in Auxiliary Theorem 4.15, so that in particular f is holomorphic on $\overline{\Delta}(0, \rho)$. Taking $k \geq 1$ arbitrary this time, the equation

$$(3a) \qquad g(w) = f(z', w) = 0 \quad \text{[with arbitrary fixed z}' \text{ in } \Delta(0', \rho')]$$

has precisely k roots inside the disc $\Delta_1(0, \rho_n)$, counting multiplicities [and no root on the boundary $C(0, \rho_n)$]. We may number the roots in some order or other:

$$(3b) \qquad w_1 = w_1(z'), \ldots, w_k = w_k(z'); \quad w_j(0') = 0,\ \forall j.$$

However, occasionally some roots may coincide, and in general it is not possible to define the individual roots $w_j(z')$ in such a way that one obtains smooth functions of z' throughout $\Delta(0', \rho')$. [Think of $f(z', w) = z_1 - w^k$.]

In this situation it is natural to ask if the functions ($3b$) might be the roots of a nice algebraic equation. Let us consider the product

$$(3c) \qquad (w - w_1) \ldots (w - w_k) = w^k + \sum_{j=1}^{k} a_j w^{k-j}, \quad a_j = a_j(z').$$

Apart from a $\pm$ sign, the coefficients a_j are equal to the so-called *elementary symmetric functions* of the roots:

$$a_1 = -(w_1 + \ldots + w_k),\ \ a_2 = w_1 w_2 + \ldots + w_1 w_k + w_2 w_3 + \ldots + w_{k-1} w_k,\ \ \ldots,$$
$$a_k = (-1)^k w_1 \ldots w_k.$$

Observe that $a_j(0') = 0,\ j = 1, \ldots, k$. We will show that the coefficient $a_j(z')$ depend analytically on z'. The proof may be based on an algebraic relation between symmetric functions (to be found in [Van der Waerden] for example) of which we will give an analytic proof.

Lemma 4.31. *The coefficients $a_j = a_j(z')$ in (3c) can be expressed as polynomials in the power sums*

$$s_p = s_p(z') = w_1^p + \dots w_k^p, \quad p = 1, 2, \dots$$

and (hence) they are holomorphic functions of z' on $\Delta(0', \rho')$.

PROOF. (i) it is convenient to divide by w^k in (4.33) and to set $1/w = t$. Thus

$$\Pi_{\nu=1}^k (1 - w_\nu t) = \sum_{j=0}^k a_j t^j \stackrel{\text{def}}{=} P(t), \quad a_0 = 1.$$

Taking the logarithmic derivative of both sides and multiplying by t, one obtains the two answers

$$t\frac{P'(t)}{P(t)} = \begin{cases} \sum_{j=1}^k j a_j t^j / \sum_{m=0}^k a_m t^m, \\ \\ \sum_{\nu=1}^k \frac{-w_\nu t}{1 - w_\nu t} = -\sum_{\nu=1}^k \sum_{p=1}^\infty w_\nu^p t^p = -\sum_{p=1}^\infty s_p t^p. \end{cases}$$

We now multiply through by the first denominator and find:

$$\sum_1^k j a_j t^j = -\sum_0^k a_m t^m \sum_1^\infty s_p t^p.$$

Equating coefficients of like powers of t on both sides, the result is

$$(3d) \qquad j a_j = -(a_{j-1} s_1 + a_{j-2} s_2 + \dots + a_0 s_j), \quad j = 1, \dots, k.$$

Hence by induction, a_j can be expressed as a polynomial in $s_1, \dots, s_j$.

(ii) We complete the proof of the Lemma by showing that the power sums $s_p(z')$ are holomorphic in z' on $\Delta(0', \rho')$. To this end we write $s_p(z')$ as an integral: by the residue theorem, cf. (1d),

$$(3e) \qquad s_p(z') = \sum_{\nu=1}^k w_\nu^p = \frac{1}{2\pi i} \int_{C(0,\rho_n)} w^p \frac{g'(w)}{g(w)} dw = \frac{1}{2\pi i} \int_{C(0,\rho_n)} w^p \frac{\partial f(z', w)/\partial w}{f(z', w)} dw.$$

The holomorphy now follows as usual from the holomorphy theorem for integrals 1.72.

The polynomial (3c) is called the Weierstrass polynomial belonging to the roots $w_1, \dots, w_k$ of the equation $f(z', w) = 0$. Replacing w by z_n we formulate:

Definition 4.32. *A* WEIERSTRASS POLYNOMIAL *in z_n of degree k is a holomorphic function in a neighbourhood of the origin in $\mathbf{C}^n$ of the special form*

$$(3f) \qquad W(z', z_n) = z_n^k + \sum_{j=1}^k a_j(z') z_n^{k-j}, \quad (k \geq 1)$$

where the coefficients $a_j(z')$ are holomorphic in a neighbourhood of $0'$ in $\mathbf{C}^{n-1}$ and such that $a_j(0') = 0,\ \ j = 1, \ldots, k$.

An arbitrary polynomial in z_n with coefficients that are holomorphic in z' is called a (holomorphic) *pseudopolynomial* in z_n.

4.4 The Weierstrass theorems. Let f, r and ρ be as in Auxiliary Theorem 4.15, so that in particular f is holomorphic on $\overline{\Delta}(0, \rho)$. Moreover, the equation $f(z', w) = 0$ with z' fixed in $\Delta(0', \rho')$ has precisely k roots (3b) inside the disc $\Delta_1(0, \rho_n)$ and no roots on the boundary $C(0, \rho_n)$. Dividing $g(w) = f(z', w)$ by the Weierstrass polynomial $W(w)$ (3c) with these same roots, we obtain a zero free holomorphic function $E(w)$ on the closed disc $\overline{\Delta}_1(0, \rho_n)$. Explicitly reintroducing z', we have

$$\frac{f(z', z_n)}{W(z', z_n)} = E(z', z_n), \quad (z', z_n) \in \Delta_{n-1}(0', \rho') \times \overline{\Delta}_1(0, \rho_n). \tag{4a}$$

Here $E(z', z_n)$ is holomorphic in z_n on $\overline{\Delta}_1(0, \rho_n)$ for each $z' \in \Delta(0'\rho')$. Also, $E(z', z_n)$ is different from 0 throughout $\delta(0', \rho') \times \overline{\Delta}_1(0, \rho_n)$.

We will show that E is holomorphic in $z = (z', z_n)$ on $\Delta(0, \rho)$. For this we use the one-variable Cauchy integral formula, initially with fixed z' :

$$E(z) = E(z', z_n) = \frac{1}{2\pi i} \int_{C(0,\rho_n)} \frac{f(z', w)}{W(z', w)} \frac{dw}{w - z_n}, \quad z = (z', z_n) \in \Delta(0, \rho) \tag{4b}$$

The holomorphy of $E(z)$ now follows from the holomorphy theorem for integrals 1.72. Indeed, the integrand is continuous in $(z, w) = (z', z_n, w)$ on $\Delta(0, \rho) \times C(0, \rho_n)$ since $W(z', w)(w - z_n)$ is different from zero there. Furthermore, for each $w \in C(0, \rho_n)$, the integrand is holomorphic being a product of holomorphic functions in z' and z_n on $\Delta(0', \rho') \times \Delta_1(0, \rho_n)$. Conclusion from $(4a, b)$:

Theorem 4.41 (WEIERSTRASS'S PREPARATION THEOREM). *Let f be holomorphic on a neighbourhood of the origin in $\mathbf{C}^n$. Suppose f vanishes at 0 (exactly) of order k relative to z_n. Then there is a neighbourhood of the origin in which f has a unique holomorphic factorization*

$$f(z) = E(z)W(z', z_n),$$

where W is a Weierstrass polynomial in z_n of degree k (3f) and E is zero free.

The factorization is unique because W is uniquely determined by f. For the local study of zero sets one may apparently restrict oneself to Weierstrass polynomials. The question of further decomposition of such polynomials will be taken up in Section 4.5.

There is also a preparation theorem for C^∞ functions, see [Malgrange].

There is a second (somewhat less important) Weierstrass theorem which deals with the division of an *arbitrary* holomorphy function F by a preassigned Weierstrass polynomial [division with remainder]:

Theorem 4.42 (Weierstrass's division theorem). *Let F be holomorphic in a neighbourhood of the origin in $\mathbf{C}^n$ and let W be an arbitrary Weierstrass polynomial in z_n of degree k (3f). Then F has a unique representation around 0 of the form*

$$F = QW + R, \tag{4c}$$

where Q is holomorphic and R is a (holomorphic) pseudopolynomial in z_n of degree $< k$.

We indicate a proof. Assuming that F and W are holomorphic on $\Delta(0, r)$ we choose $\rho < r$ such that $W(z) \neq 0$ for $z' \in \Delta(0', \rho')$ and $|z_n| = \rho_n$, cf. Auxiliary Theorem 4.15. Then Q may be defined by

$$Q(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{C(0,\rho_n)} \frac{F(z', w)}{W(z', w)} \frac{dw}{w - z_n}, \quad z \in \Delta(0, \rho). \tag{4d}$$

One readily shows that Q and hence $R \stackrel{\text{def}}{=} F - QW$ are holomorphic on $\Delta(0, \rho)$ and that R is a pseudopolynomial in z_n of degree $< k$, cf. exercise 4.6. For the uniqueness of the representation, cf. exercise 4.7.

4.5 Factorization in the rings $\mathcal{O}_0$ and $\mathcal{O}_0'[z_n]$. As indicated before, the symbol $\mathcal{O}_a$ or $\mathcal{O}_a(\mathbf{C}^n)$ denotes the *ring of germs of holomorphic functions* at a, or equivalently, all*power series*

$$f(z) = \sum_{\alpha \geq 0} c_\alpha (z - a)^\alpha = \sum_{\alpha \geq 0} c_{\alpha_1 \ldots \alpha_n} (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n}$$

in $z_1, \ldots, z_n$ with center a that have *nonempty domain of* (absolute) *convergence* [cf. Section 2.3].For $[f]$ and $[g]$ in $\mathcal{O}_a$ one defines the sum $[f] + [g] = [f + g]$ and the product $[f][g] = [fg]$ via representatives f, g. Product and sum are well defined at least throughout the intersection of the domains of f and g and this intersection will contain a.

As seen above, in working with germs, one strictly speaking has to take representatives, work with these on suitably shrunken neighbourhoods and pass to germs again. Usually the real work is done on the level of the representatives, while the other parts of the proces are a little tiresome. To avoid the latter, we will write $f \in \mathcal{O}_a$, indicating both a germ at a or a representative on a suitable neighborhood, or even its convergent power series at a. This will not lead to confusion. Obviously there will be no loss of generality by studying $\mathcal{O}_0$ only.

The zero element in $\mathcal{O}_0$ is the constant function 0. There is also a multiplicative *identity*, namely, the constant function 1. The ring $\mathcal{O}_0$ is commutative and free of zero divisors cf. exercise 1.20]. Thus $\mathcal{O}_0$ is an *integral domain.* A series or function $f \in \mathcal{O}_0$ has a multiplicative inverse $1/f$ in $\mathcal{O}_0$ if and only if $f(0) \neq 0$; such an f is called a *unit* in the ring. The *nonunits* are precisely the series or functions which vanish at the origin; they form a maximal ideal. For factorizations "at" the origin (around the origin) and for the local study of zero sets, units are of little interest.

DEFINITION 4.51. An element $f \in \mathcal{O}_0$ different from the zero element is called *reducible* (in or over $\mathcal{O}_0$) if it can be written as a product $g_1 g_2$, where g_1 and g_2 are nonunits of

$\mathcal{O}_0$. An element $f \neq 0$ is *irreducible* if for every factorization $f = g_1 g_2$ in $\mathcal{O}_0$, at least one factor is a unit.

In reducibility questions for $\mathcal{O}_0$ we may restrict ourselves to Weierstrass polynomials, cf. the preparation theorem 4.41.

EXAMPLES. It is clear that z_3^2 and $z_3^2 - z_1 z_2 z_3$ are reducible in $\mathcal{O}_0(\mathbf{C}^3)$, but how about

$$W(z) = z_3^2 - z_1^2 z_2\,? \tag{5a}$$

Proposition 4.52. *Every (holomorphic) factorization of a Weierstrass polynomial into nonunits of $\mathcal{O}_0$ is a factorization into Weierstrass polynomials, apart from units with product* 1.

PROOF. Let W be a Weierstrass polynomial in z_n of degree k $(3f)$ and suppose that $W = g_1 g_2$, where the factors g_j are holomorphic in a neighbourhood of 0 and $g_j(0) = 0$. Setting $z' = 0'$ we find

$$z_n^k = W(0', z_n) = g_1(0', z_n) g_2(0', z_n),$$

hence $g_j(0', z_n) \not\equiv 0$, so that the functions g_j are normalized relative to z_n at the origin [Definition 4.12]. Thus we can apply the preparation theorem to each g_j :

$$g_j = E_j W_j, \quad j = 1, 2$$

in some neighbourhood of 0. Here the W_j's are Weierstrass polynomials in z_n and the E_j's are zero free. It follows that

$$W = 1 \cdot W = E_1 E_2 W_1 W_2$$

in some neighbourhood of 0. Now $W_1 W_2$ is also a Weierstrass polynomial in Z_n and $E_1 E_2$ is zero free. The uniqueness part of the preparation theorem thus shows that $W_1 W_2 = W$ and $E_1 E_2 = 1$.

We can now show that the Weierstrass polynomial ($5a$) is irreducible (over $\mathcal{O}_0$). Otherwise there would be a decomposition of the form

$$z_3^2 - z_1^2 z_2 = (z_3 - w_1(z'))(z_3 - w_2(z'))$$

with holomorphic functions w_j at 0. This would imply $w_1 + w_2 = 0$, $w_1 w_2 = -w_1^2 = -z_1^2 z_2$, but the latter is impossible since $z_1\sqrt{z_2}$ is not holomorphic at 0. [The function W of ($5a$) *is* reducible over $\mathcal{O}_a$ for some points a where $W(a) = 0$ and $a_2 \neq 0$. Which precisely ?]

DEFINITION 4.53. An integral domain A with identity element is called a *unique factorization domain* (ufd) if every nonunit ($\neq 0$) can be written as a finite product of irreducible factors in A and this in only one way, apart from units and the order of the factors.

PROPERTIES 4.54. Suppose A is a unique factorization domain. Then:

(i) The polynomial ring $A[x]$ is also a ufd ("Gauss's lemma");

(ii) For any two *relatively prime elements* f and g in $A[x]$ (that is, any nonzero f and g which do not have a nonunit as a common factor), there are relatively prime elements S and T in $A[x]$, with $\operatorname{degree} S < \operatorname{degree} g$, $\deg T < \deg f$, and a nonzero element R in A such that

$$Sf + Tg = R \quad (\text{"\textit{resultant} of } f \text{ and } g\text{"}). \tag{5b}$$

We indicate proofs, but refer to algebra books for details. For part (i) we need only consider *primitive* polynomials f in $A[x]$, that is, polynomials whose coefficients have no common factors others than units. By looking at degrees, it becomes clear that such f can be decomposed into finitely many irreducible factors in $A[x]$. For the uniqueness one may first consider the case where A is a (commutative) field. Then the Euclidean algorithm holds for the greatest common divisor (f, g) in $A[x]$, hence $(f_1, g) = (f_2, g) = 1$ implies $(f_1 f_2, g) = 1$. It follows that irreducible decompositions $f = f_1 \ldots f_r$ in $A[x]$ must be unique. In the general case one first passes from A to the quotient field Q_A. A factorization of f in $A[x]$ gives one in $Q_A[x]$. For the converse, one observes that the product of two primitive polynomials in $A[x]$ is again primitive. It follows that any factorization of f in $Q_A[x]$ can be rewritten as a factorization into primitive polynomials in $A[x]$. Hence since $A_A[x]$ is a ufd, so is $A[x]$.

As to part (ii), relatively prime elements f and g in $A[x]$ are relatively prime in $Q_A[x]$, hence by the Euclidean algorithm for the greatest common divisor, there exist S_1 and T_1 in $Q_A[x]$, $\deg S_1 < \deg g$, $\deg T_1 < \deg f$ such that $S_1 f + T_1 g = 1$. The most economical removal of the denominators in S_1 and T_1 leads to (5b).

Theorem 4.55. *The rings $\mathcal{O}_0'[z_n]$ and $\mathcal{O}_0 = \mathcal{O}_0(\mathbf{C}^n)$ are unique factorization domains.*

[Here $\mathcal{O}_0'$ stands for $\mathcal{O}_0(\mathbf{C}^{n-1})$.]

PROOF. Concentrating on $\mathcal{O}_0$ we use induction on the dimension n. For $n = 0$ the ring $\mathcal{O}_0 = \mathbf{C}$ is a field and every nonzero element is a unit, hence there is nothing to prove. Suppose now that the theorem has been proved for $\mathcal{O}_0' = \mathcal{O}_0(\mathbf{C}^{n-1})$. It then follows from Gauss's lemma that the polynomial ring $\mathcal{O}_0'[z_n]$ of the pseudopolynomials in z_n is also a ufd.

Next let f be an arbitrary nonunit $\neq 0$ in $\mathcal{O}_0 = \mathcal{O}_0(\mathbf{C}^n)$. By a suitable linear coordinate transformation we ensure that f is normalized relative to z_n, cf. Lemma 4.14. Weierstrass's preparation theorem then gives a factorization $f = EW$, where E is a unit in $\mathcal{O}_0$ and $W \in \mathcal{O}_0'[z_n]$ is a Weierstrass polynomial. As a consequence of the induction hypothesis, W can be written as a finite product of irreducible polynomials in $\mathcal{O}_0'[z_n]$. The factors are Weierstrass polynomials (apart from units with product 1) and they are also irreducible over $\mathcal{O}_0$, cf. Proposition 4.52.

The uniqueness of the decomposition (apart from units and the order of the factors) follows by normalization from the uniqueness of the factorization $f = EW$ at 0 and the uniqueness of the factorization in $\mathcal{O}_0'[z_n]$.

Corollary 4.56 (IRREDUCIBLE LOCAL REPRESENTATION). *Let f be holomorphic at 0 in $\mathbf{C}^n$ and normalized relative to z_n. Then f has a holomorphic product repesentation at 0 [in a neighbourhood of 0] of the form*

$$f = EW_1^{p_1} \dots W_s^{p_s}. \tag{5c}$$

Here E is zero free, the W_j's are pairwise distinct irreducible Weierstrass polynomials in z_n and the p_j's are positive integers. The representation is unique up to the order of $W_1, \dots, W_s$.

We finally show that the rings $\mathcal{O}_0(\mathbf{C}^n)$ are *Noetherian*:

DEFINITION 4.57. A commutative ring A with identity element is called Noetherian if every ideal $I \subset A$ is finitely generated, that is, if there exist elements $g_1, \dots, g_k$ in I such that every $f \in I$ has a representation $f = \sum a_j g_j$ with $a_j \in A$.

The so-called Hilbert basis theorem asserts that for a Noetherian ring A, the polynomial ring $A[x]$ is also Noetherian, cf. [Van der Waerden] section 84.

Theorem 4.58. *$\mathcal{O}_0 = \mathcal{O}_0(\mathbf{C}^n)$ are Noetherian.*

PROOF. One again uses induction on the dimension n. For $n = 0$ the ring $\mathcal{O}_0 = \mathbf{C}$ is a field, so that the only two ideals are the ones generated by 0 and by 1. Suppose, therefore, that $n \geq 1$ and that the theorem has been proved for $\mathcal{O}'_0 = \mathcal{O}_0(\mathbf{C}^{n-1})$. Then by the above remark, the polynomial ring $\mathcal{O}'_0[z_n]$ is also Noetherian.

Now let I be any ideal in $\mathcal{O}_0 = \mathcal{O}_0(\mathbf{C}^n)$ which contains a nonzero element g. By change of coordinates and the Weierstrass preparation theorem we may assume that $g = EW$, where E is a unit in $\mathcal{O}_0$ and W is a Weierstrass polynomial in z_n. Observe that W will also belong to I and thus to the intersection $J = I \cap \mathcal{O}'_0[z_n]$.

This intersection J is an ideal in the ring $\mathcal{O}'_0[z_n]$, hence by the induction hypothesis, it is generated by finitely many elements $g_1, \dots, g_p$. We claim that in $\mathcal{O}_0$, the elements W and $g_1, \dots, g_p$ will generate I. Indeed, let F be any element of I. By the Weierstrass division theorem, $F = QW + R$, where $Q \in \mathcal{O}_0$ and $R \in \mathcal{O}'_0[z_n]$. Clearly R is also in I, hence $R \in J$, so that $R = b_1 g_1 + \dots + b_p g_p$ with $b_j \in \mathcal{O}'_0[z_n]$. Thus $F = QW + b_1 g_1 + \dots + b_p g_p$.

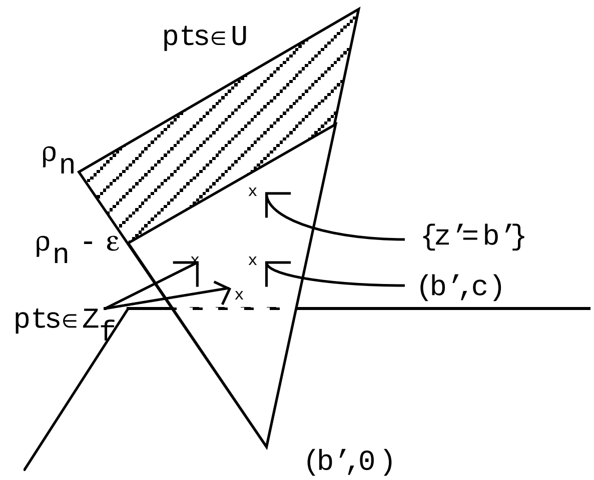
4.6 Structure of zero sets. We first discuss some global properties. Let f be a holomorphic function $\not\equiv 0$ on a connected domain D in $\mathbf{C}^n$. What sort of subset is the zero set $Z_f = Z(f)$ of f in D?

Theorem 4.61. *Z_f is closed (relative to D) and thin: it has empty interior. The zero set does not divide D even locally: to every point $a \in Z_f$ there are arbitrarily small polydiscs $\Delta(a, \rho)$ in D such that $\Delta(a, \rho) - Z_f$ is connected. $\Omega = D - Z_f$ is a connected domain.*

PROOF. It is clear that Z_f is closed [f is continuous] and that it has no interior points: if f would vanish on a small ball in D, it would have to vanish identically. [More generally, Z_f can not contain a set of uniqueness 1.55 for $\mathcal{O}(D)$.]

Next let a be any point in Z_f. We may assume that $a = 0$ and that f vanishes at 0 of order k relative to z_n. By Auxiliary Theorem 4.15, there will be arbitrarily small polydiscs $\Delta(0, \rho) \subset D$ and $\epsilon > 0$ such that $f(0', z_n) \neq 0$ for $0 < |z_n| \leq \rho_n$ and

$$f(z', z_n) \neq 0 \quad \text{throughout} \quad U = \Delta(0', \rho') \times \{\rho_n - \epsilon < |z_n| < \rho_n\}.$$



Observe that the subset $U \subset \Delta(0,\rho) - Z_f$ is connected. Furthermore, every point (b', c) of $\Delta(0,\rho) - Z_f$ may be connected to U by a straight line segment in $\Delta(0,\rho)$ lying in the complex plane $z' = b'$ but outside Z_f. Indeed, the disc $z' = b',\ \ |z_n| < \rho_n$ contains at most k distinct points of Z_f (fig 4.3). Thus $\Delta(0,\rho) - Z_f$ is connected.

Any two point p and q in Ω can be joined by a polygonal path in D. Such a path may be covered by finitely many polydiscs $\Delta(a,\rho) \subset D$ such that $\Delta(a,\rho) - Z_f$ is connected. The latter domains will connect p and q in Ω.

We now turn our attention to the local form of Z_f. We have already encountered regular points of Z_f, that is, points a around which Z_f is homeomorphic to a domain in $\mathbf{C}^{n-1}$. Regularity of a point $a \in Z_f$ is assured if f vanishes at a of order exactly 1, see Corollary 4.22.

Suppose from here on that f vanishes of order k at a, we may again take $a = 0$ and normalize relative to z_n to obtain the local irreducible representation ($5c$) for f. We will now consider Z_f purely as a set without regard to multiplicities. In that case it may be assumed that f is a Weierstrass polynomial in z_n of the form

$$f = W_1 \ldots W_s, \tag{6a}$$

where the factors are distinct and irreducible.

Theorem 4.62 (LOCAL FORM OF THE ZERO SET)**.** *Let f be a Weierstrass polynomial in z_n of degree k that is either irreducible or equal to a product ($6a$) of distinct irreducible Weierstrass polynomials $W_1, \ldots, W_s$. Then there is a neighbourhood $\Delta(0,\rho) = \Delta(0',\rho') \times \Delta_1(0,\rho_n)$ of the origin in $\mathbf{C}^n$ in which the zero set Z_f may be described as follows. There exists a holomorphic function $R(z') \not\equiv 0$ on $\Delta(0',\rho')$ such that for every point z' in $\Delta(0',\rho') - Z_R$, there are precisely k distinct points of Z_f in $\Delta(0,\rho)$ which lie above z'; all those points are regular points of Z_f. For z' in Z_R some roots of the equation $f(z', z_n) = 0$ in $\Delta_1(0,\rho_n)$ will coincide. We say that the local zero set*

$$Z_f \cap \Delta(0,\rho)$$

is a k-SHEETED COMPLEX ANALYTIC HYPERSURFACE above $\Delta(0',\rho') \subset \mathbf{C}^{n-1}$ which branches (and can have nonregular points) only above the thin subset Z_R of $\Delta(0',\rho,)$.

PROOF. The Weierstrass polynomial f in z_n of degree k ($6a$) and its partial derivative

$$(6b)\qquad \frac{\partial f}{\partial z_n} = \sum_{i=1}^{s} W_1 \ldots W_{i-1} \frac{\partial W}{\partial z_n} W_{i+1} \ldots W_s$$

of degree $k-1$ must be relatively prime in $\mathcal{O}'_0[z_n]$. Indeed, none of the irreducible factor W_j of f can divide $\partial f/\partial z_n$. This is so because W_j divides all the terms in the sum ($6b$) with $i \neq j$, but not the term with $i = j$: degree $\partial W_j/\partial z_n < \deg W_j$.

The greatest common divisor representation ($5b$) for the ufd $\mathcal{O}'_0[z_n]$ now provides a relation

$$(6c)\qquad Sf + T\frac{\partial f}{\partial z_n} = R = R(z')$$

relatively prime elements S and T in $\mathcal{O}'_0[z_n]$, $\deg S < k-1$, $\deg T < k$, and a nonzero element $R(z')$ in $\mathcal{O}'_0$. [A resultant R of f and $\partial f/\partial z_n$ is also called a *discriminant* of f as a pseudopolynomial in z_n. The special case $k = 1$ is trivial, but fits in if we take $S = 0,\ T = 1$ to yield $R = 1$.]

Relation ($6c$) may be interpreted as a relation among holomorphic functions on some polydisc $\Delta(0, r)$. We now choose $\Delta(0, \rho)$ as in auxiliary theorem 4.15. For any point $b' \in \Delta(0', \rho')$, the equation

$$(6d)\qquad f(b', z_n) = 0$$

then has precisely k roots in $\Delta_1(0, \rho_n)$, counting multiplicities. Suppose b' is such that some of these roots coincide, in other words, equation ($6d$) has a root $z_n = c$ of multiplicity ≥ 2. Then

$$(6e)\qquad f(b', c) = \frac{\partial f}{\partial z_n}(b', c) = 0,$$

hence by ($6c$), $R(b') = 0$.

Conclusion: the k roots z_n of the equation $f(z', z_n) = 0$ in $\Delta_1(0, \rho_n)$ are all distinct whenever $R(z') \neq 0$ or $z' \in \Delta(0', \rho') - Z_R$. The corresponding points (z', z_n) of Z_f are regular: at those points $\partial f/\partial z_n \neq 0$ because of ($6c$); now see Corollary 4.22.

If $R(b') = 0$, the k roots of ($6d$) must satisfy $T\partial f/\partial z_n = 0$, hence if $T(b', z_n) \not\equiv 0$, ($6e$) must hold for some $c \in \Delta_1(0, \rho_n)$. However, even then (b', c) *may* be a regular point of Z_f :

EXAMPLE 4.63. For $f(z) = z_3^2 - z_1^2 z_2$ in $\mathcal{O}_0(\mathbf{C}^3)$ we have $\partial f/\partial z_3 = 2z_3$, hence

$$R(z') = 2f(z) - z_3 \partial f/\partial z_3 = -2z_1^2 z_2$$

will be a resultant. The zero set Z_f has two sheets over $\mathbf{C}^2$, given by $z_3 = \pm z_1 \sqrt{z_2}$; the sheets meet above Z_R. The points of Z_R have the forms $z' = (a, 0)$ and $z' = (0, b)$; the corresponding points $(a, 0, 0)$ and $(0, b, 0)$ of Z_f are of different character. Around $(0, b, 0)$ with $b \neq 0$, Z_f decomposes into two separate zero sets that meet along the complex line $z_1 = z_3 = 0$. However, the points $(a, 0, 0)$ with $a \neq 0$ are regular for Z_f, as is shown by the local representation $z_2 = z_3^2/z_1^2$!

Theorem 4.62 has various important consequences such as the so-called "Nullstellensatz", cf. exercises 4.17, 4.18.

ANALYTIC SETS 4.64. A subset X of a domain $D \subset \mathbf{C}^n$ is called an analytic set if throughout D, it is locally the set of common zeros of a family of holomorphic functions. [Since the rings $\mathcal{O}_a$ are Noetherian, one may limit oneself to *finite* families.] A point $a \in X$ is called *regular* if the intersection of X with a (small) polydisc $\Delta(a, r)$ is homeomorphic to a domain in a space $\mathbf{C}^k$; the number k is called the complex dimension of X at a. By $\dim X$ one means the maximum of the dimensions at the regular points.

Taking D connected and $f \in \mathcal{O}(D)$, $f \not\equiv 0$, the zero set Z_f is an analytic set of complex dimension $n-1$. The set of the nonregular points of Z_f is locally contained in the intersection of the zero sets of two relatively prime holomorphic functions, in the preceding proof, f and $\partial f/\partial z_n$. The nonregular points belong to an analytic set of complex dimension $n-2$: locally, there are at most a fixed number of nonregular points above each point of a zero set Z_R in $\mathbf{C}^{n-1}$. Cf. [Gunning-Rossi], [Hervé]. In the case $n = 2$, Z_f is a complex analytic surface (real dimension 2) and the local sets Z_R in $\mathbf{C}$ consist of isolated points; in this case, the nonregular points of Z_f in D also lie isolated.

4.7 Zero sets and removable singularities. For $g \in \mathcal{O}(D)$ $g \not\equiv 0$, the zero set $Z_g \subset D \subset \mathbf{C}^n$ is at the same time a singularity set: think of $h = 1/g$ on the domain $D - Z_g$. However, we will see that Z_g can not be the singularity set of a *bounded* holomorphic function on $D - Z_g$. For $n = 1$ this is Riemann's theorem on removable singularities in $\mathbf{C}$. The latter is a consequence of the following simple lemma.

Lemma 4.71. *A bounded holomorphic function f on a punctured disc $\Delta_1(0, \rho) - \{0\}$ in $\mathbf{C}$ has an analytic extension to the whole disc $\Delta_1(0, \rho)$.*

PROOF. In the Laurent series $\sum_{-\infty}^{\infty} c_k w^k$ for $f(w)$ with center 0, all coefficients c_k with negative index must be zero. Indeed, for $k < 0$ and $0 < r < \rho$.

$$(7a) \qquad c_k = \left| \frac{1}{2\pi i} \int_{C(0,r)} f(w) w^{-k-1} dw \right| \leq \sup |f| \cdot r^{|k|} \to 0 \quad \text{as} \quad r \downarrow 0.$$

Thus the Laurent series is actually a power series which furnishes the desired extension.

The corresponding $\mathbf{C}^n$ result is also called the *Riemann removable singularities theorem*:

Theorem 4.72. *Let D be a connected domain in $\mathbf{C}^n$ and let Z_g be the zero set of a nonzero function $g \in \mathcal{O}(D)$. Let f be homomorphic on the domain $\Omega = D - Z_g$ and bounded on a neighbourhood in Ω) of every point $a \in Z_g$. Then f has an analytic extension F to the whole domain D.*

PROOF. Take $n \geq 2$ and choose $a \in Z_g$, then adjust the coordinate system so that $a = 0$ while g vanishes at 0 of some finite order k relative to z_n. Next choose $\overline{\Delta}(0, \rho) \subset D$ such that f is bounded on $\overline{\Delta}(0,\rho) - Z_g$ and $g(0',z_n) \neq 0$ for $0 < |z_n| \leq \rho_n$, $g(z',z_n) \neq 0$ on $\Delta(0',\rho') \times C(0,\rho_n)$, cf. auxiliary theorem 4.15. For *fixed* $z' \in \Delta(0',\rho')$, the function $g(z',z_n)$ then has precisely k zeros $w_1(z'),\ldots,w_k(z')$ in $\Delta_1(0,\rho_n)$ and no zero on $C(0,\rho_n)$. By the hypothesis, $f(z',z_n)$ will be holomorphic and bounded on $\overline{\Delta}_1(0,\rho_n)-\{w_1,\ldots,w_k\}$. Hence by Riemann's one-variable theorem, $f(z',z_n)$ has an analytic extension $F(z',z_n)$ to the disc $\overline{\Delta}_1(0,\rho_n)$. Since $F(z',w) = f(z',w)$ in particular for $w \in C(0,\rho_n)$, the one-variable Cauchy integral formula gives the representation

$$(7b) \qquad F(z) = F(z',z_n) = \frac{1}{2\pi i}\int_{C(0,\rho_n)} \frac{f(z',w)}{w-z_n}dw, \quad z_n \in \Delta_1(0,\rho_n), \quad z' \in \Delta(0',\rho').$$

How will F behave as a function of $z = (z',z_n) \in \Delta(0,\rho)$? The function $f(z',w)$ is holomorphic and hence continuous on the set $\Delta(0',\rho') \times C(0,\rho_n)$, on which $g(z',w) \neq 0$. Thus the integrand in (7b) is continuous in $(z,w) = (z',z_n,w)$ on the set $\Delta(0,\rho)\times C(0,\rho_n)$. Moreover, for fixed $w \in C(0,\rho_n)$, the integrand is holomorphic in $z = (z',z_n)$, cf. the proof at the beginning of Section 4.4. So it follows as usual from the holomorphy theorem for integrals 1.72 that $F(z)$ is holomorphic on $\Delta(0,\rho)$; naturally, $F = f$ outside Z_g on $\Delta(0,\rho)$.

We know now that f extends analytically to some polydisc $\Delta(a,\rho)$ around each point $a \in Z_g$. The uniqueness theorem will show that the various extensions $F = F_a$ are compatible: if the polydiscs $\Delta(a,\rho)$ and $\Delta(b,\sigma)$ for F_a and F_b overlap, the intersection contains a small ball of $D - Z_g$ and there $F_a = f = F_b$, hence $F_a = F_b$ throughout the intersection.

REMARKS 4.73. In Theorem 4.72, the zero set Z_g may be replaced by an aribtrary analytic set X in D of complex dimension $\leq n-1$ [thus, a set X which is locally contained in the zero set of a nonzero function, cf. 4.64]. If X has *complex dimension* $\leq n-2$, it will be a *removable singularity set* for *every* holomorphic function on $D-X$. This is clear when $n=2$, since an analytic set of dimension 0 consists of isolated points. For the general case, see exercise 4.26.

Another remarkable result on removable singularities is *Rado's theorem*, see exercises 4.27, 4.28.

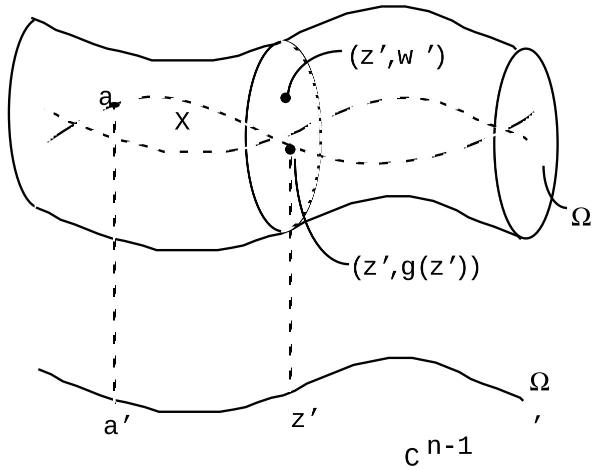
4.8 Hartogs' singularities theorem. Roughly speaking, the theorem asserts that singularity sets X in $\mathbf{C}^n$ of complex dimension $n-1$ are zero sets of analytic functions. The setup is as follows, cf. fig 4.4. The basic domain Ω will

$$\Omega = \Omega' \times \Delta_1(0,R),$$

where Ω' is a connected domain in $\mathbf{C}^{n-1}$. The subset $X \subset \Omega$ will be the graph of an arbitrary function $g : \Omega' \to \Delta_1(0,R)$:

$$X = \{z = (z',z_n) = (z',w) : z' \in \Omega',\ z_n = w = g(z')\}.$$

We will say that a holomorphic function f on $\Omega - X$ becomes *singular* at the point $a \in X$ [and that a is a singular point for f] if f has no analytic continuation to a neighbourhood for a. Under a mild restriction on X, a function $f \in \mathcal{O}(\Omega - X)$ will either become singular everywhere on X or nowhere on X:



Proposition 4.81. *Let* $sup_K|g(z')| = R_K < R$ *for every compact subset* $K \subset \Omega'$*. [This is certainly the case if* g *is continuous.] Let* f *in* $\mathcal{O}(\Omega - X)$ *become singular at some point* $a \in X$*. Then* f *becomes singular at every point of* X *and* g *is continuous.*

PROOF. Let $E \subset \Omega'$ consist of all points z' such that $(z', g(z'))$ is a singular point for $f(z', w)$. Then E is nonempty and closed in Ω' and the restriction $g \mid E$ is continuous. Indeed, let $\{z'_\nu\}$ be any sequence in E with limit $b' \in \Omega'$ and let c be any limit point of the sequence $\{g(z'_\nu)\}$. Then $|c| \leq \limsup|g(z'_\nu)| \leq R_K < R$ where $K = \{z'_\nu\}_1^\infty \cup \{b'\}$. Thus $(b'c)$ belongs to Ω and as a limit point of singular points, (b', c) must be a singular point for f. Hence $c = g(b')$ and $b' \in E$. The argument shows that E is closed and that $g \mid E$ is continuous at b'.

Using Hartogs' continuity theorem 2.61 we can now show that the open set $\Omega'_0 = \Omega' - E$ is empty. Indeed, if Ω'_0 is not empty, E and Ω'_0 must have a common boundary point z'_0 in the connected domain Ω'. Since $g \mid E$ is continuous at z'_0, there is a polydisc $\Delta(z'_0, r')$ in Ω' above which the singular points $(z', g(z'))$ of f have $g(z')$ very close to $w_0 = g(z'_0)$. It follows that $f(z', w)$ is analytic on a subdomain of Ω of the form

$$\Delta(z'_0, r') \times \{\rho_n < |w - w_0| < r_n\} \cup D'_0 \times \Delta_1(w_0, r_n),$$

where $D'_0 = \Delta(z'_0, r') \cap \Omega'_0$ is nonempty. But then f has an analytic continuation to the neighbourhood $\Delta(z'_0, r') \times \Delta_1(w_0, r_n)$ of (z'_0, w_0)! This contradiction proves that E is all of Ω' and that g is continuous.

Theorem 4.82. *Let* Ω, g *and* X *be as described at the beginning of the Section. Suppose that there is a holomorphic function* f *on* $\Omega - X$ *which becomes singular at every point of* X*. Then* g *is holomorphic on* Ω'*, hence the singularity set* X *is the zero set of the holomorphic function* $h(z) = z_n - g(z')$ *in* Ω*.*

PROOF. We will sketch how to show that g is smooth; if one knows that g is of class C^1, the recessed edge theorem 3.52 may be used to prove that g is holomorphic, see part

(v) below. The smoothness proof depends on the smoothness of continuous functions that possess the mean value property for circles or spheres: such functions are harmonic. In order to prove that a certain continuous auxiliary function has the mean value property, it will first be shown that it has the sub mean value property, in other words, that it is subharmonic. Readers who have not encountered subharmonic functions before may wish to postpone the proof until they have studied Chapter 8.

(i) The function g is *continuous*. Indeed, let $z' \to b'$ in Ω'. Then one limit point of $g(z')$ must be $c = g(b')$: the singular point $(b', g(b'))$ can not be isolated. If there are other limit points w of $g(z')$, they must have $|w| = R$, since $|w| < R$ would imply that there would be more than one singular point of f above b'. Thus for small $\epsilon > 0$ there is a small polydisc $\Delta(b', r')$ such that for any z' in it, either $|g(z') - c| \le \epsilon$ or $|g(z') - c| \ge 2\epsilon$. Denoting the corresponding subsets of $\Delta(b', r')$ by E and Ω'_0, respectively, the argument of the preceding proof shows that E is closed, that $g \mid E$ is continuous and that Ω'_0 is empty.

(ii) The function $-\log|g(z') - w|$ will be *subharmonic* in z'. We give a proof for $n = 2$, taking $g(0) = 0$ and writing z instead of z' for the time being. Working close to the origin, it will be shown that the continuous function $\Omega' \to \mathbf{R} \cup \{-\infty\}$ given by

$$G_w(z) = G(z, w) = -\log|g(z) - w| \tag{8a}$$

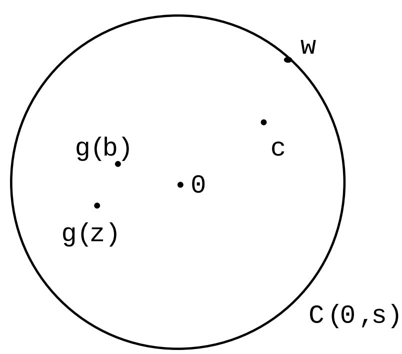
is subharmonic in z around 0 whenever $|w| = s$ is not too small and not too large. We have to prove then that G_w has the sub mean value property for small $r > 0$:

$$G(z, w) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} G(z + re^{it}, w)dt. \tag{8b}$$

For $c \neq 0$ fixed, w close to c and z near 0, [so that $|g(z)|$ is small], one may repesent $f(z, w)$ by a power series in $w - c$ with holomorphic coefficients $a_k(z)$:

$$f(z, w) = \sum_{k \ge 0} a_k(z)(w - c)^k. \tag{8c}$$

Cf. Section 2.6: our function f is holomorphic on a neighbourhood of the point $(0, c)$ in $\mathbf{C}^2$.]



For *fixed* z, the point $w = g(z)$ may be a singular point for $f(z, w)$, but other singularities must be as far away as the boundary of Ω. Hence if $|c|$ is not too large, $f(z, w)$ will be analytic in w (at least) for $|w-c| < |g(z)-c|$. Thus by the Cauchy-Hadamard formula for the radius of convergence of a power series in one variable,

$$1/\limsup |a_k(z)|^{1/k} \geq |g(z)-c|,$$

so that

$$A(z) \stackrel{\text{def}}{=} \limsup_{k\to\infty} \frac{1}{k} \log |a_k(z)| \leq -\log |g(z)-c| = G(z,c). \tag{8d}$$

In the following, we will show that for b near 0,

$$A^*(b) \stackrel{\text{def}}{=} \limsup_{z\to b} A(z) = \lim_{\rho\downarrow 0} \sup_{|z-b|<\rho} A(z) equals\ G(b,c). \tag{8e}$$

By the holomorphy of the coefficients $a_k(z)$, the functions $(1/k)\log |a_k(z)|$ are subharmonic around 0. There they are uniformly bounded from above, hence their sub mean value property is inherited by the lim sup, $A(z)$ in (8d). [Use Fatou's lemma.] In the same way, the sub mean value property carries over to the lim sup $A^*(b)$ in (8e), considered as a function of b.

Now suppose for a moment that $A^*(b) < G(b,c)$ for some b. Then there are small δ and $\epsilon > 0$ such that $A(z) < G(b,c) - 2\delta$ for $|z-b| < 2\epsilon$. At this stage we appeal to a lemma of Hartogs on sequences of subharmonic functions with a uniform upper bound [exercise 8.31]. It implies that the subharmonic functions $(1/k)\log |a_k(z)|$ with lim sup $< G(b,c) - 2\delta$ must satisfy the fixed inequality

$$(1/k)\log |a_k(z)| < G(b,c) - \delta \quad \text{for} \quad |z-b| < \epsilon$$

for all k which exceed some index k_0. By simple estimation, it would then follow that the series in (8c) is uniformly convergent on the product domain $|z-b| < \epsilon$, $|w-c| < (1+\delta)|g(b)-c|$. Thus f would have an analytic continuation to a neighbourhood of the singular point $(b, g(b))$.

This contradiction shows that $G(b,c) = A^*(b)$. Being continuous, it follows that $G(b,c)$ is subharmonic as a function of b.

(iii) Actually, $-\log |g(z') - w|$ will be *harmonic* in z'. Indeed, since $(1/2\pi)\int_\pi^\pi \log |\zeta - se^{i\theta}| d\theta = \log s$ whenever $|\zeta| < s$, integration of (8b) over a suitable circle $|w| = s$ leads to the result

$$-\log s = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(z, se^{i\theta}) d\theta \leq \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G(z + re^{it}, se^{i\theta}) dt d\theta = -\log s.$$

It follows that one must have equality in (8b) for all small $r > 0$ and $|w| = s$. the resulting mean value property implies that $G_w(z)$ is harmonic and in particular also C^∞ smooth, cf. Section 8.1 and exercise 8.14.

(iv) The function g is *smooth*. Indeed, by exponentiation it follows from (iii) that $(g-w)(\overline{g}-\overline{w})$ is smooth for each w of absolute value s. Choosing $w = \pm s$, subtraction will show that $\operatorname{Re} g$ is smooth. The choices $w = \pm is$ will show that $\operatorname{Im} g$ is smooth.

(v) We finally show that g is *holomorphic.* Setting $\varphi = \operatorname{Re}(g - z_n)$, $\psi = \operatorname{Im}(g - z_n)$, our set X is the intersection of the smooth real hypersurfaces $V : \varphi = 0$ and $W : \psi = 0$ in Ω. The gradients of φ and ψ are linearly independent:

$$\partial\varphi/\partial x_n = -1,\ \partial\varphi/\partial y_n = 0 \quad \text{while} \quad \partial\psi/\partial x_n = 0,\ \partial\psi/\partial y_n = -1.$$

Now the hypothesis of the theorem implies that the restriction of f to Ω_0, the part of Ω where $\min(\varphi, \psi) < 0$, can not be continued analytically to a neighbourhood of any point of X. Thus part (ii) of the recessed edge theorem 3.52 shows that g is holomorphic, cf. exercise 3.19.

Remark. Hartogs' original proof [Hartogs 1909] had different final steps, cf. exercises 4.32, 4.33 and [Narasimhan].

Exercises

4.1. Suppose that f vanishes of order $k \geq 2$ relative to z_n at 0. Show that f need not vanish of order k relative to z at 0.

4.2. Carry out an invertible linear transformation of $\mathbf{C}^3$ in order to make $f(z) = z_1 z_2 z_3$ vanish of order 3 relative to the new third coordinate at 0.

4.3. Let f and g be holomorphic at the origin of $\mathbf{C}^n$ and not identically zero. Prove that f and g can be simultaneously normalized relative to z_n at 0. (A single linear coordinate transformation will normalize both functions.)

4.4. Determine a polydisc $\Delta(0, \rho)$ as in Auxiliary Theorem (4.16) for the function $f(z) = 2z_1^2 + z_2 z_3 + 2z_3^2 + 2z_3^3$ on $\mathbf{C}^3$. How many zeros does $f(z', z_3)$ have in $\Delta_1(0, \rho_3)$ for $z' \in \Delta(0', \rho')$?

4.5. Apply Weierstrass's factorization theorem to $f(z) = z_1 z_2 z_3 + z_3(e^{z_3} - 1)$ in $\mathcal{O}_0(\mathbf{C}^3)$. [Determine both $W(z', z_3)$ and $E(z', z_3)$.]

4.6. Prove Weierstrass's division formula (4c) . [Defining Q as in formula (4d), show that

$$R(z) \stackrel{\text{def}}{=} F(z) - Q(z)W(z) = \frac{1}{2\pi i} \int_{C(0,\rho_n)} \frac{F(z', w)}{W(z', w)} \frac{W(z', w) - W(z', z_n)}{w - z_n} dw,$$
$$z \in \Delta(0, \rho)$$

is a pseudopolynomial in z_n of degree$< k = \deg W$.]

4.7. Let F and W be as in Weierstrass's division theorem. Prove that there is only one holomorphic representation $F = QW + R$ around 0 with a pseudopolynomial R of degree$< k = \deg W$. [If also $F = Q_1 W + R_1$, then $(Q_1 - Q)W = R - R_1$.]

4.8. Suppose that $P = QW$ in $\mathcal{O}_0$ where P is a pseudopolynomial in z_n and W a Weierstrass polynomial in z_n. Prove that Q is a pseudopolynomial in z_n.

4.9. Prove that the pseudopolynomial $z_2^2 - z_1^2$ in z_2 is divisible by the pseudopolynomial $z_1 z_2^2 - (1 + z_1^2)z_2 + z_1$ in $\mathcal{O}_0$, but that the quotient is not a pseudopolynomial.

4.10. Prove that a power series f in $\mathcal{O}_0$ has a multiplicative inverse in $\mathcal{O}_0$ if and only if $f(0) \neq 0$.

4.11. Characterize the irreducible and the reducible elements in $\mathcal{O}_0(\mathbf{C}^1)$.

4.12. Prove directly that $\mathcal{O}_0(\mathbf{C}^1)$ is a unique factorization domain.

4.13. Determine a resultant of $f(z) = z_3^2 - z_1$ and $g(z) = z_3^2 - z_2$ as elements of $\mathcal{O}_0'[z_3]$.

4.14. Let A be a ufd. Prove that nonconstant polynomials f and g in $A[x]$ have a (nonconstant) common factor in $A[x]$ if and only if there are nonzero polynomials S and T, $\deg S < \deg g,\ \deg T < \deg f$ such that $Sf + Tg = 0$.

4.15. The *Sylvester resultant* $R(f, g)$ of two polynomials

$$f(x) = a_0 x^k + \ldots + a_k,\ g(x) = b_0 x^m + \ldots + b_m$$

with coefficients in a commutative ring A with identity is defined by the determinant indicated in fig 4.6.

$$\left|\begin{array}{cccccc}
a_0 & a_1 & \dots & a_k & & \\
 & & & & & 0 \\
 & a_0 & \dots & \dots & a_k & \\
 & & \dots & \dots & \dots & \\
0 & & & & & \\
 & & a_0 & \dots & \dots & a_k \\
b_0 & b_1 & \dots & b_m & & \\
 & & & & & 0 \\
 & b_0 & \dots & \dots & b_m & \\
 & & \dots & \dots & \dots & \\
0 & & & & & \\
 & & b_0 & \dots & \dots & b_m
\end{array}\right|
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\end{array}}\right\} \begin{array}{c} m \\ rows \end{array} \\
\left.\vphantom{\begin{array}{c}b\\b\\b\\b\\b\\b\end{array}}\right\} \begin{array}{c} k \\ rows \end{array}
\end{array}$$

fig 4.6

Denote the cofactors of the elements in the last column by $c_0, \dots, c_{m-1},\ d_0, \dots, d_{k-1}$ and set

$$c_0 x^{m-1} + \dots + c_{m-1} = S(x),\ d_0 x^{k-1} + \dots + d_{k-1} = T(x).$$

Prove that $Sf + Tg = R(f,g)$. [Add to the last column x^{k+m-1} times the first, plus x^{k+m-2} times the second, etc. Expand.]

4.16. Describe the zero sets of the Weierstrass polynomials $z_3^2 - z_1 z_2$ and $z_3^4 - z_1 z_2 z_3$ around 0 in $\mathbf{C}^3$. Identify the nonregular points.

4.17. Let f and g be relatively prime in $\mathcal{O}_0$ and normalized relative to z_n. Prove that around 0, the zero sets Z_f and Z_g can coincide only above the zero set Z_R of a nonzero holomorphic function $R(z')$, defined around $0'$ in $\mathbf{C}^{n-1.}$

4.18. (*Nullstellensatz*) Let f be irreducible over $\mathcal{O}_0$ with $f(0) = 0$ and suppose that $g \in \mathcal{O}_0$ vanishes everywhere on Z_f around 0. Prove that f is a divisor of g in $\mathcal{O}_0$. Extend to the case where f is a product of pairwise relatively prime irreducible factors.

4.19. Let f and g be relatively prime in $\mathcal{O}_0$. Prove that they are also relatively prime in $\mathcal{O}_a$ for all points a in a neighbourhood of 0.

4.20. Describe the ideals in $\mathcal{O}_0(\mathbf{C}^1)$ and verify that $\mathcal{O}_0(C^1)$ is a Noetherian ring.

4.21. Prove that an analytic set is locally the set of common zeros of *finitely* many holomorphic functions.

4.22. Let Ω', Ω, g and X be as at the beginning of Section 4.8. Let f be holomorphic on Ω and zero free on $\Omega - X$. Suppose that f vanishes at a point $a \in X$. Prove (without using the results of Section 4.8) that $f = 0$ everywhere on X and that g is holomorphic on Ω'.

4.23. Let X be an analytic subset of a connected domain $D \subset \mathbf{C}^n$ of complex dimension $\leq n-1$. Let f be holomorphic on $\Omega = D - X$ and bounded on a neighbourhood (in Ω) of every point $a \in X$. Prove that f has an analytic extension F to the whole domain D.

4.24. Use the preceding removable singularities theorem to verify that $\Omega = D - X$ is connected. [If $\Omega = \Omega_0 \cup \Omega_1$ with disjoint open Ω_j and $f = 0$ on Ω_0, $f = 1$ on Ω_1, then $\ldots$].

4.25. Let f be continuous on $D \subset \mathbf{C}^2$ and holomorphic on $D - V$, where V is a *real* hyperplane, for example, $\{y_1 = 0\}$. Prove that f is holomorphic on D.

4.26. (*An analytic singularity set in* $\mathbf{C}^n$ *of complex dimension* $\leq n-2$ *is removable*) Let X be an analytic subset of $D \subset \mathbf{C}^n$ which is locally contained in the set of common zeros of two relatively prime holomorphic functions. Suppose that f is holomorphic on $D - X$. Prove that f has an analytic extension to D. Begin by treating the case $n = 2$! [Taking $a \in X$ equal to 0, one may assume that X is locally contained in, or equal to, $Z_g \cap Z_h$, where g is a Weierstrass polynomial in z_n with coefficients in $z' = (z_1, \ldots, z_{n-1})$ and h (obtained via a resultant) a Weierstrass polynomial in z_{n-1} with coefficients in $z'' = (z_1 \ldots, z_{n-2})$. Choose $\rho > 0$ such that $g(z', z_n) \neq 0$ for $|z_n| = \rho_n$, $z' \in \overline{\Delta}(0', \rho')$ and $h(z'', z_{n-1} \neq 0$ for $|z_{n-1}| = \rho_{n-1}$, $z'' \in \Delta(0'', \rho'')$. Extend $f(z) = f(z'', z_{n-1}, z_n)$ analytically to the closed bidisc $|z_{n-1}| \leq \rho_{n-1}$, $|z_n| \leq \rho_n$ for each $z'' \subset \Delta(0'', \rho'')$. How can one represent the analytic extension $F(z'', z_{n-1}, z_n)$? Show that $F(z)$ is holomorphic on $\Delta(0, \rho)$.]

4.27. (Special case of Rado's theorem) Let f be continuous on the closed disc $\overline{\Delta}(0,1) \subset \mathbf{C}$ and holomorphic on $\Omega = \Delta(0,1) - Z_f$. Let F be the Poisson integral of f on the disc. Prove that

(i) $F = f$ on Ω. [Take $|f| \leq 1$ and apply the maximum principle to harmonic functions such as $\mathrm{Re}\,(F - f) + \epsilon \log |f|$ on Ω.]

ii) F provides an analytic extension of f to $\Omega(0,1)$. [$G = \overline{D}F = \partial F/\partial \overline{z}$ will be antiholomorphic: $DG = \partial G/\partial z = \partial^2 F/\partial z \partial \overline{z} = 0$ and on $\Omega, \ldots$]

4.28. (*Rado's removable singularities theorem*) Let $\Omega \subset D \subset \mathbf{C}^n$ be open and suppose that f is holomorphic on Ω and such that $f(z) \to 0$ whenever z tends to a boundary point ζ of Ω in D. Prove that f has an analytic extension to D, obtained by setting $f = 0$ on $D - \Omega$.

4.29. Let D be a connected domain in $\mathbf{C}^n$, let V be a complex hyperplane intersecting D and let f be holomorphic on $D - V$. Give two proofs for the following assertion: If f has an analytic continuation to a neighbourhood of some point $a \in V \cap D$, then f has an analytic continuation to the whole domain D.

4.30. For $D \subset \mathbf{C}^2$ and $X = D \cap \mathbf{R}^2$, let f be analytic on $D - X$. Prove that f has an analytic extension to D. [One approach is to set $z_1 + iz_2 = z_1'$, $z_1 - iz_2 = z_2'$, so that X becomes a graph over $\mathbf{C}$.]

4.31. Proposition 4.81 has sometimes been stated without the restriction $\sup_K |g(z')| = R_K < R$. Show by an example that some restriction is necessary.

4.32. (Proof of Hartogs' theorem for $n = 2$ without appeal to the recessed edge theorem) For z in a small neighbourhood of 0, let $g = g(z)$ have its values close to 0. Suppose one knows that $-\log|g - w|$ is harmonic in z for every w in a neighbourhood of the circle $C(0, s)$. Deduce that

$$\frac{g_{z\overline{z}}}{g - w} - \frac{g_z g_{\overline{z}}}{(g - w)^2} = \text{constant}.$$

Conclude that $g_{z\overline{z}} = 0$ and $g_z g_{\overline{z}} = 0$, so that either $D_g = g_z \equiv 0$ or $\overline{D}_g \equiv 0$.

4.33. (Continuation) Rule out the possibility $D_g \equiv 0$ in the proof of Hartog' theorem for $n = 2$ by a coordinate transformation, $(z, w) = (z, z + \tilde{w})$. [The singularity set X becomes $\tilde{w} = \tilde{g}(z) = g(z) - z$.]

CHAPTER 5

Holomorphic mappings and complex manifolds

Holomorphic mappings $\zeta = f(z)$ from a connected domain D in a space $\mathbf{C}^m$ to some space $\mathbf{C}^p$ are a useful tool in many problems. They are essential for the definition and study of complex manifolds [Section 5.5-5.7]. Holomorphic maps may be defined by a system of equations

$$(0) \qquad \zeta_j = f_j(z_1, \ldots, z_m),\ j = 1, \ldots, p \quad \text{with} \quad f_j \in \mathcal{O}(D).$$

is a basic property that compositions of such maps are again holomorphic, cf. exercise 1.5.

One often encounters $1-1$ holomorphic maps. In the important case $m = p = n$, such a map will take $D \subset \mathbf{C}^n$ onto a domain D' in $\mathbf{C}^n$, and the inverse map will also be holomorphic (the map f is "biholomorphic"), see Section 5.2. In this case the domains D and D' are called analytically isomorphic, or (bi)holomorphically equivalent; the classes of holomorphic functions $\mathcal{O}(D)$ and $\mathcal{O}(D')$ are closely related.

In $\mathbf{C}$ (but not in $\mathbf{C}^n$), there is a close connection between $1-1$ holomorphic and conformal mappings. A famous result, the Riemann mapping theorem, asserts that any two simply connected planar domains, different from $\mathbf{C}$ itself, are conformally or holomorphically equivalent. However, in $\mathbf{C}^n$ with $n \geq 2$, different domains are rarely holomorphically equivalent, for example, the polydics and the ball are not. Similarly, $\mathbf{C}^n$ domains rarely have nontrivial analytic automorphisms. However, if they do, the automorphism groups give important information. We will discuss some of the classical results of H. Cartan on analytic isomorphisms in $\mathbf{C}^n$ which make it possible to determine the automorphism groups of various special domains [Section 5.3, 5.4].

One-to-one holomorphic maps continue to be an active subject of research. In recent years the main emphasis has been on boundary properties of such maps. Some of the important developments in the area are indicated in Section 5.8; see also the references given in that Section.

5.1 Implicit mapping theorem. The level set (where f=constant) or zero sets of holomorphic maps (0) are a key to their study and applications. The level set of f through the point $a \in D$ is the solution set of the equation $f(z) = f(a)$ or of the system

$$(1a) \qquad 0 = f_j(z) - f_j(a) = \sum_{k=1}^{m} \frac{\partial f_j}{\partial z_k}(a)(z_k - a_k) + \text{higher order terms}, \quad j = 1, \ldots, p.$$

The interesting case is that where the number m of unknowns is at least as large as the number p of equations.

An approximation to the level set is provided by the zero set of the linear part or *differential mapping,*

$$(1b) \qquad df\,\big|_a : df_j = \sum_{k=1}^{m} \frac{\partial f_j}{\partial z_k}(a) dz_k, \quad j = 1, \ldots, p.$$

We will assume that our holomorphic map f is *nonsingular* at a. By that one means that the linear map $df\,\Big|_a$ is nonsingular, that is, it must be of *maximal rank.* Taking $m \ge p$, the (rectangular) *Jacobi matrix* or Jacobian

$$J_f(a) \stackrel{\text{def}}{=} \left[\frac{\partial f_j}{\partial z_k}(a)\right], \quad j = 1, \dots, p; \quad k = 1, \dots, m$$

thus will have *rank equal to p.* The solution set of the linear system $df\,\Big|_a = 0$ will then be a linear subspace of $\mathbf{C}^m$ of complex dimension $n = m - p$.

We now turn to a more precise description of the level set of f when $m - p = n \ge 1$. It is convenient to renumber the variables z_k in such a way that the final $p \times p$ submatrix of $J_f(a)$ becomes invertible. Renaming the last p variables $w_1, \dots, w_p$ and setting $a = 0,\ f(a) + 0$, the system $(1a)$ for the level set becomes

$$(1c) \qquad f_j(z, w) = f_j(z_1, \dots, z_n,\ w_n, \dots, w_p) = 0, \quad j = 1, \dots, p$$

with $f_j(0) = 0$ and

$$(1d) \qquad \det J(0) \stackrel{\text{def}}{=} \det\left[\frac{\partial f_j}{\partial w_k}(0)\right] \neq 0.$$

Under these conditions one has the following extension of the Implicit function theorem 4.21:

Theorem 5.11 (IMPLICIT MAPPING THEOREM)**.** *Let $f = (f_1, \dots, f_p),\ f_j = f_j(z, w)$ be a holomorphic map of the polydisc $\Delta(0, r) \subset \mathbf{C}^n_z \times \mathbf{C}^p_w$ to $\mathbf{C}^p$ such that*

$$f(0) = 0, \quad \det J(0) \neq 0.$$

Then there exist a polydisc $\Delta(0, \rho) = \Delta_n(0, \rho') \times \Delta_p(0, \rho'')$ in $\Delta(0, r)$ and a unique holomorphic map $w = \varphi(z) = (\varphi_1, \dots, \varphi_p)$ from $\Delta_n(0, \rho') \subset \mathbf{C}^n_z$ to $\Delta_p(0, \rho'') \subset \mathbf{C}^p_w$ such that $\varphi(0) = 0$ and

$$f(z, w) = 0 \quad \text{at a point} \quad (z, w) \in \Delta(0, \rho)$$

if and only if

$$w = \varphi(z) \quad \text{with} \quad z \in \Delta_n(0, \rho').$$

COROLLARY 5.12 (*Local form of the zero set for the map f*). In $\Delta(0, \rho) \subset \mathbf{C}^{n+p}$, the zero of the holomorphic map f is the graph of the holomorphic map φ on $\Delta_n(0, \rho')$. Equivalently, the zero set of f in $\Delta(0, \rho)$ is the image of the $1 - 1$ holomorphic map $\psi = (id, \varphi)$ on $\Delta(0, \rho') \subset \mathbf{C}^n$. This map is bicontinuous, hence the zero set of f around the origin is homeomorphic to a domain in $\mathbf{C}^n$ and hence has complex dimension n.

PROOF of Theorem 5.11. In the following, a map φ from $\Delta_n(0, \rho')$ to $\Delta_p(0, \rho'')$ will be loosely referred to as a map with associated polydisc $\Delta_n(0, \rho') \times \Delta_p(0, \rho'')$.

We will use the Implicit function theorem 4.21 and apply induction on the number p of equations $(1c)$. By hypothesis $(1d)$ at least one of the partial derivatives $D_p f_j =$

$\partial f_j/\partial w_p,\ \ j=1,\ldots,p$ must be $\neq 0$ at the origin, say $D_p f_p(0)\neq 0$. One may then solve the corresponding equation

$$f_p(z,w_1,\ldots,w_p)=0$$

for w_p: around 0, it will have a holomorphic solution

$$w_p=\chi(z,w')=\chi(z,w_1,\ldots,w_{p-1}) \tag{1e}$$

with $\chi(0)=0$ and associated polydisc $\Delta_{n+p-1}(0,s')\times\Delta_1(0,s'')$.

Substituting the solution $(1e)$ into the other equations, one obtains a new system of $p-1$ holomorphic equations in $p-1$ unknown functions on some neighbourhood of the origin:

$$g_j(z,w')\overset{\text{def}}{=} f_j(z,w',\,\chi(z,w'))=0,\quad j=1,\ldots,p-1 \tag{1f}$$

with $g_j(0)=0$. The new Jacobian J' will have the elements

$$\frac{\partial g_j}{\partial w_k}=\frac{\partial f_j}{\partial w_k}+\frac{\partial f_j}{\partial w_p}\frac{\partial\chi}{\partial w_k}=\frac{\partial f_j}{\partial w_k}-\frac{\partial f_j}{\partial w_p}\left(\frac{\partial f_p}{\partial w_k}\Big/\frac{\partial f_p}{\partial w_p}\right)\quad j,\,k=1,\ldots,p-1. \tag{1g}$$

In the final step we have used the identity $f_p(z,w',\,\chi(z,w'))=0$ to obtain the relations

$$\frac{\partial f_p}{\partial w_k}+\frac{\partial f_p}{\partial w_k}\frac{\partial\chi}{\partial w_k}\equiv 0,\quad k=1,\ldots,p-1.$$

By $(1g)$ the k-th column of J' is obtained by taking the k-th column of J and subtracting from it a multiple of the final column of J. The zeros which then appear in the last row of J are omitted in forming J', but taken into account for the evaluation of $\det J$:

$$\det J=(\det J')\cdot\frac{\partial f_p}{\partial w_p}.$$

Conclusion: $\det J'\neq 0$ at the origin.

If we assume now that the theorem had been proved already for the case of $p-1$ equations in $p-1$ unknown functions, it follows that the new system $(1f)$ has a holomorphic solution $w'=(\varphi_1,\ldots,\varphi_{p-1})$ around 0 which vanishes at 0 and has associated polydisc $\Delta_n(0,\sigma')\times\Delta_{p-1}(0,\sigma'')$. Combination with $(1e)$ finally furnishes a holomorphic solution $w=(\varphi_1,\ldots,\varphi_p)$ of our system $(1c)$ around 0 which vanishes around 0 and has associated polydisc $\Delta_n(0,\rho')\times\Delta_p(0,\rho'')$. That the map φ is unique follows from the observation that the corresponding map $\psi=(id,\varphi)$ on $\Delta_n(0,\rho')$ is uniquely determined by the zero set of f in $\Delta(0,\rho)$.

REMARKS 5.13. Theorem 5.11 may also be derived from a corresponding implicit mapping theorem of real analysis. Indeed, the system of p holomorphic equations $(1c)$ in p unknown complex functions $w_j=u_j+iv_j$ of $z_1,\ldots,z_n$ can be rewritten as a system of $2p$ real equations in the $2p$ unknown real functions u_j,v_j of $x_1,y_1,\ldots,x_n,y_n$. The new Jacobian $J_{\mathbf{R}}(0)$ will also have nonzero determinant, cf. exercise 5.3, hence the real system is *uniquely solvable* and will furnish *smooth solutions* $u_j(x,y),\ v_j(x,y)$ around the origin. Writing $\varphi_j(x) = u_j(z) + iv_j(z)$, the identities

$$f_j(z_1, \ldots, z_n,\, \varphi_1(z), \ldots, \varphi_p(z)) \equiv 0, \quad j = 1, \ldots, p$$

may be differentiated with respect to each $\overline{z}_\nu$ to show that the functions φ_j satisfy the Cauchy-Riemann equations, hence they are holomorphic.

Actually, the contemporary proofs of the real analysis theorem involve successive approximation or a fixed point theorem, and such techniques may be applied directly to the holomorphic case as well, cf. exercise 5.7.

Inverse maps. We first prove a theorem on the existence of a local holomorphic inverse when the given map has nonvanishing Jacobi determinant. The derivation will be based on the preceding result, but it will be more natural now to interchange the roles of z and w.

Theorem 5.21 (LOCAL INVERSE)**.** *Let g be a holomorphic map of a neighbourhood of* 0 *in $\mathbf{C}^n$ to $\mathbf{C}^n$ such that $g(0) = 0$ and $\det J_g(0) \neq 0$. Then there is a (connected open) neighbourhood U of* 0 *on which g is invertible. More precisely, there is a holomorphic map h of a $\mathbf{C}^n$ neighbourhood V of* 0 *onto U which inverts $g\,|_U$:*

$$w = g(z) \quad \textit{for} \quad z \in U \Longleftrightarrow z = h(w) \quad \textit{for} \quad w \in V.$$

PROOF. Letting w vary over all of $\mathbf{C}^n$ and z over a suitable neighbourhood of 0 in $\mathbf{C}^n$, the equation

$$\zeta = f(w,z) \stackrel{\text{def}}{=} w - g(z) \quad [\text{ or } \zeta_j = w_j - g_j(z),\ \forall j]$$

will define a holomorphic map of a polydisc $\Delta(0,r)$ in $\mathbf{C}^{2n}$ to $\mathbf{C}^n$. This map will satisfy the conditions of the Implicit mapping theorem 5.11 with $p = n$ and (w,z) instead of (z,w) :

$$f(0) = 0, \quad \det\left[\frac{\partial f_j}{\partial z_k}(0)\right] = \pm\det\left[\frac{\partial g_j}{\partial z_k}(0)\right] \neq 0.$$

Hence there are a polydisc $\Delta(0,\rho) = \Delta_n(0,\rho') \times \Delta_n(0,\rho'')$ in $\mathbf{C}^n_w \times \mathbf{C}^n_z$ and a unique holomorphic map $z = h(w)$ from $\Delta_n(0,\rho')$ to $\Delta_n(0,\rho'')$ such that $h(0) = 0$ and

$$\begin{aligned} &f(w,z) \equiv w - g(z) = 0 \quad \text{for} \quad (w,z) \in \Delta(0,\rho) \\ &\Longleftrightarrow z = h(w) \quad \text{for} \quad w \in \Delta_n(0,\rho') \end{aligned} \qquad (2a)$$

We still have to identify suitable sets U and V. For U one may take any (connected open) neighbourhood of 0 in $\Delta_n(0,\rho'')$ for which $V \stackrel{\text{def}}{=} g(U)$ belongs to $\Delta_n(0,\rho')$. Indeed, for such a choice of U and any $z \in U$, the point $(g(z), z)$ lies in $\Delta(0,\rho)$, hence by (2a) $z = h \circ g(z)$, so that $h \mid V$ is the inverse of $g \mid U$ and conversely. Finally, by the arrow pointing to the left, $g = h^{-1}$ on $h(\Delta_n(0,\rho'))$, hence since h is continuous, $V = h^{-1}(U)$ will be open.

We can now prove the fundamental result that a $1-1$ holomorphic map in $\mathbf{C}^n$ (with n-dimensional domain) is *biholomorphic*, that is, the inverse is also holomorphic (Clements 1912):

Theorem 5.22 (HOLOMORPHY OF GLOBAL INVERSE). *Let $\Omega \subset \mathbf{C}^n$ be a connected domain and let $w = f(z)$ be a $1-1$ holomorphic map of Ω onto a set Ω' in $\mathbf{C}^n$. Then Ω' is also a connected domain and the Jacobi determinant,* $\det J_f(z)$ *is different from zero throughout Ω, hence f^{-1} will be a holomorphic map of Ω' onto Ω.*

PROOF. The proof is a nice application of the local theory of zero sets and will use induction on the dimension n. In view of Theorem 5.21 we need only show that $\det J_f(a) \neq 0$, $\forall a \in \Omega$; it will follow that Ω' is open. Whenever convenient, we may take $a = f(a) = 0$.

(i) for $n = 1$ it is well known that the map $w = f(z)$ is $1-1$ around the origin (if and) only if $f'(0) \neq 0$. Indeed, if

$$f(z) = bz^k + \text{higher order terms}, \quad b \neq 0,\ k \geq 2,$$

the $f(z)$ will assume all nonzero values c close to 0 at k different points z near the origin. Cf. the proof of the Open mapping theorem 1.81; the k roots will be distinct because $f'(z)$ cannot vanish for small $z \neq 0$.

(ii) Now the induction step – first an outline. We have to prove that the analytic function $\det J_f(z)$ on $\Omega \subset \mathbf{C}^n$, $n \geq 2$ is zero free. Supposing on the contrary that for our $1-1$ map f, the zero set $Z = Z(\det J_f)$ is nonempty, the induction hypothesis will be used to show that all elements of the matrix J_f must vanish on Z. From this it will be derived that f=constant on Z around the regular points, contradicting the hypothesis that f is $1-1$.

For simplicity we focus on the typical case $n = 3$, assuming the result for $n = 2$. Thus, let f :

$$w_j = f_j(z_1, z_2, z_3), \quad j = 1, 2, 3 \tag{2b}$$

be a $1-1$ holomorphic map on $\Omega \subset \mathbf{C}^3$, $0 \in \Omega$, with $f(0) = 0$ and suppose that

$$\det J_j = \begin{vmatrix} D_1 f_1 & D_2 f_1 & D_3 f_1 \\ D_1 f_2 & D_2 f_2 & D_3 f_2 \\ D_1 f_3 & D_2 f_3 & D_3 f_3 \end{vmatrix} = 0 \quad \text{for} \quad z = 0 \tag{2c}$$

(a) We first assume that the Jacobi matrix $J_j(0)$ contains a nonzero element; renumbering coordinates we may take $D_3 f_3(0) \neq 0$. Replacing w_i by $w_i - c_i w_3$ with suitable c_i, $i = 1, 2$ we can ensure that for the representation of our map, $D_3 f_i(0) = 0$, $i = 1, 2$. Then by $(2c)$ also

$$\begin{vmatrix} D_1 f_1 & D_2 f_2 \\ D_1 f_2 & D_2 f_2 \end{vmatrix} = 0 \quad \text{for} \quad z = 0. \tag{2d}$$

Around the origin the zero set $Z(f_3)$ will be the graph of a holomorphic function $z_3 = \varphi(z_1, z_2)$ with $\varphi(0) = 0$ [Implicit function theorem 4.21]. The restriction of f to $Z(f_3)$ must be $1-1$; around 0 this restriction is given by

$$w_i = h_i(z_1, z_2) \stackrel{\text{def}}{=} f_i(z_1, z_2, \varphi(z_1, z_2)), \quad i = 1, 2;\ w_3 = 0.$$

It follows that the holomorphic map h must be $1-1$ around $0 \in \mathbf{C}^2$, hence by the induction hypothesis, $\det J_h(0) \neq 0$. However, since $D_3 f_i(0) = 0$,

$$D_j h_i(0) = D_j f_i(0) + D_3 f_i(0) D_j \varphi(0) = D_j f_i(0), \quad i, j = 1, 2$$

so that by $(2d)$, $\det J_h(0) = 0$. This contradiction proves that all elements in $J_f(0)$ must vanish.

(b) By the preceding argument, all elements $D_k f_j$ of the Jacobian J_f must vanish at every point of the zero set $Z = Z(\det J_f)$ in Ω. This zero set can not be all of Ω, for otherwise $D_k f_j \equiv 0$, $\forall j, k$ and then f would be constant on Ω, hence not $1-1$.

Thus $\det J_f \not\equiv 0$ and the zero set Z will contain a regular point a [cf. Theorem 4.62]. By suitable manipulation we may assume that Z is the graph of a holomorphic function $z_3 = \psi(z_1, z_2)$ around a. Then the restriction $f \mid Z$ is locally given by

$$w_i = k_i(z_1, z_2) \stackrel{\text{def}}{=} f_i(z_1, z_2, \psi(z_1, z_2)), \quad i = 1, 2, 3.$$

However, the derivatives $D_j k_i$ will all vanish around $a' = (a_1, a_2)$. Indeed, they are linear combinations of $D_j f_i$ and $D_3 f_i$ on Z, hence equal to zero. The implication is that k=constant around a' hence $f \mid Z$ is constant around a, once again a contradiction.

The final conclusion is that $\det J_f \neq 0$ throughout Ω, thus completing the proof for $n = 3$. The proof for general n is entirely similar.

REMARKS 5.23. Let us first consider holomorphic maps f from $\Omega \subset \mathbf{C}^m$ to $\Omega' \subset \mathbf{C}^p$. In the $1-1$ case such a map f is biholomorphic if $p = m$, but if $p > m$, the inverse map need not be holomorphic on $f(\Omega)$ [it need not even be continuous!], cf. exercise 5.9.

For $p = m$ biholomorphic maps $f : \Omega \to f(\Omega)$ are examples of so-called proper maps. A map $f : \Omega \to \Omega'$ is called *proper* if for any compact subset $K \subset \Omega'$ the pre-image $f^{-1}(K)$ is compact in Ω. When Ω and Ω' are bounded, this means that for any sequence of points $\{z^{(\nu)}\}$ in Ω which tends to the boundary $\partial\Omega$, the image sequence $\{f(z^{(\nu)})\}$ must tend to the boundary $\partial\Omega'$.

5.3 Analytic isomorphisms I. In Sections 5.3 and 5.4, D will always denote a connected domain in $\mathbf{C}^n$.

Definition 5.31. *A $1-1$ holomorphic (hence biholomorphic) map of D onto itself is called an (analytic) AUTOMORPHISM of D. The group of all such automorphisms is denoted by* Aut D.

Domains that are analytically isomorphic must have isomorphic automorphism groups. Indeed, if f establishes an analytic isomorphism of D onto $D' \subset \mathbf{C}^n$ and h runs over the automorphisms of D, then $f \circ h \circ f^{-1}$ runs over the automorphisms of D'. H. Cartan's 1931 theorem below will make it possible to determine the automorphism groups of some simple domains and to establish the non-isomorphy of certain pairs of domains, cf. Section 5.4.

EXAMPLES 5.32. What are the automorphisms f of the *unit disc* $\Delta(0, 1)$ in $\mathbf{C}$? Suppose first that $f(0) = 0$. Schwarz's lemma will show that such an automorphism must have the form

$$f(z) = e^{i\theta} z \quad \text{for some} \quad \theta \in \mathbf{R}.$$

[Indeed, by the maximum principle $|f(z)/z|$ must be bounded by 1 on Δ and similarly, using the inverse map, $|z/f(z)| \le 1$. Thus $|f(z)/z| = 1$, so that $f(z)/z$ must be constant.]

There also are automorphisms f that take the origin to an arbitrary point $a \in \Delta(0,1)$, or that take such a point a to 0. An example of the latter is given by

$$f(z) = \frac{z-a}{1-\overline{a}z}. \tag{3a}$$

[Formula (3a) defines a $1-1$ holomorphic map on $\mathbf{C} - \{1/\overline{a}\}$ and $|f(z)| = 1$ for $|z| = 1$, hence $|f(z)| < 1$ for $|z| < 1$. Every value $w \in \Delta(0,1)$ is taken on by f on $\Delta(0,1)$.]

On the unit bidisc $\Delta_2(0,1) = \Delta(0,0;\,1,1)$ in $\mathbf{C}^2$ the formulas

$$w_j = g_j(z) = \frac{z_j - a_j}{1 - \overline{a}_j z_j}, \quad j = 1,2 \tag{3b}$$

define an automorphism that carries $a = (a_1, a_2) \in \Delta_2(0,1)$ to the origin.

EXAMPLES 5.33. The *unitary transformations* of $\mathbf{C}^n$ are the linear transformations

$$w = Az \quad \text{or} \quad w_j = \sum_{k=1}^{n} a_{jk} z_k, \quad j = 1, \ldots, n$$

that leave the scalar product invariant [and hence all norms and all distances]:

$$(Az, Az') = (z, \overline{A}^T Az') = (z, z'), \quad \forall z, z'.$$

[Thus they may also be described by the condition $\overline{A}^T A = I_n$ or $\overline{A}^T = A^{-1}$.] In particular $|Az| = |z|$, $\forall z$: unitary transformations define automorphisms of the *unit ball* $B(0,1)$ in $\mathbf{C}^n$.

There are also automorphisms of the ball that carry an arbitrary point $a \in B(0,1)$ to the origin. First carrying out a suitable unitary transformation, it will be sufficient to consider the case where $a = (c, 0, \ldots, 0)$ with $c = |a| > 0$. If $n = 2$ one may then take

$$w_1 = \frac{z_1 - c}{1 - cz_1}, \quad w_2 = \frac{(1-c^2)^{\frac{1}{2}}}{1 - cz_1} z_2. \tag{3c}$$

Cf. also exercises 5.21, 5.22 and [Rudin 4].

Theorem 5.34. *Let $D \subset \mathbf{C}^n$ be bounded and let $w = f(z)$ be a holomorphic map of D into itself with fixed point $a : f(a) = a$. Suppose furthermore that the Jacobian $J_f(a)$ is equal to the $n \times n$ identity matrix I_n, so that the development of f around a can be written in vector notation as*

$$f(z) = a + (z-a) + P_2(z-a) + \ldots + P_s(z-a) + \ldots,$$

where $P_s(\zeta)$ is a vector [n-tuple] of homogeneous polynomials P_{sj} in $\zeta_1, \ldots, \zeta_n$ of degree s. Then f is the identity map:

$$f(z) \equiv z.$$

PROOF. The essential idea of the proof is to iterate the map f. The iterates $f \circ f,\ f \circ f \circ f, \ldots$ will also be holomorphic maps $D \to D$ with fixed point a. Taking $a = 0$ as we may, the components of f become

$$f_j(z) = \sum_{\alpha \geq 0} c_\alpha^{(j)} z^\alpha = z_j + P_{2j}(z) + \ldots + P_{sj}(z) + \ldots,$$

where P_{sj} is a homogeneous polynomial of degree s. We choose positive vectors $r = (r_1, \ldots, r_n)$ and $R = (R_1, \ldots, R_n)$ such that

$$\Delta(0, r) \subset D \subset \Delta(0, R).$$

Then f_j will in particular be holomorphic on $\overline{\Delta}(0, r)$ and $|f_j|$ will be bounded by R_j. Hence by the Cauchy inequalities 1.65:

$$|c_\alpha^{(j)}| \leq R_j / r^\alpha, \quad \forall \alpha, \quad j = 1, \ldots, n. \tag{3d}$$

Now let s be the smallest integer ≥ 2 such that

$$f(z) = z + P_s(z) + h(igher)\, o(rder)\, t(erms)$$

with $P_s \not\equiv 0$ [if there is no such s we are done]. Then the composition $f \circ f$ has the expansion

$$\begin{aligned} f \circ f(z) &= f(z) + P_s \circ f(z) + h.o.t. \\ &= z + P_s(z) + h.o.t. + P_s(z) + h.o.t. \\ &= z + 2P_s(z) + h.o.t. \end{aligned}$$

[It is convenient to use components and to begin with the cases $n = 1$ and $n = 2$.] Quite generally, the k times iterated map will have the expansion

$$f^{\circ k}(z) = f \circ f \circ \ldots \circ f(z) = z + kP_s(z) + h.o.t.$$

[Use induction.] This is also a holomorphic map of D into itself, hence inequality (3d) may be applied to the coefficients in kP_{sj} :

$$|kc_\alpha^{(j)}| \leq R_j / r^\alpha,\ |\alpha| = s; \quad j = 1, \ldots, n; \quad k = 1, 2, \ldots.$$

The conclusion for $k \to \infty$ is that $P_s \equiv 0$ and this contradiction shows that $f(z) \equiv z$.

5.4 Analytic isomorphisms II: circular domains.

DEFINITION 5.41. $D \subset \mathbf{C}^n$ is called a *circular domain* if $a \in D$ implies that $e^{i\theta} a = (e^{i\theta} a_1, \ldots, e^{i\theta} a_n)$ belongs to D for every real number θ.

For $a \neq 0$, the points $z = e^{i\theta} a,\ \theta \in R$ for a circle with center 0 in the complex line through 0 and a. (Circular domains need not be multicircular!) Circular domains admit the one-parameter family of *automorphisms* $\{k_\theta\}$, given by the formula

$$k_\theta(z) = e^{i\theta} z,\ z \in D. \tag{4a}$$

Observe that *linear* mappings *commute* will transformations k_θ :

$$Ak_\theta(z) = Ae^{i\theta} z = e^{i\theta} Az = k_\theta(Az). \tag{4b}$$

The proof of the main theorem will depend on the following fact:

Lemma 5.42. *Linear mappings are the* only *holomorphic mappings $f = (f_1, \ldots, f_n)$ of a neighbourhood of* 0 *in* $\mathbf{C}^n$ *that commute with all* k_θ*'s.*

PROOF. Indeed, suppose that

$$f(e^{i\theta}z) \equiv e^{i\theta} f(z) \quad \text{or} \quad f_j(e^{i\theta}z) \equiv e^{i\theta} f_j(z), \quad \forall j.$$

Expanding $f_j(z) = \sum_{\alpha \geq 0} b_\alpha z^\alpha$, it follows that

$$f_j(e^{i\theta}z) = \sum b_\alpha (e^{i\theta}z_1)^{\alpha_1} \ldots (e^{i\theta}z_n)^{\alpha_n} = \sum b_\alpha e^{i|\alpha|\theta} z^\alpha \equiv e^{i\theta} \sum b_\alpha z^\alpha,$$

hence by the uniqueness of the power series representation,

$$(e^{i|\alpha|\theta} - e^{i\theta}) b_\alpha = (e^{i(|\alpha|-1)\theta} - 1) e^{i\theta} b_\alpha = 0.$$

If this holds for all θ's [or for a suitable subset!], the conclusion is that $b_\alpha = 0$ whenever $|\alpha| \neq 1$, and then f_j is linear.

Theorem 5.43. *Let D and D' be bounded circular domains in $\mathbf{C}^n$ containing the origin. Suppose that $f = (f_1, \ldots, f_n)$ is an analytic isomorphism of D onto D' such that $f(0) = 0$. Then the map f must be linear:*

$$f_j(z) = a_{j1}z_1 + \ldots + a_{jn}z_n, \quad j = 1, \ldots, n.$$

PROOF. The proof will involve a number of holomorphic maps φ [of a neighbourhood of 0 in $\mathbf{C}^n$ to $\mathbf{C}^n$] with $\varphi(0) = 0$. We will represent the differential or *linear part* of such a φ at the origin by

$$d\varphi = d\varphi\big|_0 : w_j = \sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial z_k}(0) z_k.$$

Observe that such linear parts obey the following rules:

$$d(\varphi \circ \psi) = d\varphi \circ d\psi, \quad d\varphi^{-1} \circ d\varphi = d(\varphi^{-1} \circ \varphi) = id, \quad dk_\theta = k_\theta \tag{4c}$$

[cf. (4*a*); the differential of a linear map is the map itself].

To the given analytic isomorphism f we associate the auxiliary map

$$g = k_{-\theta} \circ f^{-1} \circ k_\theta \circ f, \quad \theta \in \mathbf{R} \quad \text{fixed.} \tag{4d}$$

This will be an automorphism of D with $g(0) = 0$. Linearization gives

$$dg = dk_{-\theta} \circ df^{-1} \circ dk_\theta \circ df = k_{-\theta} \circ k_\theta \circ df^{-1} \circ df = id,$$

because k_θ commutes with linear maps. Thus the development of g around the origin has the form

$$g(z) = z + P_2(z) + h.o.t.$$

Applying Theorem 5.32 to g we find that $g(z) \equiv z$ or $g = id$. Returning to the definition of g (4*d*), the conclusion is that

$$f \circ k_\theta = k_\theta \circ f, \quad \forall \theta \in \mathbf{R},$$

hence by Lemma 5.42, f is linear.

As an application one may verify a classical result of Poincaré:

Theorem 5.44. *The unit polydisc $\Delta(0,1)$ and the unit ball $B(0,1)$ in $\mathbf{C}^2$ are not analytically isomorphic.*

PROOF. Suppose that f is an analytic isomorphism of Δ onto B. It may be assumed that $f(0)=0$. Indeed, if f initially carried $\zeta \in \delta$ to 0 in B, we could replace f by $f \circ g^{-1}$ where g is an automorphism of Δ that takes ζ to 0, cf. $(3a)$.

Theorem 5.43 now shows that $f=(f_1, f_2)$ must be linear:

$$f_1(z) = az_1 + bz_2, \quad f_2(z) = cz_1 + dz_2.$$

Here $|f_j(z)|$ must be <1 for $|z_\nu| < 1$. Setting $z_1 = re^{it}$ and $z_2 = r$, it follows for $r \uparrow 1$ and suitable choices of t that

$$(4e) \qquad |a| + |b| \leq 1, \quad |c| + |d| \leq 1.$$

We also know that $z \to \partial\Delta$ must imply $f(z) \to \partial B$ [the map f must be proper, cf. 5.23]. Setting $z = (r,0)$ or $(0,r)$ it follows for $r \uparrow 1$ that

$$(4f) \qquad |a|^2 + |c|^2 = 1, \quad |b|^2 + |d|^2 = 1.$$

Combination of $(4f)$ and $(4e)$ shows that

$$2 = |a|^2 + |c|^2 + |b|^2 \leq (|a| + |b|)^2 + (|c| + |d|)^2 \leq 2,$$

hence $|a||b| + |c||d| = 0$, so that

$$(4g) \qquad ab = cd = 0.$$

If $b=0$ we must have $|d|=1$ $(4f)$, hence $c=0$ $(4g)$ and thus $|a|=1$ $(4f)$; if $a=0$ we must have $|c|=1$, $d=0$ and $|b|=1$. In conclusion, the matrix of the linear transformation f must have one of the following forms:

$$\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & e^{i\theta_2} \\ e^{i\theta_1} & 0 \end{bmatrix}$$

However, the corresponding maps take Δ onto itself, not onto B! This contradiction shows that there is no analytic isomorphism of Δ onto B.

A different proof that readily extends to $\mathbf{C}^n$ is indicated in exercise 5.13. For further results on Aut D, see [Behnke-Thullen].

5.5 Complex submanifolds of $\mathbf{C}^n$. It is useful to start with a discussion of local holomorphic coordinates.

DEFINITION 5.51. Suppose we have a system of functions

$$(5a) \qquad w_1 = g_1(z), \ldots, w_n = g_n(z)$$

which defines a $1-1$ holomorphic map g of a neighbourhood U of a in $\mathbf{C}^n_z$ onto a neighbourhood V of $b = g(a)$ in $\mathbf{C}^n_w$. Such a system is called a *local coordinate system for* $\mathbf{C}^n$ at a, or a *holomorphic coordinate system for* U.

The reasons for the names are: (i) there is a [holomorphic] $1-1$ correspondence between the points $z \in U$ and the points $w = g(z)$ of the neighbourhood V of $b = g(a)$; (ii) every holomorphic function of z on U can be expressed as a holomorphic function of w on V (and conversely), cf. Section 5.2. By the same Section, holomorphic functions (5a) will form a local coordinate system for $\mathbf{C}^n$ at a if and only if

$$(5b) \qquad \det J_g(a) \neq 0.$$

Lemma 5.52. *Suppose we have a family of* $p < n$ *holomorphic functions* $\{w_1 = g_1(z), \ldots, w_p = g_p(z)\}$ *with*

$$\operatorname{rank} J(g_1, \ldots, g_p) = p$$

at a. *Such a family can always be augmented to a local coordinate system* (5a) *for* $\mathbf{C}^n$ *at* a.

PROOF. Indeed, the vectors

$$c_1 = \left[\frac{\partial g_1}{\partial z_k}(a)\right]_{k=1,\ldots,n}, \ldots, c_p = \left[\frac{\partial g_p}{\partial z_k}(a)\right]_{k=1,\ldots,n}$$

will be linearly independent in $\mathbf{C}^n$. Thus this set can be augmented to a basis of $\mathbf{C}^n$ by adding suitable constant vectors

$$c_q = [c_{qk}]_{k=1,\ldots,n}, \quad q = p+1, \ldots, n.$$

Defining $g_q(z) = \sum_k c_{qk} z_k$ for $p+1 \leq q \leq n$, the holomorphic functions

$$w_1 = g_1(z), \ldots, w_n = g_n(z)$$

will satisfy condition (5b), hence they form a coordinate system for $\mathbf{C}^n$ at a.

DEFINITION 5.53. A subset M of $\mathbf{C}^n$ is called a complex submaifold if for every point $a \in M$, there are a neighbourhood U of a in $\mathbf{C}^n$ and an associated system of holomorphic functions $g_1(z), \ldots, g_p(z)$, with $\operatorname{rank} J(g_1, \ldots, g_p)$ equal to p on U, such that

$$(5c) \qquad M \cap U = \{z \in U : g_1(z) = \ldots = g_p(z) = 0\}.$$

All values of $p \geq 0$ and $\leq n$ are allowed; it is not required that p be the same everywhere on M.

Examples. The zero set Z_f of a holomorphic function f on open $\Omega \subset \mathbf{C}^n$ is in general not a complex submanifold, but the subset Z_f^* of the regular points of Z_f is one, cf. Section 4.6. Any open set $\Omega \subset \mathbf{C}^n$ is a complex submanifold. The solution set of a system of $p \leq n$ linear equations over $\mathbf{C}^n$ with nonsingular coefficient matrix is a complex submanifold.

Locally, a complex submanifold M is *homeomorphic to a domain in some space* $\mathbf{C}^s$, $0 \leq s \leq n$. In fact, the Implicit mapping theorem 5.11 will give an effective dual representation. Using the defining equations ($5c$) for M around a, one can express p of the coordinates z_j in terms of the other $n - p$ with the aid of a holomorphic map φ. One thus obtains the following

DUAL REPRESENTATION 5.54. Up to an appropriate renumbering of the coordinates, the general point $a \in M$ in Definition 5.53 will have a neighbourhood $\Delta(a, \rho) \subset U$ such that

$$(5d) \qquad M \cap \Delta(a, \rho) = \{z \in \mathbf{C}^n : z = \psi(z') = (z', \varphi(z')), \quad z' \in \Delta_s(a', \rho') \subset \mathbf{C}^s\}.$$

Here $\psi = (id, \varphi)$ is a $1-1$ holomorphic map on $\Delta(a', \rho')$ such that $g_1 \circ \psi = \ldots = g_p \circ \psi \equiv 0$ and $s = n - p$.

Such a map ψ is called a *local* (holomorphic) *parametrization* of M at a and the number s is called the (complex) *dimension* of M at a. The dimension will be locally constant; the maximum of the local dimensions is called $\dim M$. If M is connected, $\dim_a M = \dim M$ for all $a \in M$.

One may use the dual representation ($5d$) to define holomorphic functions on a complex submanifold M of $\mathbf{C}^n$:

DEFINITION 5.55. A function $f : M \to \mathbf{C}$ is called *holomorphic* at (or around) $a \in M$ if for some local holomorphic parametrization ψ of M at a, the composition $f \circ \psi$ is holomorphic in the ordinary sense.

In order to justify this definition, one has to show that different local parametrizations of M at a will give the same class of holomorphic functions on M at a. We do this by proving the following characterization:

Theorem 5.56. *A function f on a complex submanifold M of $\mathbf{C}^n$ is holomorphic at $a \in M$ if and only if it is locally the restriction of a holomorphic function on some $\mathbf{C}^n$ neighbourhood of a.*

PROOF. (i) The difficult part is to extend a given holomorphic function f on M to a holomorphic function on a $\mathbf{C}^n$ neighbourhood of $a \in M$. The idea is simple enough: start with a set of local defining functions $w_1 = g_1, \ldots, w_n = g_n$ for $\mathbf{C}^n$ at a [lEMMA 5.52]. In the w-coordinates M is locally given by $w_1 = \ldots = w_p = 0$ and one can use $w_{p+1}, \ldots, w_n$ as local coordinates for M. It turns out that the given f on M can be considered locally as a holomorphic function $F(w_{p+1}, \ldots, w_n)$. The latter actually defines a holomorphic function on a $\mathbf{C}^n$ neighbourhood of a which is independent of $w_1, \ldots, w_p$.

We fill in some details. In a small neighbourhood $U = \Delta(a, \rho)$ of a we will have two representations for M. There is a certain local parametrization ψ ($5d$) which was used to define f as a holomorphic function on M at a :

$$f \circ \psi(z') \in \mathcal{O}\{\Delta(a', \rho')\}.$$

We also have the initial representation ($5c$). Using the augmentation of Lemma 5.52 and taking U small enough, the neighbourhood $V = g(U)$ of $b = g(a)$ in $\mathbf{C}^n_w$ will give us representations

$$(5e) \qquad \begin{aligned} U = h(V) = \{z \in \mathbf{C}^n : z = h(w),\, w \in V\}, \quad h = g^1, \\ M \cap U = h(V \cap \{w_1 = \ldots = w_p = 0\}) = \{z \in \mathbf{C}^n : z = h(0, \tilde{w}),\, (0, \tilde{w}) \in V\}. \end{aligned}$$

Here $(0, \tilde{w}) = (0, \ldots, 0, w_{p+1}, \ldots, w_n)$. In view of ($5d$) we obtain from ($5e$) a holomorphic map of the $(0, \tilde{w}$-part of V onto $\Delta(a', \rho')$:

$$z' = h'(0, \tilde{w}) : \text{ written out, } z_j = h_j(0, \tilde{w}), \quad j = 1, \ldots, s.$$

Thus in terms of the w-coordinates, $f \mid M \cap U$ is given by

$$(5f) \qquad F(w) \stackrel{\text{def}}{=} f \circ \psi \circ h'(0, \tilde{w}),$$

$(0, \tilde{w}) \in V$. If we now let w run over all of V, formula ($5f$) furnishes a holomorphic function $F(w)$ on all of U which is independent of $w_1, \ldots, w_p$.

(ii) The proof in the other direction is simple. Indeed, if $f^*(z)$ is any holomorphic function on a $\mathbf{C}^n$ neighbourhood of $a \in M$ and ψ is any local parametrization ($5d$) of M at a, then the restriction $f^* \mid M$ is holomorphic at a since

$$f^*\big|_M \circ \psi = f^* \circ \psi$$

will be holomorphic on $\Delta(a', \rho') \subset \mathbf{C}^s$ for small ρ'.

5.6 Complex manifolds. A topological *manifold* X of (real) dimension n is a Hausdorff space, in which every point has a neighbourhood that is homeomorphic to a (connected) domain in $\mathbf{R}^n$. Further structure may be introduced via an *atlas* for X, that is, a family $\mathcal{U}$ of *coordinate systems* (U, ρ), consisting of domains U which jointly cover X and associated homeomorphisms ρ ("projections") onto domains in $\mathbf{R}^n$. If $(U, \rho), (V, \sigma) \in \mathcal{U}$, $U \cap V \neq \emptyset$ are "overlapping coordinate systems", the composition

$$(6a) \qquad \sigma \circ \rho^{-1} : \rho(U \cap V) \to \sigma(U \cap V)$$

must be a homeomorphism between domains in $\mathbf{R}^n$.

DEFINITION 5.61. A *complex (analytic) manifold* X, of complex dimension n, is a topological manifold of real dimension $2n$ with a *complex* structure. The latter is given by a complex (analytic) atlas $\mathcal{U} = \{(U, \rho)\}$, that is, an atlas for which the projections $\rho(U)$ are domains in $\mathbf{C}^n$ while the homeomorphisms ($6a$) are $1 - 1$ (BI)HOLOMORPHIC maps.

The complex structure makes it possible to define holomorphic functions on X :

DEFINITION 5.62. Let X be a complex manifold, Ω a domain in X. A function $f : \Omega \to \mathbf{C}$ is called holomorphic if for some covering of Ω by coordinate systems (U, ρ) of the complex atlas, the functions

$$(6b) \qquad f \circ \rho^{-1} : \rho(\Omega \cap U) \to \mathbf{C}$$

are ordinary holomorphic functions on domains in a space $\mathbf{C}^n$.

The property of holomorphy of f at $a \in X$ will not depend on the particular coordinate system that is used around a [the maps ($6a$) are biholomorphic.] Many results on ordinary holomorphic functions carry over to the case of complex manifolds, for example, the Uniqueness theorem (1.54) and the Open mapping theorem (1.81). Thus if X is connected and compact, an everywhere holomorphic function f on X must be constant. [Indeed, $|f|$ will assume a maximum value somewhere on X.]

Holomorphic functions on a complex manifold are holomorphic maps from the manifold to $\mathbf{C}$. In general holomorphic maps are defined in much the same way:

DEFINITION 5.63. Let X_1, X_2 be complex manifolds with atlanta $\mathcal{U}^1$, $\mathcal{U}^2$, respectively and let $\Omega_1 \subset X_1$, $\Omega_1 \subset X_1$ be domains. A map $f : \Omega_1 \to \Omega_2$ is called holomorphic if for any $(U, \rho) \in \mathcal{U}^1$, $(V, \sigma) \in \mathcal{U}^2$ with $V \cap f(U \cap \Omega_1) \neq \emptyset$ the map

$$\sigma \circ f \circ \rho : \rho(U \cap \Omega_1) \to \sigma(V)$$

is holomorphic.

One similarly tranfers notions like biholomorphic map and analytically equivalence to complex manifolds. It should be noticed that a topological manifold may very well carry different complex structures, leading to complex manifolds that are not analytically equivalent, cf. exercise 5.34, 5.35.

EXAMPLE 5.64. Let $\mathbf{C}_e$ be the extended complex plane $\mathbf{C} \cup \{\infty\}$ or the *Riemann sphere* with the standard topology. We may define a complex structure by setting

$$(U, \rho) = \mathbf{C}_e - \{\infty\}, \rho(z) = z), \quad (V, \sigma) = (\mathbf{C}_e - \{0\},\ \sigma(z) = 1/z).$$

Both U and V are homeomorphic to the complex plane. Clearly $U \cap V = \mathbf{C} - \{0\}$ and the same holds for $\rho(U \cap V)$ and $\sigma(U \cap V)$;

$$\sigma \circ \rho^{-1}(z) = 1/z$$

is a $1 - 1$ holomorphic map of $\rho(U \cap V)$ onto $\sigma(U \cap V)$.

A function f on a domain $\Omega \subset \mathbf{C}_e$ containing ∞ will be holomorphic at ∞ if $f \circ \sigma^{-1}(z) = f \circ \rho^{-1}(1/z)$ is holomorphic at $\sigma(\infty) = 0$ [or can be extended to a function holomorphic at 0]. A function which is holomorphic everywhere on $\mathbf{C}_e$ must be constant.

One may also obtain the Riemann sphere $\mathbf{C}_e$ from $\mathbf{C}$ via the introduction of *homogeneous coordinates.* Starting with the collection of nonzero complex pairs (w_0, w_1), one introduces the equivalence relation

$$(w_0, w_1) \sim (w_0', w_1') \Longleftrightarrow w_0' = \lambda w_0, \quad w_1' = \lambda w_1$$

for some nonzero $\lambda \in \mathbf{C}$. The point $z \in \mathbf{C}$ is represented by $(1, z)$ and equivalent pairs. Points far from the origin have the form $(1, \mu)$ with large (complex μ and they are also conveniently represented by $(1/\mu,\ 1)$. The point at ∞ will be represented by the limit pair $(0,1)$. This approach leads to the complex projective plane $\mathbf{P}^1$ which is analytically isomorphic to the Riemann sphere, cf. Section 5.7 below.

EXAMPLE 5.65. Let $\mathcal{R}$ be the *Riemann surface* for the complete analytic function $\log z$ on $\mathbf{C}-\{0\}$, cf. Section 2.1. All possible local power series for $\log z$ may be obtained from the special function elements (a_k, H_k, f_k), $k \in \mathbf{Z}$ defined below, where H_k is a half-plane containing a_k and $f_k(z)$ a corresponding holomorphic branch of $\log z$:

$$\begin{cases} a_k & = \; e^{ik\pi/2}, \quad H_k = \{z \in \mathbf{C} : (k-1)\pi/2 < \arg z < (k+1)\pi/2\}, \\ \\ f_k(z) & = \; \text{branch of } \log z \text{ on } H_k \text{ with } (k-1)\pi/2 < \operatorname{Im} f_k(z) < (k+1)\pi/2. \end{cases}$$

The points of $\mathcal{R}$ have the form

$$\begin{cases} p & = \; (b, g_b), \quad b \in \mathbf{C}\{0\}, \\ \\ g_b & = \; \text{power series at } b \text{ for a branch } g(z) \text{ of } \log z \text{ on, say, a} \\ & \quad\; \text{convex neighbourhood } V \text{ of } b \text{ in } \mathbf{C}-\{0\}. \end{cases}$$

Corresponding *basic neighbourhoods* $\mathcal{N}(p, V, g)$ in $\mathcal{R}$ consist of all points $q = (z, h_z)$ such that $z \in V$ and $h_z = g_z$. There is a *projection* ρ of $\mathcal{R}$ onto $\mathbf{C}-\{0\}$ given by

$$\rho(p) = \rho((b, g_b)) = b.$$

It is not difficult to verify that $\mathcal{R}$ is a Hausdorff space and that the restriction of ρ to $\mathcal{N}(p, V, g)$ is a homeomorphism onto $V \subset \mathbf{C}$. Finally, the multivalued function $\log z$ on $\mathbf{C}-\{0\}$ may be redefined as a single-valued function $\mathcal{L}\mathrm{og}$ on $\mathcal{R}$:

$$\mathcal{L}\mathrm{og}\, p = \mathcal{L}\mathrm{og}\, g(b, g_b) = g(b). \tag{6c}$$

We now use the special basic neighbourhoods $\mathcal{N}(a_k, H_k, f_k)$ and the projection ρ to define a complex structure on $\mathcal{R}$:

$$U_k = \mathcal{N}(a_k, H_k, f_k), \quad \rho_k = \rho \mid U_k, \quad \forall k \in \mathbf{Z}. \tag{6d}$$

For nonempty $U_j \cap U_k$, the map $\rho_k \circ \rho_j^{-1}$ is simply the identity map on $\rho_j(U_j \cap U_k) = \rho_k(U_j \cap U_k)$. We will verify that the function $\mathcal{L}\mathrm{og}$ is holomorphic on $\mathcal{R}$ in the sense of Definition 5.62. Indeed, ρ_k is a homeomorphism of U_k onto H_k and the points $q = (z, h_z)$ in U_k have the form $\rho_k^{-1}(z)$, $z \in H_k$, implying that $h_z = (f_k)_z$. Hence

$$\mathcal{L}\mathrm{og}\, \rho_k^{-1}(z) = \mathcal{L}\mathrm{og}\, q = \mathcal{L}\mathrm{og}\,(z, h_z) = h(z) = f_k(z), \quad \forall z \in H_k.$$

REMARK 5.66. The equation $e^w - z = 0$ defines a complex submanifold M of $\mathbf{C}^2$. One can show that this M is analytically isomorphic to the Riemann surface $\mathcal{R}$ for $\log z$. In fact, every Riemann domain over $\mathbf{C}^n$ (even when defined in a more general way than in Section 2.1) is analytically isomorphic to a submanifold of some space $\mathbf{C}^N$, cf. [Hörmander 1].

5.7 Complex projective space $\mathbf{P}^n$. Geometrically one may think of $\mathbf{P}^n$ as the collection of all complex lines through the origin in $\mathbf{C}^{n+1}$. Such a line is determined by an arbitrary point $w = (w_0, w_1, \ldots, w_n) \neq 0$; equivalently, one can use any other point $w = (\lambda w_0, \ldots, \lambda w_n),\ \lambda \in \mathbf{C},\ \lambda \neq 0$.

DEFINITION 5.71. The elements of $\mathbf{P}^n$ are equivalence classes $[w]$ of points w in $\mathbf{C}^{n+1} - \{0\}$:

$$w' \sim w \quad \text{if} \quad w' = \lambda w \quad \text{for some} \quad \lambda \in \mathbf{C} - \{0\}.$$

Neighbourhoods of $[w]$ in $\mathbf{P}^n$ are obtained from neighbourhoods of a representing point w in $\mathbf{C}^{n+1} - \{0\}$ by identifying equivalent elements.

For the topology, it is convenient to choose a representing point w and a neighbourhood of w on the unit sphere S in $\mathbf{C}^{n+1}$.

A *complex structure* is defined on $\mathbf{P}^n$ by the following coordinate systems (U_j, ρ_j), $j = 0, 1, \ldots, n$:

U_j consists of the classes $[w]$ in which w has $(j+1)st$ coordinate $w_j \neq 0$,

$$[w] = [(w_0, \ldots, w_n)] = \left[\left(\frac{w_0}{w_j}, \ldots, \frac{w_{j-1}}{w_j}, 1, \frac{w_{j+1}}{w_j}, \ldots, \frac{w_n}{w_j}\right)\right],$$

and

$$\rho_j \circ [w] = \left(\frac{w_0}{w_j}, \ldots, \frac{w_{j-1}}{w_j},\ \frac{w_{j+1}}{w_j}, \ldots, \frac{w_n}{w_j}\right), \quad [w] \in U_j. \tag{7a}$$

Every element $[w]$ of U_j has precisely one representative in $\mathbf{C}^{n+1} - \{0\}$ with $(j+1)st$ coordinate w_j equal to 1; the elements of U_j are in $1-1$ correspondence with the points of the affine hyperplane $H_j : \{w_j = 1\}$ in $\mathbf{C}^{n+1}$. This correspondence is a homeomorphism, hence U_j is topologically the same as $\mathbf{C}^n$. We will check the holomorphy of the composite map $\rho_k \circ \rho_j^{-1}$ when $j < k$. If $[w]$ is any element of $U_j \cap U_k$, where w denotes a $\mathbf{C}^{n+1}$ representative, then $w_j \neq 0$ and $w_k \neq 0$. By $(7a)$, $\rho_j(U_j \cap U_k)$ consists of the points $z = (z_1, \ldots, z_n)$ with

$$z_1 = \frac{w_0}{w_j}, \ldots, z_j = \frac{w_{j-1}}{w_j},\ z_{j+1} = \frac{w_{j+1}}{w_j}, \ldots, z_k = \frac{w_k}{w_j}, \ldots, z_n = \frac{w_n}{w_j},$$

that is, we get all points $z \in \mathbf{C}^n$ with $z_k \neq 0$. Applying $\rho_k \circ \rho_j^{-1}$ to such z we find, cf. $(7a)$,

$$\begin{aligned} \rho_k \circ \rho_j^{-1}(z_1, \ldots, z_n) &= \rho_k \circ [(z_1, \ldots, z_j,\ 1,\ z_{j+1}, \ldots, z_n)] \\ &= \left(\frac{z_1}{z_k}, \ldots, \frac{z_j}{z_k}, \frac{1}{z_k}, \frac{z_{j+1}}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_n}{z_k}\right). \end{aligned} \tag{7b}$$

This formula indeed defines a $1-1$ holomorphic map of $\rho_j(U_j \cap U_k)$ onto $\rho_k(U_j \cap U_k)$ [the $(j+1)st$ coordinate is $\neq 0$]. For $j > k$ the proof is similar, although there are minor differences.

Conclusion: $\mathbf{P}^n$ is a complex manifold of dimension n.

EXAMPLE 5.72. The complex projective plane $\mathbf{P}^1$ is covered by two coordinate systems (U_0, ρ_0) and (U_1, ρ_1). Here

$$\begin{aligned} &U_0 = \{[(1,z)] : z \in \mathbf{C}\}, \quad \rho_0 \circ [(1,z)] = z, \\ &U_1 = \{[(w,1)] : w \in \mathbf{C}\} = \{[(1,z)] : z \in \mathbf{C} - \{0\}\} \cup [(0,1)], \\ &\rho_1 \circ [(w,1)] = \rho_1 \circ [(1,z)] = w = \frac{1}{z} \quad \text{for} \quad w = \frac{1}{z} \neq 0, \quad \rho_1 \circ [(0,1)] = 0. \end{aligned}$$

Cf. Example 5.64 !

A function f will be holomorphic at a point $[a]$ of $\mathbf{P}^n$, with $a_j \neq 0$, if $f \circ \rho_j^{-1}$ is holomorphic at the point $\rho_j \circ [a]$ of $\mathbf{C}^n$. For $\mathbf{P}^1$ the rule agrees with the standard definition of holomorphy at a point of the Riemann sphere.

We observe that $\mathbf{P}^n$ is *compact* for every n. Indeed, the formula

$$\varphi(w_0, w_1, \ldots, w_n) = [(w_0, w_1, \ldots, w_n)], \quad |w| = 1 \tag{7c}$$

defines a continuous map of the unit sphere S in $\mathbf{C}^{n+1}$ onto $\mathbf{P}^n$; the image of a compact set under a continuous map is compact.

It is useful to consider $\mathbf{P}^n$ as a *compactification* of $\mathbf{C}^n$. Starting with $\mathbf{C}^n$ one introduces homogeneous coordinates:

$$z = (z_1, \ldots, z_n) \quad \text{is represented by} \quad (\lambda, \lambda z_1, \ldots, \lambda z_n)$$

for any nonzero $\lambda \in \mathbf{C}$. This gives an imbedding of $\mathbf{C}^n$ in $\mathbf{P}^n$. In order to obtain the whole $\mathbf{P}^n$ one has to add the elements

$$[(w_0, w_1, \ldots, w_n)] \quad \text{with} \quad w_0 = 0.$$

It is reasonable to interpret those elements as complex directions $(w_1, \ldots, w_n)$ in which one can go to infinity in $\mathbf{C}^n$, or as points at infinity for $\mathbf{C}^n$. Indeed, if $z = \mu w,\ w \in \mathbf{C}^n - \{0\}$, then

$$\begin{aligned} (z_1, \ldots, z_n) \longleftrightarrow [(1, \mu w_1, \ldots, \mu w_n)] &= [(\frac{1}{\mu}, w_1, \ldots, w_n)] \\ &\longrightarrow [(0, w_1, \ldots, w_n)] \quad \text{as} \quad \mu \to \infty. \end{aligned}$$

Observe that the imbedding of $\mathbf{C}^n$ in $\mathbf{P}^n$ reveals a complex hyperplane $\{w_0 = 0\}$ of "points at infinity" for $\mathbf{C}^n$; strictly speaking, it is a copy of $\mathbf{P}^{n-1}$.

By a projective transformation

$$w_j = \sum_{k=0}^{n} a_{jk} w'_k, \quad j = 0, 1, \ldots, n, \quad \det[a_{jk}] \neq 0,$$

any point of $\mathbf{P}^n$ can be mapped onto any other point of $\mathbf{P}^n$; the hyperplane $\{w_0 = 0\}$ can be mapped onto any other (complex) hyperplane.

Let f be a meromorphic function on $\Omega \subset \mathbf{C}$. We can write $f = g/h$ with g, h, holomorphic and without common zeroes. Thus we can associate to f the map $F : \Omega \to \mathbf{P}$, $F(z) = [(h(z), g(z)]$. On the other hand, let F be a map from Ω to $\mathbf{P}$. It follows from the definitions that we may write $F(z) = [(f_1(z), f_2(z))]$, with (composition with ρ_0) f_2/f_1 holomorphic if $f_1 \neq 0$ and (composition with ρ_1) f_1/f_2 holomorphic if $f_2 \neq 0$. In other words, f_1/f_2 has singular points precisely at the zeroes of the function f_2/f_1, and thus is meromorphic. Its associated map to $\mathbf{P}$ is again F. In the higher dimensional case mappings to $\mathbf{P}$ will form meromorphic functions, but meromorphic functions may have intersecting zero and polar set and then don't give rise to mappings to $\mathbf{P}$.

Theorem 5.73. *Any holomorphic map f from $\mathbf{P}^n$ to $\mathbf{P}$ is of the form*

$$f[(z_0, \ldots, z_n)] = [P(z_0, \ldots, z_n), Q(z_0, \ldots, z_n)]$$

where P and Q are homogeneous polynomials of the same degree on $\mathbf{C}^{n+1}$.

PROOF. We leave the case $n = 1$ to the reader and assume from now on $n \geq 2$. Define $F : \mathbf{C}^{n+1} \setminus \{0\} :\to \mathbf{P}$ by

$$F(z) = f \circ \pi(z) = [(g(z), h(z))],$$

where $\pi : \mathbf{C}^{n+1} \to \mathbf{P}^n$ is the projection $\pi(z) = [z]$. By the definition of holomorphic map, we find that $\{g = 0\} \cap \{h = 0\} = \emptyset$ and g/h is holomorphic when $h \neq 0$ and also $\frac{1}{g/h} = h/g$ is holomorphic when $g \neq 0$. Thus g/h is meromorphic on $\mathbf{C}^{n+1} \setminus \{0\}$. As we shall see in Chapter 12, if $n \geq 2$, $\mathbf{C}^{n+1} \setminus \{0\}$ is special in the sense that every meromorphic function on it is the quotient of two globally defined holomorphic functions. Thus $g/h = g'/h'$ with g' and h' holomorphic on $\mathbf{C}^{n+1} \setminus \{0\}$. Now we use the sperical shell theorem to extend g' and h' analytically to all of $\mathbf{C}^{n+1}$. Since F factorizes through $\mathbf{P}^n$. We also have homogeneity:

$$(7d) \qquad g'(\lambda z)/h'(\lambda z) = g'(z)/h'(z), \text{ for } z \in \mathbf{C}^{n+1} \setminus \{0\} \text{ and } \lambda \in \mathbf{C} \setminus \{0\}.$$

Now we expand $g'(z) = \sum_{j \geq j_0} P_j(z)$ and $h'(z) = \sum_{k \geq k_0} Q_k(z)$, where P_j and Q_k are homogeneous polynomials of degree j and k and $P_{j_0}, Q_{k_0} \not\equiv 0$. We can regard (7d) as an identity for holomorphic functions in λ:

$$h'(z) \sum_{j \geq j_0} P_j(z)\lambda^j = g'(z) \sum_{k \geq k_0} Q_k(z)\lambda^k.$$

It follows that $j_0 = k_0$ and that for every j and every $z \in \mathbf{C}^{n+1} \setminus \{0\}$

$$h'(z)P_j(z) \equiv g'(z)Q_j(z).$$

Comparing degrees this implies that for all j and k

$$(7e) \qquad P_j Q_k = Q_j P_k.$$

For (7e) to be true,we must have for each j: either $P_{j_0}/Q_{j_0} = P_j/Q_j$ or $P_j \equiv Q_j \equiv 0$. In other words $g'/h' = P_{j_0}/Q_{j_0}$.

Theorem 5.74. *Every map $f : \mathbf{P}^n \to \mathbf{P}^m$ can be written in the form*

$$f([z]) = [P_0(z), P_1(z), \ldots, P_n(z)],$$

where the P_j are homogeneous of the same degree.

PROOF. We clearly may write f in the above form with P_j suitable (not necessarily holomorphic !) functions, which don't have a common zero and satisfy P_i/P_j is holomorphic outside the zeroes of P_j. Hence outside the zeroes of P_k, k arbitrary, we find that

$$\frac{P_i}{P_0} = \frac{P_i/P_k}{P_0/P_k},$$

hence P_i/P_0 is meromorphic. It follows as in the proof of the previous theorem that P_i/P_0 is a quotient of homogeneous polynomials of the same degree and we are done.

5.8 Recent results on biholomorphic maps. The (unit) ball B in $\mathbf{C}^n$ is homogeneous in the sense that the group $\operatorname{Aut} B$ acts transitively: any point of B can be taken to any other point by an analytic automorphism. For $n = 1$ it follows from the Riemann mapping theorem that all simply connected planar domains are homogeneous (also true for $\mathbf{C}$ itself). However, from a $\mathbf{C}^n$ point of view, homogeneous domains turn out to be rare. Limiting oneself to bounded domains with C^2 boundary and ignoring holomorphic equivalence, the ball is in fact the only connected domain with transitive automorphism group. For $n \geq 2$ almost all small perturbations of the ball lead to inequivalent domains. Furthermore, there are no proper holomorphic mappings of the ball to itself besides automorphisms when $n \geq 2$. [For $n = 1$, all finite products of fractions as in $(3a)$ define proper maps $B \to B$.]

Suppose now that D_1 and D_2 are holomorphically equivalent (bounded connected) domains in $\mathbf{C}^n$ with smooth boundary, Question: Can every biholomorphic map f of D_1 onto D_2 be extended to a smooth map on $\overline{D}_1$? In the case $n = 1$ a classical result of Kellogg implies that the mapping functions are [nearly] as smooth up to the boundary as the boundaries themselves. Since 1974 there are also results of such type for $n \geq 2$. The major breakthrough was made by C. Fefferman: If D_1 and D_2 are strictly pseudoconvex domains [for this notion, see Section 9.3] with C^∞ boundary, then any biholomorphic map between them extends C^∞ to the boundary. Subsequently, the difficult proof has been simplified, while at the same time the condition of strict pseudoconvexity could be relaxed. In particular, it follows from this work that strict pseudoconvexity is a biholomorphic invariant. It has also become possible to prove relatively sharp results for the case of C^k boundaries. Finally, many of the results have been extended to proper mappings.

Conversely one may ask under what conditions maps from [part of] ∂D_1 onto [part of] ∂D_2 can be extended to biholomorphic maps. This problem has led to the discovery of important differential invariants of boundaries (N. Tanaka, Chern, Moser). Lately there has been much activity in the area by the Moscow school of complex analysis.

References: [Diederich-Lieb], [Krantz], [Range], [Rudin 4], [Shabat] and Encyclopaedia of the Mathematical Sciences vol. 7, 9.

Exercises

5.1. Let $f = (f_1, \ldots, f_p)$ be a holomorphic map from a connected domain $D \subset \mathbf{C}^m$ to $\mathbf{C}^p$ such that $D_k f_j \equiv 0, \ \forall j, k$. What can you say [and prove] about f?

5.2. Write out a complete proof of the Implicit mapping theorem 5.11 for the case $p = 2$.

5.3. Let $f_j(z, w), \quad j = 1, \ldots, p$ be a family of holomorphic functions of (z, w) on a neighbourhood of 0 in $\mathbf{C}^n \times \mathbf{C}^p$ such that

$$\det J(O) = \left| \frac{\partial(f_1, \ldots, f_p)}{\partial(w_1, \ldots, w_p)}(0) \right| \neq 0.$$

Write $f_j = g_j + ih_j, \ w_k = u_k + iv_k$ and show that the unique solvability of the system of equations

$$\tilde{d}f\big|_0 = 0 : \sum_{k=1}^{p} \frac{\partial f_j}{\partial w_k}(0) dw_k = 0, \quad j = 1, \ldots, p$$

for $dw_1, \ldots, dw_p$ implies the unique solvability of the related real system $\tilde{d}g|_0 = \tilde{d}h|_0 = 0$ (variables $du_1, \ dv_1, \ldots, dv_p$). Deduce that

$$\det J_{\mathbf{R}}(0) = \left| \frac{\partial(g_1, h_1, \ldots, g_p, h_p)}{\partial(u_1, v_1, \ldots, u_p, v_p)}(0) \right| \neq 0.$$

[One can show more precisely that $\det J_R(0) = |\det J_0)|^2$.]

5.4. (Continuation). Let $w_j = \varphi_j(z) = \varphi_j(x + iy), \ j = 1, \ldots, p$ be a C^1 solution of the system of equations $f_j(z, w) = 0, \ j = 1, \ldots, p$ (where $f_j(0) = 0$) around the origin. Prove that the functions $\varphi_j(z)$ must be holomorphic.

5.5. Let g be an infinitely differentiable map $\mathbf{R} \to \mathbf{R}$ with $g(0) = 0, \ g'(0) \neq 0$. Prove that g is invertible in a neighbourhood of 0 and that $h = g^{-1}$ is also of class C^∞ around 0.

5.6. Give an example of a $1 - 1$ map f of $\mathbf{R}$ onto $\mathbf{R}$ with $f(0) = 0$ which is of class C^∞ while f^{-1} is not even of class C^1.

5.7. Use successive approximation to give a direct proof of Theorem 5.21 on the existence of a local holomorphic inverse. [By suitable linear coordinate changes it may be assumed that $J_g(0) = I_n$, so that the equation becomes $w = g(z) = z - \varphi(z)$, where φ vanishes at 0 of order ≥ 2. For small $|w|$ one may define

$$z^{(0)} = w, \quad z^{(\nu)}(w) = w + \varphi(z^{(\nu-1)}), \quad \nu = 1, 2, \ldots.]$$

5.8. Give a complete proof of Theorem 5.22 on the holomorphy of the global inverse.

5.9. (i) Let $w = f(z)$ be the holomorphic map of $D = \mathbf{C} - \{1\}$ $\mathbf{C}^2$ given by $w_1 = z(z-1), \ w_2 = z^2(z-1)$. Prove that f is $1-1$ but that f^{-1} is not continuous on $f(D)$.

(ii) Prove that a $1-1$ holomorphic map f of $\Omega \subset \mathbf{C}^n$ onto $\Omega' \subset \mathbf{C}^n$ is proper.

5.10. Construct biholomorphic maps of
   (i) the right half-plane $H : \{\text{Re } z > 0\}$ in $\mathbf{C}$ onto the unit disc $\Delta(0,1)$;
   (ii) the product $H \times H : \{\text{Re } z_1 > 0, \ \text{Re } z_2 > 0\}$ in $\mathbf{C}^2$ onto the unit bidisc $\Delta_2(0,1)$.

5.11. Determine all analytic automorphisms of $\Delta_2(0,1)$ and of $B_2(0,1)$ that leave the origin fixed.

5.12. Let D_1 and D_2 be connected domains in $\mathbf{C}^n$ containing the origin and suppose that there is a $1-1$ holomorphic map of D_1 onto D_2 which takes 0 to). Let $\text{Aut}_0 D_j$ denote the subgroup of the automorphisms of D_j that leave 0 fixed. Prove that $\text{Aut}_0 D_1$ is isomorphic to $\text{Aut}_0 D_2$.

5.13. Use exercise 5.12 to verify that $\Delta_2(0,1)$ and $B_2(0,1)$ are not analytically isomorphic. Also compare the groups $\text{Aut}_0 D_j$ for $D_1 = \Delta_n(0,1), \ D_2 = B_n(0,1)$.

5.14. Let $D \subset \mathbf{C}$ be a bounded connected domain and let f be a holomorphic map of D into itself with fixed point a. Prove:
   (i) $|f'(a)| \leq 1$;
   (ii) If f is an automorphism of D, then $|f'(a)| = 1$;
   (iii) If $f'(a) = 1$ then $f(z) \equiv z$.

5.15. Let D_1 and D_2 be bounded connected domains in $\mathbf{C}^n$. Prove that for given $a \in D_1, \ b \in D_2$ and $n \times x$ matrix A, there is at most one biholomorphic map f of D_1 onto D_2 such that $f(a) = b$ and $J_f(a) = A$.

5.16. Let D_1 and D_2 be bounded connected domain in $\mathbf{C}, \ a \in D_1, \ b \in D_2$. Prove that ther is at most one biholomorphic map f of D_1 onto D_2 such that $f(a) = b$ and $f'(a) > 0$.

5.17. Determine all analytic automorphisms of
   (i) the disc $\Delta(0,1) \subset \mathbf{C}$;
   (ii) the bidisc $\Delta_2(0,1) \subset \mathbf{C}^2$.

5.18. (i) Prove that the (analytic) automorphisms of $\mathbf{C}$ have the form $w = az + b$.
   (ii) The situation in $\mathbf{C}^2$ is more complicated. Verify that the equations $w_1 = z_1, \ w_2 = g(z_1) + z_2$ define an automorphism of $\mathbf{C}^2$ for any entire function g on $\mathbf{C}$. Cf. Theorem 5.43 and exercise 5.32.

5.19. Prove that the *Cayley transformation*:

$$w_1 = \varphi_1(z) = \frac{z_1}{1+z_2}, \quad w_2 = \varphi_2(z) = i\frac{1-z_2}{1+z_2}$$

furnishes a $1-1$ holomorphic map of the unit ball $B_2 = B_2(0,1)$ in $\mathbf{C}^2$ onto the *Siegel upper half-space*:

$$D_2 \stackrel{\text{def}}{=} \{(w_1, w_2) \in \mathbf{C}^2 : \text{Im } w_2 > |w_1|^2\}.$$

5.20. (Continuation) The boundary $\partial D_2 = \{(w, t + i|w|^2) : w \in \mathbf{C}, \ \ t \in \mathbf{R}\}$ is parametrized by $\mathbf{C} \times \mathbf{R}$. Show that ∂D_2 becomes a nonabelian group (*Heisenberg group*) under the multiplication

$$(w,t) \cdot (w',t') = (w + w', \ t + t' + 2\,\text{Im } w \cdot \overline{w}').$$

5.21. (Continuation) Show that the "translation" $(w_1, w_2) \to (w_1, w_2 + t)$, $t \in \mathbf{R}$ is an automorphism of D_2. Deduce that the ball B_2 admits an automorphism that carries the origin to a point a at prescribed distance $|a| = c$ from the origin.

5.22. Derive the automorphism (3c) of the unit ball $B_2(0,1) \subset \mathbf{C}^2$ that takes $a = (c, 0)$ to the origin by trying $z_1' = \varphi(z_1)$, $z_2' = \psi(z_1)z_2$. [Set $z_2 = 0$ to determine the form of φ.]

5.23. What is the difference between a complex submanifold M of $\mathbf{C}^n$ and an analytic set V (4.64)?

5.24. Prove that a connected complex submanifold M of $\mathbf{C}^n$ has the same dimension m at each of its points. Verify that such an M is a complex manifold of dimension m in the sense of Definition 5.61. Finally, show that there exist nonconstant holomorphic functions on such an M, provided M contains more than just one point.

5.25. (Riemann domain over $D \subset \mathbf{C}^n$). Let (a, U, f), $a \in \mathbf{C}^n$ be a function element, $\mathcal{F}$ the classical complete analytic function generated by the element. Describe the Riemann domain $\mathcal{R}$ for $\mathcal{F}$ and show that it is a Hausdorff space. Verify that $\mathcal{R}$ can be made into a complex manifold of dimension n and describe how $\mathcal{F}$ becomes a holomorphic function on $\mathcal{R}$. [Cf. Section 2.1 and Example 5.65.]

5.26 Prove that the equation $e^w - z = 0$ defines a complex submanifold M of $\mathbf{C}^2$. Show that M is analytically isomorphic to the Riemann surface $\mathcal{R}$ for $\log z$ described in Example 5.65.

5.27. Prove the statements about holomorphic functions on a complex manifold made right after Definition 5.62.

5.28. This is an exercise about $\mathbf{P}^n$. The notations are as in Section 5.7.
   (i) Describe the map $\rho_k \circ \rho_j^{-1}; \rho_j(U_j \cap U_k) \to \rho_k(U_j \cap U_k)$ also when $j > k$ and verify that it is $1-1$ holomorphic.
   (ii) Prove that the map φ (7c) of the unit sphere S in $\mathbf{C}^{n+1}$ to $\mathbf{P}^n$ is continuous.
   (iii) Describe $\varphi^{-1} \circ [w]$ for $[w] \in \mathbf{P}^n$. Conclusion: There is a $1-1$ correspondence between the points of $\mathbf{P}^n$ and ... on S.

5.29. Let f be defined on a domain in $\mathbf{P}^1$. What does analyticity of f at the point $[a] = [(a_0, a_1)]$ of $\mathbf{P}^1$ mean? Show that f is analytic at the point $[(0,1)]$ of $\mathbf{P}^1$ (the point ∞ for $\mathbf{C}$) if and only if

$$f \circ \rho_1^{-1}(w) = f \circ [(w, 1)] \quad (= f \circ \rho_0^{-1}\left(\frac{1}{w}\right) \quad \text{when} \quad w \neq 0)$$

is analytic at $w = 0$.

5.30. Let f be analytic and bounded on a 'conical set' $|(z_2/z_1) - b| < \delta$, $\quad |z_1| > A$ around the direction $(1, b)$ in $\mathbf{C}^2_z$. Prove that f can be continued analytically to a neighbourhood of the "infinite point" $[(0, 1, b)]$. That is, using $\mathbf{P}^2$, $f \circ \rho_1^{-1}(\zeta)$ has an analytic continuation to a neighbourhood of $0, b)$ in $\mathbf{C}^2_\zeta$. [Where will $f \circ \rho_1^{-1}(\zeta)$ be analytic and bounded?]

5.31. (Behaviour of entire functions at infinity) Let f be an entire function on $\mathbf{C}^2$ which is analytic at (that is, in a neighbourhood of) one infinite point, say $[(0,1,b)]$. Prove that f is constant. Show quite generally that a nonconstant entire function on $\mathbf{C}^n$ must become singular at every infinite point. [Cf. Hartogs' singularities theorem 4.82.]

5.32. Determine the automorphisms of $\mathbf{P}^1$ and $\mathbf{P}^n$ (cf. exercise 5.18).

5.33. Let $A \in Gl(n+1, \mathbf{C})$ such that A leaves the quadratic form $-|z_0|^2+|z_1|^2+|z_2|^2+\cdots+|z_n|^2$ invariant. Show that A gives rise to an automorphism of $\mathbf{P}^n$ which leaves the unit ball in the coordinate system $U_0 = \{z_0 \neq 0\}$ invariant. Describe all automorphisms of the unit ball in $\mathbf{C}^n$.

5.34. Consider the topological manifold $D = \{(x,y) \in \mathbf{R}^2 : x^2+y^2 < 1\}$. For $0 \leq t \leq 1$ we put different complex structures on D, each consisting of one coordinate system (D, ρ_t):

$$\rho_t : D \to \mathbf{C}, \quad \rho_t(x,y) = \frac{r}{1-tr}(x+iy),$$

where $x^2+y^2 = r^2$. Thus we have for each $0 \leq t \leq 1$ a complex manifold. Which ones are biholomorphically equivalent?

5.35. Consider the torus:

$$T = \{F(s,t)) \in \mathbf{R}^3 : \; F(s,t) = 2(\cos s, \sin s, 0) + (\cos t \cos s, \cos t \sin s, \sin t), \; s,t \in \mathbf{R}\}$$

Let $V_1 = \{1 < s,t < 6\}$, $V_2 = \{1 < s < 6, -2 < t < 2\}$, $V_3 = \{-2 < s < 2, 1 < t < 6\}$, $V_4 = \{-2 < s,t < 2\}$ and $F_j = F|_{V_j}$, $j = 1, \ldots, 4$. We describe coordinate systems for T locally inverting F: For $j = 1 \ldots, 4$, let

$$U_j = \{F(s,t) \in T : (s,t) \in V_j\}, \; \rho_j(x,y,z) = g_{a,b}(F_j^{-1}(x,y,z)),$$

where $g_{a,b}(s,t) = as+ibt$. Show that for $a,b \in \mathbf{R}\backslash\{0\}$ this defines an analytic structure on T. Which values of a and b give rise to analytically equivalent manifolds?

5.36. Show that the projection

$$\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n, \quad (z_0, \ldots, z_n) \mapsto [(z_0, \ldots z_n)]$$

is a holomorphic mapping. Relate this to the Spherical shell theorem 2.8.

CHAPTER 6

Domains of holomorphy

As was indicated in Section 1.9, there are many areas of complex analysis where it is necessary or advantageous to work with domains of holomorphy, cf. Chapters 7, 11, 12. In $\mathbf{C}$, every domain is a domain of holomorphy, but in $\mathbf{C}^n$, $n \geq 2$, the situation is quite different. One reason is that holomorphic functions in $\mathbf{C}^n$ can not have isolated singularities: singularities are "propagated" in a certain way.

In order to get insight into the structure of domains of holomorphy, we will study several different characterizations, most of them involving some kind of (generalized) convexity. In fact, there are striking parallels between the properties of convex domains and those of domains of holomorphy. To mention the most important one, let $d(\cdot, \partial\Omega)$ denote the boundary distance function on Ω. Convex domains may be characterized with the aid of a mean value inequality for the function $\log 1/d$ on (real) lines. Domains of holomorphy are characterized by so-called *pseudoconvexity*; the latter may be defined with the aid of a circular mean value inequality for $\log 1/d$ on complex lines.

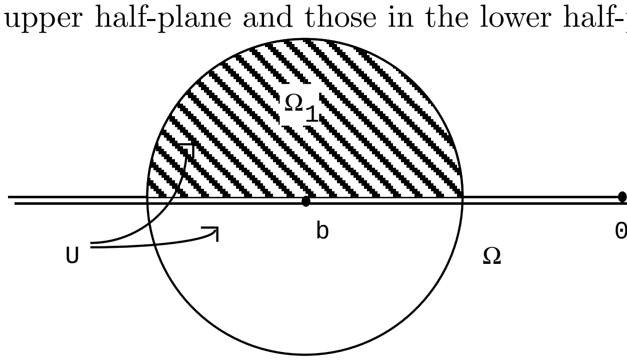
In the present chapter it is shown that domains of holomorphy are pseudoconvex. One form of that result goes back to Levi (about 1910), who then asked if the converse is true. His question turned out to be very difficult. A complete proof that every pseudoconvex domain is indeed a domain of holomorphy was found only in the 1950's. Although different approaches have been developed, the proof remains rather complicated, cf. Chapters 7, 11.

A special reference for domains of holomorphy is [Pflug].

6.1 Definition and examples. For $n = 1$ the example

$$\Omega = \mathbf{C} - (-\infty, 0], \quad f(z) = p(\text{rincipal})\ v(\text{alue})\ \log z$$

shows that one has to be careful in defining a domain of existence or a domain of holomorphy. Indeed, the present function f could not be continued analytically to a neighbourhood U of any boundary point b on the negative real axis if one would simultaneously consider the values of f in the upper half-plane and those in the lower half-plane (fig 6.1).



Of course, one should only pay attention to the values of f on one side of $\mathbf{R}$, those on Ω_1, say , and then one will (for small U) obtain an analytic continuation "above" the original domain of definition. There are similar examples in $\mathbf{C}^n$, cf. Section 2.9.

DEFINITION 6.11. A domain (open set) Ω in $\mathbf{C}^n$ is called a DOMAIN OF HOLOMORPHY if for every (small) connected domain U that intersects the boundary $\partial\Omega$ and for every component Ω_1 of $U \cap \Omega$, there is a function f in $\mathcal{O}(\Omega)$ whose restriction $f\,|_{\Omega_1}$ has no (direct) analytic continuation to U.

An open set Ω will be a domain of holomorphy if and only if all its connected components are domains of holomorphy.

SIMPLE CRITERION 6.12. The following condition is clearly sufficient for Ω to be a domain of holomorphy: for every point $b \in \partial\Omega$ and every sequence of points $\{\zeta_\nu\}$ in Ω with limit b, there is a function f in $\mathcal{O}(\Omega)$ which is unbounded on the sequence $\{\zeta_\nu\}$. [Actually, this condition is also necessary, see Exercise 6.22.]

EXAMPLES 6.13. (i) In $\mathbf{C}$ every domain Ω is a domain of holomorphy, just think of $f(z) = 1/(z-b),\ b \in \partial\Omega$. [What if $\Omega = \mathbf{C}$?]

(ii) In $\mathbf{C}^n$ every "polydomain" $\Omega = \Omega_1 \times \ldots \times \Omega_n$ with $\Omega_j \subset \mathbf{C}$, is a domain of holomorphy, just consider functions $f(z) = 1/(z_j - b_j),\ b_j \in \partial\Omega_j$.

(iii) In $\mathbf{C}^n$ with $n \geq 2$ no connected domain $D - K$ ($K \subset D$ compact) is a domain of holomorphy, think of the Hartogs-Osgood-Brown continuation theorem 4.41.

CONVEX DOMAINS 6.14. Every convex domain $D \subset \mathbf{C}^n = \mathbf{R}^{2n}$ is a domain of holomorphy. Indeed, for any given boundary point b of D there is a supporting real hyperplane V, that is, a hyperplane through b in $\mathbf{R}^{2n}$ which does not meet D (so that D lies entirely on one side of V, fig 6.2).

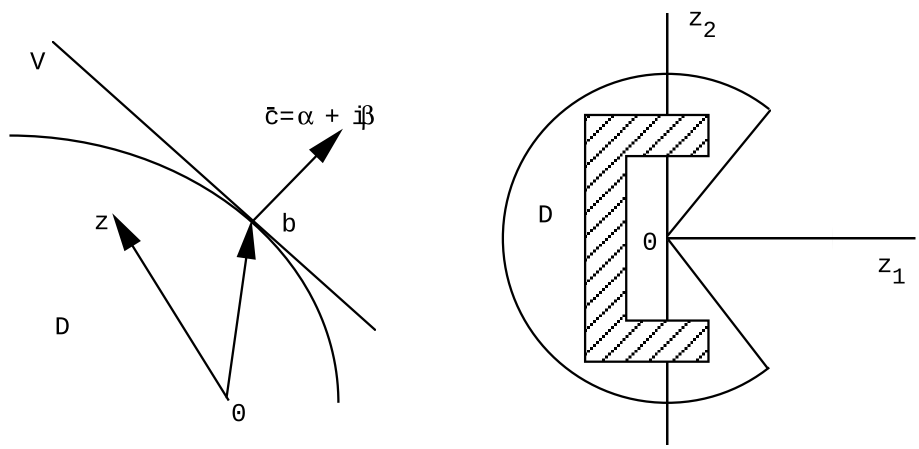


fig 6

We introduce the unit normal $(\alpha_1, \beta_1, \alpha_2, \ldots, \beta_n)$ to V at b which points away from D; in complex notation: $\alpha + i\beta = \bar{c}$, say. The component of the vector $z = x + iy$ in the direction of $\bar{c} = \alpha + i\beta$ will be given by

$$\alpha_1 x_1 + \beta_1 y_1 + \alpha_2 x_2 + \ldots + \beta_n y_n = \mathrm{Re}\,(\alpha - i\beta)(x + iy) = \mathrm{Re}\, c \cdot z.$$

Thus the hyperplane V has the equation $\mathrm{Re}\, c \cdot z = \mathrm{Re}\, c \cdot b$ and throughout D one has $\mathrm{Re}\, c \cdot z < \mathrm{Re}\, c \cdot b$. It follows that the function

$$f(z) = \frac{1}{c \cdot (z - b)}$$

is holomorphic on D and tends to infinity as $z \to b$. [Observe that this f becomes singular at all points of the supporting complex hyperplane $c \cdot (z-b) = 0$ through b which is contained in V.]

A domain of holomorphy need not be convex in the ordinary sense: think of the case $n = 1$ and of the case of logarithmically convex complete multicircular domains in $\mathbf{C}^n$, cf. fig 2.5. The latter are always domains of holomorphy, see Sections 6.3 and 6.4. On the other hand, we have:

EXAMPLE 6.15. Let 0 be a boundary point of a connected domain $D \subset \mathbf{C}^2$ which contains a punctured disc $z_1 = 0,\ 0 < |z_2| \le R$ as well as full discs $z_1 = -\delta,\ |z_2| \le R$ arbitrarily close to the punctured disc (fig 6.3). Then D can not be a domain of holomorphy. Indeed, by Hartogs' continuity theory 2.61, every $f \in \mathcal{O}(D)$ has an analytic continuation to a neighbourhood of 0.

6.2 Boundary distance functions and ordinary convexity. In characterizations of domains of holomorphy, boundary distance functions play an essential role. It is instructive to begin with characterizations of convex domains in terms of such functions.

DEFINITION 6.21. Let Ω be a domain in $\mathbf{R}^n$ or $\mathbf{C}^n$. For the *points* $x \in \Omega$, the *distance to the boundary* is denoted by

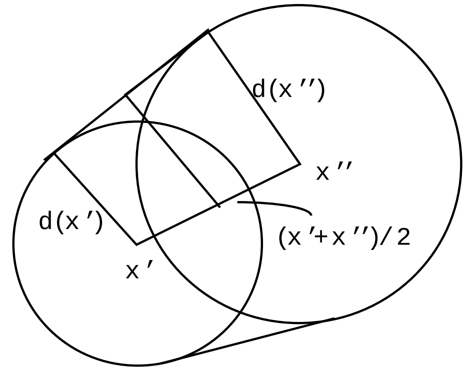
$$(2a) \qquad d(x) \stackrel{\text{def}}{=} d(x, \partial\Omega) \stackrel{\text{def}}{=} \inf_{\xi \in \partial\Omega} d(x, \xi).$$

For a nonempty *part* K of Ω, the *distance to the boundary* is denoted by

$$(2b) \qquad d(K) \stackrel{\text{def}}{=} d(K, \partial\Omega) \stackrel{\text{def}}{=} \inf_{x \in K} d(x, \partial\Omega).$$

If Ω is not the whole space, the infimum in (2a) is attained for some point $b \in \partial\Omega$. Observe that the function $d(x)$ is continuous. If K is compact and $\partial\Omega$ nonempty, the distance $d(K)$ is also attained. Note that $d(x)$ is the radius of the largest ball about x which is contained in Ω. Similarly, $d(K)$ is the largest number ρ such that Ω contains the ball $B(x, \rho)$ for every point $x \in K$.

Suppose now that D is a *convex* domain and that x' and x'' belong to D. Then D will contain the balls $B(x', d(x'))$ and $B(x'', d(x''))$ and also their convex hull. The latter will contain the ball about the point $\frac{1}{2}(x' + x'')$ with radius $\frac{1}{2}\{d(x') + d(x'')\}$ [geometric exercise, cf. fig 6.4], hence



$$(2c)\qquad d(\tfrac{1}{2}(x'+x'')) \geq \tfrac{1}{2}\{d(x')+d(x'')\} \geq \sqrt{d(x')d(x'')}.$$

It follows that the function

$$(2d)\qquad v(x) \stackrel{\text{def}}{=} \log 1/d(x) = -\log d(x)$$

satisfies the following *mean value inequality* on the line segments in D :

$$(2e)\qquad v(\tfrac{1}{2}(x'+x'')) \leq \tfrac{1}{2}\{v(x')+v(x'')\};$$

the value of v at the midpoint of a line segment is majorized by the mean of the values at the end points.

A continuous function v on a domain D with property $(2e)$ is a so-called *convex function*: the graph on line segments in D lies below [never comes above] the chords. In formula:

$$(2f)\qquad v((1-\lambda)x'+\lambda x'') \leq (1-\lambda)v(x')+\lambda v(x''), \quad \forall\lambda\in[0,1], \quad \forall[x',x'']\subset D.$$

For dyadic fractions $\lambda = p/2^k$ this follows from the mean value inequality by repeated bisection of segments; for other λ one uses continuity. [For our special function v one can also derive $(2f)$ from fig 6.4 and properties of the logarithm, cf. Exercise 6.6.] We have thus proved:

Proposition 6.22. *On a convex domain D, the function $v = \log 1/d$ is convex.*

Conversely, one can show that convexity of the function $v = \log 1/d$ on a connected domain D implies convexity of the domain, cf. Exercises 6.7, 6.8.

We still remark that on any bounded domain Ω, the function $\log 1/d$ is a so-called exhaustion function:

DEFINITION 6.23. Let Ω in $\mathbf{R}^n$ or $\mathbf{C}^n$ be open. A continuous real function α on Ω is called EXHAUSTION FUNCTION for Ω if the open sets

$$(2g)\qquad \Omega_t = \{z\in\Omega : \alpha(z) < t\}, \quad t\in\mathbf{R}$$

have compact closure [are "relatively compact"] in Ω.

Observe that the sets Ω_t jointly exhaust $\Omega : \cup\Omega_t = \Omega$. For Ω equal to the whole space $\mathbf{R}^n$, the function $|x|^2$ is a convex exhaustion function:

$$\left|\tfrac{1}{2}(x'+x'')\right|^2 \leq \tfrac{1}{2}(|x'|^2+|x''|^2).$$

Every convex domain has a convex exhaustion function, and every connected domain with a convex exhaustion function is convex, cf. Exercises 6.7, 6.8.

ANOTHER CHARACTERISTIC PROPERTY OF CONVEX DOMAINS. Let D be a connected domain in $\mathbf{R}^n$ or $\mathbf{C}^n$ and let K be a nonempty compact subset of D. To start out we again suppose that D is convex. Then the convex hull $CH(K)$ will also be a compact subset of D. We will in fact show that it has the same boundary distance as K itself:

$$d(CH(K)) = d(K). \tag{2h}$$

Indeed, as we know [Section 2.2], any point $x \in CH(K)$ can be represented as a finite sum

$$x = \sum_1^m \lambda_j s_j \quad \text{with} \quad s_j \in K,\ \lambda_j \geq 0 \quad \text{and} \quad \sum \lambda_j = 1.$$

Now by $(2f)$ or fig 6.4, all points $y = (1-\lambda)s_1 + \lambda s_2,\ 0 \leq \lambda \leq 1$ of a segment $[s_1, s_2] \subset D$ satisfy the inequality

$$d(y) \geq \min\{d(s_1),\ d(s_2)\}.$$

Hence for our point x, using induction,

$$d(x) \geq \min\{d(s_1), \ldots, d(s_m)\} \geq d(K).$$

Thus $d(CH(K)) \geq d(K)$ and $(2h)$ follows.

Conversely, let D be any domain with property $(2h)$, or simply a domain such that $CH(K)$ is a compact subset of it whenever K is one. Then D must be convex. Indeed, for any two points $x', x'' \in D$ one may take $K = \{x', x''\}$. Then $CH(K)$ is the line segment $[x', x'']$ and by the hypothesis it belongs to D.

We summarize as follows:

Proposition 6.24. *The following conditions on a connected domain D in $\mathbf{R}^n$ or $\mathbf{C}^n$ are equivalent:*
(i) D is convex;
(ii) $d(CH(K)) = d(K)$ for every compact subset K of D;
(iii) $CH(K)$ is a compact subset of D for every compact $K \subset D$.

In Section 6.4 we will prove an analogous characterization for domains of holomorphy Ω in $\mathbf{C}^n$. It will involve the so-called *holomorphically convex hull* of K in Ω.

6.3 Holomorphic convexity. As an introduction we characterize the ordinary convex hull $CH(K)$ for compact $K \subset \mathbf{C}^n$ with the aid of holomorphic functions. One may describe $CH(K)$ as the intersection of all closed half-spaces containing K [cf. Section 2.2]. It is of course sufficient to take the minimal half-spaces containing K — those that are bounded by a supporting hyperplane. In $\mathbf{C}^n$ those half-spaces are given by the inequalities

$$\operatorname{Re} c \cdot z \leq \operatorname{Re} c \cdot b = \sup_{\zeta \in K} \operatorname{Re} c \cdot \zeta, \quad c \in \mathbf{C}^n - \{0\}, \tag{3a}$$

where b is an appropriate boundary point of K associated with the direction $\bar{c}$ (cf. 6.14). Thus the *convex hull* of $K \subset \mathbf{C}^n$ may be described as follows:

$$(3b)\qquad \begin{aligned} CH(K) &= \{z \in \mathbf{C}^n : \operatorname{Re} c \cdot z \le \sup_{\zeta \in K} \operatorname{Re} c \cdot \zeta, \quad \forall c \in \mathbf{C}^n\} \\ &= \{z \in \mathbf{C}^n : |e^{c \cdot z}| \le \sup_{\zeta \in K} |e^{c \cdot \zeta}|, \quad \forall c \in \mathbf{C}^n\}. \end{aligned}$$

In the last line, $CH(K)$ is described with the aid of the special class of entire functions $f(z) = \exp(c \cdot z),\ c \in \mathbf{C}^n$. If one uses a larger class of holomorphic functions, one obtains a smaller [no larger] hull for K, depending on the class [see for example Exercise 6.16]. In the following definition, the class of admissible holomorphic functions and the resulting hull are determined by a domain Ω containing K.

DEFINITION 6.31. Let $\Omega \subset \mathbf{C}^n$ be a domain $K \subset \Omega$ nonempty and compact (or at least bounded). The $\mathcal{O}(\Omega)$-convex hull $\hat{K}_\Omega$, or HOLOMORPHICALLY CONVEX HULL of K relative to Ω, is the set

$$(3c)\qquad \hat{K}_\Omega \stackrel{\text{def}}{=} \{z \in \Omega : |f(z)| \le \|f\|_K = \sup_{\zeta \in K} |f(\zeta)|, \quad \forall f \in \mathcal{O}(\Omega)\}.$$

Before turning to examples we give another definition.

DEFINITION 6.32. Let Ω be a domain in $\mathbf{C}^n$ and let φ be a continuous map from the closed unit disc $\overline{\Delta}_1(0,1) \subset \mathbf{C}$ to Ω which is holomorphic on the open disc $\Delta_1 = \Delta_1(0,1)$. Then φ, or rather

$$\overline{\Delta} = \overline{\Delta}_\varphi \stackrel{\text{def}}{=} \varphi(\overline{\Delta}_1)$$

is called a (closed) *analytic disc* in Ω. The image $\Gamma = \Gamma_\varphi = \varphi(C)$ of the boundary $C = \partial\overline{\Delta}_1$ will be called the *edge* of the analytic disc:

$$\operatorname{edge} \overline{\Delta}_\varphi \stackrel{\text{def}}{=} \Gamma_\varphi = \varphi(\partial\overline{\Delta}_1).$$

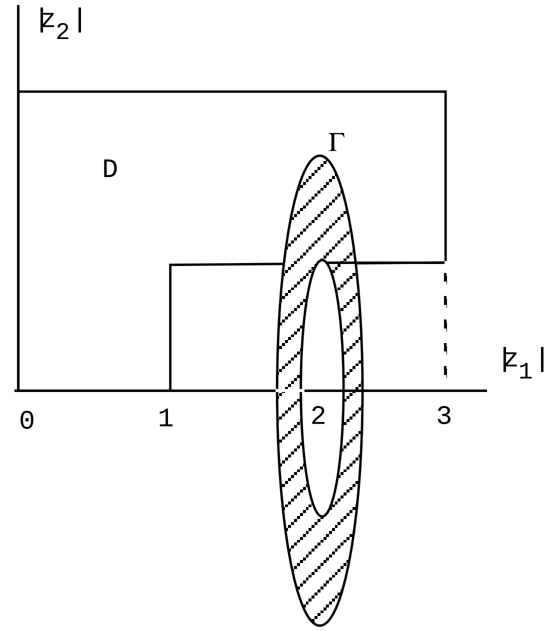
EXAMPLES 6.33. (i) Let $\overline{\Delta} = \overline{\Delta}_1(a,r)$ be a closed disc in $\Omega \subset \mathbf{C}$ and let $\Gamma = \partial\overline{\Delta}$. Then by the maximum principle

$$|f(w) \le \|f\|_\Gamma \quad \forall w \in \overline{\Delta}, \quad \forall f \in \mathcal{O}(\Omega),$$

hence the holomorphically convex hull $\hat{\Gamma}_\Omega$ contains the disc $\overline{\Delta}$. The function $f(w) = w - a$ shows that $\hat{\Gamma}_\Omega = \overline{\Delta}$. Compare Exercise 6.12, however.

(ii) More generally, let Ω be a domain in $\mathbf{C}^n$ and let $\overline{\Delta}_\varphi$ be an analytic disc in Ω. Now let f be in $\mathcal{O}(\Omega)$. Applying the maximum principle to the composition $f \circ \varphi$ on $\overline{\Delta}_1$, we find that the hull $\hat{\Gamma}_\Omega$ of the edge Γ must contain the *whole* analytic disc $\overline{\Delta}_\varphi$:

$$|f(z)| = |f \circ \varphi(w)| \le \|f \circ \varphi\|_{C(0,1)} = \|f\|_\Gamma, \quad \forall z = \varphi(w) \in \overline{\Delta}_\varphi.$$



(iii) Let $D \subset \mathbf{C}^2$ be the multicircular domain (fig 6.5)

$$D = \{|z_1| < 1,\ |z_2| < 3\} \cup \{|z_1| < 3,\ 1 < |z_2| < 3\},$$

Γ the circle $\{z_1 = 2,\ |z_2| = 2\}$ in D. Every function f in $\mathcal{O}(D)$ has an analytic continuation to the equiradial bidisc $\Delta_2(0,3)$, cf. Section 2.5. The holomorphically convex hull of Γ relative to the *bidisc* will be the disc $\{z_1 = 2,\ |z_2| \le 2\}$ [why not more?]. Hence $\hat{\Gamma}_D$ is the part of that disc which belongs to D :

$$\hat{\Gamma}_D = \{z \in \mathbf{C}^2 : z_1 = 2, \quad 1 < |z_2| \le 2\}.$$

PROPERTIES 6.34. (a) $\hat{K}_\Omega$ is *closed* relative to Ω since we are dealing with continuous functions f in Definition 6.31. Also, $\hat{K}_\Omega$ is a *bounded* set even if Ω is not, since by $(3b)$

$$\hat{K}_\Omega \subset CH(K) \subset B(0,R) \quad \text{whenever} \quad K \subset B(0,R).$$

However, $\hat{K}_\Omega$ need not be compact, cf. Example (iii) above. We will see in Section 6.4 that noncompactness of $\hat{K}_\Omega$ can occur only if Ω fails to be a domain of holomorphy.

(b) for any point $z_0 \in \Omega - \hat{K}_\Omega$ and arbitrary constants $A \in \mathbf{C}$, $\varepsilon > 0$ there is a function g in $\mathcal{O}(\Omega)$ such that

$$(3d) \qquad g(z_0) = A, \quad \|g\|_K < \varepsilon.$$

Indeed, there must be a function f in $\mathcal{O}(\Omega)$ for which $|f(z_0)| > \|f\|_K$. Now take $g = A\, f^p / f(z_0)^p$ with sufficiently large p.

DEFINITION 6.35. A domain $\Omega \subset \mathbf{C}^n$ is called HOLOMORPHICALLY CONVEX if the $\mathcal{O}(\Omega)$-convex hull $\hat{K}_\Omega$ is compact for every compact subset K of Ω.

Holomorphic convexity will provide a characterization for domains of holomorphy [Section 6.4].

EXAMPLES 6.36. (i) Every domain $\Omega \subset \mathbf{C}$ is holomorphically convex. Indeed, for any compact $K \subset \Omega$, the bounded, relatively closed subset $\hat{K}_\Omega$ of Ω must be closed in $\mathbf{C}$ [and hence compact]. Otherwise $\hat{K}_\Omega$ would have a limit point b in $\partial\Omega$. The function $1/(z-b)$ which is bounded on K would then fail to be bounded on $\hat{K}_\Omega$.

(ii) Every logarithmically convex complete multicircular domain $D \subset \mathbf{C}^n$ is holomorphically convex. We sketch the proof, taking $n = 2$ for convenience. Let $K \subset D$ be compact. Enlarging K inside D, we may assume that K is the union of finitely many closed polydiscs. Now let b be any point in ∂D. In the plane of $|z_1|,\ |z_2|$, there will be a curve $\alpha_1 \log|z_1| + \alpha_2 \log|z_2| = c$ with $\alpha_j \geq 0$ that separates the point $(|b_1|, |b_2|)$ from the trace of K. To verify this, one may go to the plane of $\log|z_1|,\ \log|z_2|$ in which $\log \operatorname{tr} D$ is a convex domain. It may finally be assumed that the numbers α_j are rational or, removing denominators, that they are nonnegative integers. The monomial $f(z) = z_1^{\alpha_1} z_2^{\alpha_2}$ will then satisfy the inequality $|f(b)| > \|f\|_K$, hence b can not be a limit point of $\hat{K}_D$.

6.4 The Cartan-Thullen characterizations of domains of holomorphy. We begin with an important auxiliary result on simultaneous analytic continuation (see also Exercise 6.26 !):

Proposition 6.41. *Let K be a compact subset of $\Omega \subset \mathbf{C}^n$, let a be a point of the $\mathcal{O}(\Omega)$-convex hull $\hat{K}_\Omega$ and let f be any holomorphic function on Ω. Then the power series for f with center a converges (at least) throughout the ball $B(a, d(K))$ and uniformly on every compact subset of that ball. More generally, if g is any function in $\mathcal{O}(\Omega)$ which is majorized by the boundary distance function d on K:*

$$|g(\zeta)| \leq d(\zeta), \quad \forall \zeta \in K, \tag{4a}$$

then the power series for f around a converges throughout the ball $B(a, |g(a)|)$.

PROOF. The first result is the special case $g \equiv d(K)$ of the second. We will prove the first result and then indicate what has to be done to obtain the more general one.

Observe that the unit ball $B(0,1)$ is the union of the maximal polydiscs $\Delta(0, r)$ which it contains, that is, the polydiscs for which $r = (r_1, \ldots, r_n)$ has length 1. Taking $0 < \lambda < d(K)$, let K_λ be the λ-neighbourhood of K, that is, the set of all points in $\mathbf{C}^n$ at a distance $< \lambda$ from K. The closure $\overline{K}_\lambda$ will be a compact subset of Ω; note that we may represent it in the form

$$\overline{K}_\lambda = \bigcup_{\zeta \in K} \overline{B}(\zeta, \lambda) = \bigcup_{\zeta \in K,\, |r|=1} \overline{\Delta}(\zeta, \lambda r).$$

Naturally, $M_\lambda = \sup|f|$ on $\overline{K}_\lambda$ will be finite. Applying the Cauchy inequalities 1.65 to f on $\overline{\Delta}(\zeta, \lambda r),\ \zeta \in K,\ |r| = 1$, we obtain

$$(4b) \qquad \left|D^\alpha f(\zeta)\right| \le \frac{M_\lambda \alpha!}{(\lambda r)^\alpha} = \frac{M_\lambda \alpha!}{(\lambda r_1)^{\alpha_1} \ldots (\lambda r_n)^{\alpha_n}}, \quad \forall \alpha \ge 0.$$

For given α, the right-hand side furnishes a uniform bound for the modulus of the holomorphic function $D^\alpha f$ throughout K, hence a bound for $\|D^\alpha f\|_K$. Since a belongs to $\hat{K}_\Omega$, the same bound must be valid for $|D^\alpha f(a)|$. [Use (3c) for $D^\alpha f$.] It follows that the power series for f with center a,

$$(4c) \qquad \sum_{\alpha \ge 0} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha,$$

will converge at every point z with $|z_j - a_j| < \lambda r_j,\ j = 1, \ldots, n$. In other words, it converges throughout the polydisc $\Delta(a, \lambda r)$. This holds for all $\lambda < d(K)$ and all r with $|r| = 1$, hence the series converges throughout the union $B(a, d(K))$ of those polydiscs, and it converges uniformly on every compact subset of that ball. [Cf. Theorem 2.42.]

For the second result one takes $0 < \lambda < 1$ and introduces the set

$$K_\lambda^* = \bigcup_{\zeta \in K} \overline{B}(\zeta, \lambda|g(\zeta)|).$$

This too is a compact subset of Ω [use (4a) and the continuity of g]. Instead of (4b) one now obtains $|D^\alpha f(\zeta)| \le M_\lambda^* \alpha!/(\lambda|g(\zeta)|r)^\alpha$ or

$$(4d) \qquad \left|D^\alpha f(\zeta) \cdot g(\zeta)^{|\alpha|}\right| \le \frac{M_\lambda^* \alpha!}{(\lambda r)^\alpha}, \quad \forall \alpha \ge 0,\ \ |r| = 1.$$

These inequalities hold throughout K [also where $g(\zeta) = 0$]; they will extend to the point $a \in \hat{K}_\Omega$. Via the convergence of the series (4c) throughout the polydiscs $\Delta(a, \lambda|g(a)|r)$ with $\lambda < 1$ and $|r| = 1$, one obtains its convergence on the union $B(a, |g(a)|)$.

One more definition and we will be ready for the main result.

DEFINITION 6.42. $\Omega \subset \mathbf{C}^n$ is called the (maximal) *domain of existence* for the function $f \in \mathcal{O}(\Omega)$ if for every (small) connected domain U that intersects the boundary of Ω and for every component Ω_1 of $U \cap \Omega$, it is impossible to continue the restriction $f\,|_{\Omega_1}$ analytically to U, cf. fig 6.1.

Theorem 6.43. (CARTAN-THULLEN). *The following conditions on a domain $\Omega \subset \mathbf{C}^n$ are equivalent:*

(i) Ω *is a domain of holomorphy;*

(ii-a) *For every compact subset $K \subset \Omega$, the holomorphically convex hull $\hat{K}_\Omega$ has the same distance to the boundary $\partial\Omega$ as K:*

$$d(\hat{K}_\Omega) = d(K);$$

(ii-b) *All holomorphic functions g on Ω which are majorized by the function d on K are majorized by d on $\hat{K}_\Omega$:*

$$(4e) \qquad |g(\zeta)| \le d(\zeta), \quad \forall \zeta \in K \Longrightarrow |g(z)| \le d(z), \quad \forall z \in \hat{K}_\Omega;$$

(iii) Ω *is holomorphically convex, that is, $\hat{K}_\Omega$ is a compact subset of Ω whenever K is;*

(iv) Ω *is the maximal domain of existence for some function $f \in \mathcal{O}(\Omega)$.*

PROOF. For the proof we may assume that Ω is connected: if Ω is a domain of holomorphy, so are all its components and conversely. We may also assume $\Omega \ne \mathbf{C}^n$ and will write $\hat{K}$ for $\hat{K}_\Omega$.

(i) $\Rightarrow$ (ii-a). Since $K \subset \hat{K}$ one has $d(\hat{K}) \le d(K)$. For the other direction, choose any point a in $\hat{K}$. For any function f in $\mathcal{O}(\Omega)$, the power series with center a converges (at least) throughout the ball $B = B(a, d(K))$ [Proposition 6.41]. The sum of the series furnishes a direct analytic continuation of f [from the component of $\Omega \cap B$ that contains a] to B and this holds for all f in $\mathcal{O}(\Omega)$. However, by the hypothesis Ω is a domain of holomorphy, hence B must belong to Ω or we would have a contradiction. It follows that $d(a) \ge d(K)$ and, by varying a. that $d(\hat{K}) \ge d(K)$.

(i) $\Rightarrow$ (ii-b). This implication also follows from Proposition 6.41. If (4a) holds for $g \in \mathcal{O}(\Omega)$, the power series for any $f \in \mathcal{O}(\Omega)$ with center $a \in \hat{K}$ will define a holomorphic extension of f to $B(a, |g(a)|)$, hence such a ball must belong to Ω. Thus $d(a) \ge |g(a)|$ and (4e) follows.

(ii-a) or (ii-b) $\Rightarrow$ (iii). Let $K \subset \Omega$ be compact. Because (ii-b) implies (ii-a) [take $g \equiv d(K)$] we may assume (ii-a). We know that $\hat{K} \subset \Omega$ is bounded and closed relative to Ω [Properties 6.34]. Since by the hypothesis $\hat{K}$ has positive distance to $\partial\Omega$, it follows that $\hat{K}$ is compact.

(iii) $\Rightarrow$ (iv). We will construct a function f in $\mathcal{O}(\Omega)$ that has zeros of arbitrarily high order associated to any boundary approach.

Let $\{a_\nu\}$ be a sequence of points that lies dense in Ω and let B_ν denote the maximal ball in Ω with center a_ν. Let $\{E_\nu\}$ be the "standard exhaustion" of Ω by the increasing sequence of compact subsets

$$E_\nu = \{z \in \Omega : |z| \le \nu, \ d(z, \partial\Omega) \ge 1/\nu\}, \quad \nu = 1, 2, \dots ;$$

$\cup E_\nu = \Omega$. By changing scale if necessary we may assume that E_1 is nonempty. By (iii) the subsets $\hat{E}_\nu \subset \Omega$ are also compact. Since B_ν contains points arbitrarily close to $\partial\Omega$, we can choose points

$$\zeta_\nu \quad \text{in} \quad B_\nu - \hat{E}_\nu, \quad \nu = 1, 2, \dots .$$

We next choose functions g_ν in $\mathcal{O}(\Omega)$ such that

$$g_\nu(\zeta_\nu) = 1, \quad \|g_\nu\|_{E_\nu} < 2^{-\nu}, \quad \nu = 1, 2, \dots$$

[cf. formula (3d)]. Our function f is defined by

$$(4f) \qquad f(z) = \prod_{\nu=1}^{\infty} \{1 - g_\nu(z)\}^\nu, \quad z \in \Omega;$$

we will carefully discuss its properties. For the benefit of readers who are not thoroughly familiar with infinite products, we base our discussion on infinite series.

We begin by showing that the infinite product in $(4f)$ is uniformly convergent on every set E_μ. Let z be any point in E_μ. Then for $\nu \geq \mu$

$$|g_\nu(z)| \leq \|g_\nu\|_{E_\mu} \leq \|g_\nu\|_{E_\nu} < 2^{-\nu}.$$

Using the power series $-\sum w^s/s$ for the principal value of $\log(1-w)$ on the unit disc $\{|w| < 1\}$, it follows that

$$\nu|\text{p.v.}\log\{1 - g_\nu(z)\}| \leq \nu|g_\nu(z) + \tfrac{1}{2}g_\nu(z)^2 + \ldots| < 2\nu 2^{-\nu}.$$

Thus the series

$$\sum_{\nu\geq\mu} \nu \ \text{p.v.}\log\{1 - g_\nu(z)\}$$

is uniformly convergent on E_μ; the sum function is holomorphic on the interior E_μ^0. Exponentiating, we find that the product

$$\prod_{\nu\geq\mu} \{1 - g_\nu(z)\}^\nu$$

is also uniformly convergent on E_μ; the product function is zero free on E_μ and holomorphic on E_μ^0.

Multiplying by the first $\mu - 1$ factors, the conclusion is that the whole product in $(4f)$ converges uniformly on E_μ. The product defines f as a holomorphic function on E_μ^0 and hence on Ω. Since f is zero free on E_1 it does not vanish identically; on E_μ it vanishes precisely where one of the first $\mu - 1$ factors of the product is equal to zero. At the point $z = \zeta_\nu \in B_\nu$ the factor $\{1 - g_\nu(z)\}^\nu$ vanishes of order $\geq \nu$, hence the same holds for f.

We will show that f can not be continued analytically across $\partial\Omega$. Suppose on the contrary that f has a direct analytic continuation F to a connected domain U intersecting $\partial\Omega$ if one starts from the component Ω_1 of $U \cap \Omega$. Now choose a point b in $\partial\Omega_1 \cap U$ and select a subsequence $\{a'_k = a_{\nu_k}\}$ of $\{a_\nu\}$ which lies in Ω_1 and converges to b. The associated balls $B'_k = B_{\nu_k}$ must also tend to b, hence for large k they lie in Ω_1 and by omitting a few, we may assume that they all do. At $z = \zeta'_k = \zeta_{\nu_k} \in B'_k$ our function f vanishes of order $\geq \nu_k \geq k$ and the same must then hold for F. Thus

$$D^\alpha F(\zeta'_k) = 0 \quad \text{for all} \quad \alpha\text{'s} \quad \text{with} \quad |\alpha| < k.$$

Since $\zeta'_k \to b$ it follows by continuity that $D^\alpha F(b) = 0$ for every multi-index α, hence $F \equiv 0$. By the uniqueness theorem this would imply $f \equiv 0$, but that is a contradiction.

(iv) $\Rightarrow$ (i): clear.

REMARK 6.44. One can show that a result like 6.43(ii-a) is also valid for other distance like functions, e.g., as introduced in exercise 6.27. , cf. [Pflug].

6.5 Domains of holomorphy are pseudoconvex. On a convex domain Ω the function $v = \log 1/d$ is convex: it satisfies the linear mean value inequality ($2e$), or with different letters,

$$v(a) \leq \tfrac{1}{2}\{v(a-\xi) + v(a+\xi)\} \tag{5a}$$

for every straight line segment $[a-\xi,\, a+\xi]$ in Ω. Pseudoconvexity of a domain Ω in $\mathbf{C}^n$ may be defined in terms of a weaker mean value inequality for the function $v = \log 1/d$. In the case $n = 1$ this will be the inequality that characterizes subharmonic functions:

DEFINITION 6.51. A continuous SUBHARMONIC function v on $\Omega \subset \mathbf{C}$ is a continuous real valued function that satisfies for every point $a \in \Omega$ and all sufficiently small vectors $\zeta \in \mathbf{C} - \{0\}$, v the CIRCULAR MEAN VALUE INEQUALITY:

$$v(a) \leq \overline{v}(a;\zeta) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} v(a + e^{it}\zeta)dt, \quad 0 < |\zeta| < r_a. \tag{5b}$$

In the present case of $\mathbf{C}$ one may write $\zeta = re^{i\varphi}$ and thus $\overline{v}(a;\zeta) = \overline{v}(a;r)$, the mean value of v over the circle $C(a,r)$. For subharmonic v as defined here, the mean value inequality ($5b$) will automatically hold for every ζ with $0 < |\zeta| < d(a)$ [one may take $r_a = d(a)$, Section 8.2].

In the case of $\mathbf{C}^n$, the mean value inequality relative to circles in complex lines leads to the class of plurisubharmonic functions:

DEFINITION 6.52. A continuous PLURISUBHARMONIC (psh) function v on $\Omega \subset \mathbf{C}^n$ is a continuous real function with the property that its restrictions to the intersections of Ω with complex lines are subharmonic. Equivalently, it is required that for every point $a \in \Omega$ and every vector $\zeta \in \mathbf{C}^n - \{0\}$, the function $v(a + w\zeta)$, $w \in \mathbf{C}$ satisfy circular mean value inequalities at the point $w = 0$. The condition may also be expressed by formula ($5b$), but now for vectors $\zeta \in \mathbf{C}^n$.

It may be deduced from ($5a$) [by letting ξ run over a semicircle] that convex functions on $\Omega \subset \mathbf{C}^n$ are plurisubharmonic. An important example is given by the function $|z|^2$. Observe also that the sum of two *psh* functions is again *psh*

More general [not necessarily continuous] subharmonic and plurisubharmonic functions will be studied in Chapter 8. The following lemma is needed to prove circular mean value inequalities for continuous functions.

Lemma 6.53. *Let f be a continuous real functions on the closed unit disc $\overline{\Delta}_1(0,1) \subset \mathbf{C}$ with the following special property:*

Π *For every polynomial $p(w)$ such that $\operatorname{Re} p(w) \geq f(w)$ on the circumference $C(0,1)$, one also has $\operatorname{Re} p(0) \geq f(0)$. Then f satisfies the mean value inequality at 0 relative to the unit circle:*

$$f(0) \leq \overline{f}(0;1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})dt.$$

PROOF. By Weierstrass's theorem on trigonometric approximation, any 2π-periodic continuous real function on $\mathbf{R}$ can be uniformly approximated, within any given distance ε, by real trigonometric polynomials

$$\sum_{k\geq 0}(a_k \cos kt + b_k \sin kt) = \operatorname{Re} \sum_{k\geq 0}(a_k - ib_k)e^{ikt} = \operatorname{Re} p(e^{it}).$$

Here $p(w)$ stands for the polynomial $\sum_{k\geq 0}(a_k - ib_k)w^k$. For our given f, we now approximate $f(e^{it}) + \varepsilon$ with error $\leq \varepsilon$ by $\operatorname{Re} p(e^{it})$ on $\mathbf{R}$:

$$-\varepsilon \leq f(e^{it}) + \varepsilon - \operatorname{Re} p(e^{it}) \leq \varepsilon.$$

Then

$$f(w) \leq \operatorname{Re} p(w) \leq f(w) + 2\varepsilon \quad \text{on} \quad C(0,1).$$

Hence by property (Π) of f,

$$\begin{aligned} f(0) \leq \operatorname{Re} p(0) = a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re} p(e^{it})dt \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})dt + 2\varepsilon = \overline{f}(0,1) + 2\varepsilon. \end{aligned}$$

The proof is completed by letting $\varepsilon \downarrow 0$.

DEFINITION 6.54. A domain (open set) Ω in $\mathbf{C}^n$ is called PSEUDOCONVEX if the function

$$(5c) \qquad v(z) = \log 1/d(z) = \log 1/d(z, \partial\Omega)$$

is plurisubharmonic on Ω.

There are also other definitions of pseudoconvexity possible, cf. Remark 6.57 and Theorem 9.34.

Theorem 6.55. *Every domain of holomorphy in $\mathbf{C}^n$ is pseudoconvex.*

PROOF. Let $\Omega \subset \mathbf{C}^n$ be a domain of holomorphy. Choose any point a in Ω. We will show that the function $v = -\log d$ satisfies the mean value inequality $(5b)$ for every $\zeta \in \mathbf{C}^n$ with $0 < |\zeta| < d(a)$. Fixing such a ζ, the flat analytic disc

$$\overline{\Delta} = \{z \in \mathbf{C}^n : z = a + w\zeta, \ |w| \leq 1\}$$

will belong to Ω. We set

$$(5d) \qquad f(w) = v(a + w\zeta) = -\log d(a + w\zeta), \quad w \in \overline{\Delta}_1(0,1)$$

and get ready to apply Lemma 6.53 to this continuous real f.

Thus, let $p(w)$ be any polynomial in w such that

$$\text{(5}e\text{)} \qquad \operatorname{Re} p(w) \geq f(w) = -\log d(a + w\zeta), \quad \forall w \in C(0,1).$$

In order to exploit the fact that Ω is a domain of holomorphy, we have to reformulate (5e) as an inequality for a holomorphic function on Ω. This is done by choosing a polynomial $q(z)$ in z such that

$$\text{(5}f\text{)} \qquad q(a + w\zeta) = p(w), \quad \forall w \in \mathbf{C};$$

singling out a nonzero coordinate ζ_j of ζ, one may simply take $q(z) = p\{(z_j - a_j)/\zeta_j\}$. Then for $z = a + w\zeta$ with $w \in C(0,1)$, (5e) gives

$$\operatorname{Re} q(z) = \operatorname{Re} p(w) \geq -\log d(z),$$

or equivalently,

$$\text{(5}g\text{)} \qquad |e^{-q(z)}| \leq d(z), \quad \forall z \in \Gamma = \operatorname{edge} \overline{\Delta}.$$

We know that $\overline{\Delta} \subset \Omega$ belongs to the holomorphically convex hull of Γ relative to Ω, cf. 6.33. Now Ω is a domain of holomorphy, hence by the Cartan-Thullen theorem 6.43, inequality (5g) must also hold everywhere on $\overline{\Delta} \subset \hat{\Gamma}_\Omega$, cf. (4$e$). It will hold in particular for $z = a$, hence $|e^{-q(a)}| \leq d(a)$ or

$$\text{(5}h\text{)} \qquad \operatorname{Re} p(0) = \operatorname{Re} q(a) \geq -\log d(a) = f(0).$$

Summing up, (5e) always implies (5h), so that f has property ($\prod$) of Lemma 6.53. Conclusion:

$$f(0) \leq \overline{f}(0;1) \quad \text{or} \quad v(a) \leq \overline{v}(a;\zeta).$$

We close with an important auxiliary result for the solution of the Levi problem in Chapters 7, 11.

Proposition 6.56. *Every pseudoconvex domain Ω has a plurisubharmonic exhaustion function: It is "PSH EXHAUSTIBLE" The intersection $\Omega' = \Omega \cap V$ of a psh exhaustible domain Ω with a complex hyperplane V is also psh exhaustible.*

PROOF. (i) If $\Omega = \mathbf{C}^n$, then the function $|z|^2$ will do. For other pseudoconvex Ω, the function

$$\alpha(z) = \log \frac{1}{d(z)} + |z|^2, \quad z \in \Omega$$

will be a psh exhaustion function. Indeed, α is a sum of psh functions, hence psh, cf. Definition 6.54 and the lines following Definition 6.52. The term $|z|^2$ ensures the compactness of the subsets $\overline{\Omega}_t$ of Ω when Ω is unbounded, cf. Definition 6.23.

(ii) If α is any psh exhaustion function for Ω, then $\alpha' = \alpha \mid_{\Omega'}$ will be a psh exhaustion function for $\Omega' = \Omega \cap V$. [Verify this.]

REMARK 6.57. Psh exhaustion functions are essential in the solution of the $\overline{\partial}$ problem on pseudoconvex domains as presented in Chapter 11. For that reason, one sometimes defines pseudoconvexity in terms of the existence of psh exhaustion functions. In fact, every psh exhaustible domain is also pseudoconvex in the sense of Definition 6.54 [cf. Section 9.3].

Exercises

6.1. Prove directly from the definition that $U_j = \{z \in \mathbf{C}^n : z_j \notin 0\}$ is a domain of holomorphy. Prove also that $U_j \cap U_k$ is a domain of holomorphy.

6.2. Let Ω_1 and Ω_2 be domains of holomorphy in $\mathbf{C}^m$ and $\mathbf{C}^p$, respectively. Prove that the product domain $\Omega = \Omega_1 \times \Omega_2$ is a domain of holomorphy in $\mathbf{C}^{m+p}$.

6.3. (*Analytic polyhedra*) Let P be an analytic polyhedron in $\mathbf{C}^n$, that is, $\overline{P}$ is compact and there exist a neighbourhood U of $\overline{P}$ and a finite number of holomorphic functions $f_1, \ldots, f_k$ on U such that

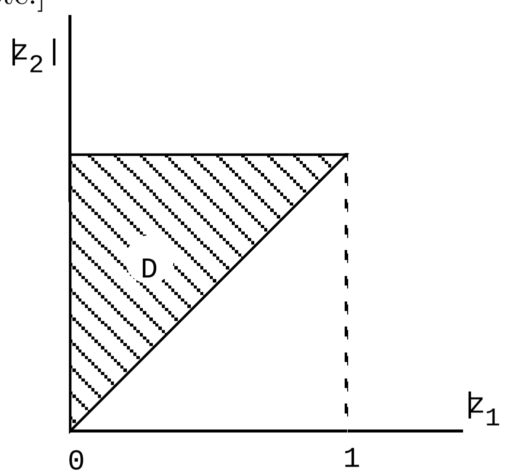
$$P = \{z \in U : |f_j(z)| < 1, \quad j = 1, \ldots, k\}.$$

Prove that P is a domain of holomorphy. [Examples: the polydisc $\Delta(a, r)$ with $r < \infty$, the multicircular domain in $\mathbf{C}^2$ given by the set of inequalities $|z_1| < 2$, $|z_2| < 2$, $|z_1 z_2| < 1$.]

6.4. Prove directly that the *Reinhardt triangle*

$$D = \{(z_1, z_2) \in \mathbf{C}^2 : 0 < |z_1| < |z_2| < 1\}$$

is a domain of holomorphy. [This is a logarithmically convex multicircular domain which, however, is not complete.]



6.5. Prove that the closure $\overline{D}$ of the Reinhardt triangle can not be the intersection of a family of domains of holomorphy. [Cf. Section 2.5.] Show in addition that every holomorphic function on D which is bounded with all its derivatives of arbitrary order, extends to the polydisc $\Delta_2(0, 1)$. [Cf. [Sibony] for related examples.]

6.6. Let D be convex and let x', x'' lie in D. Show that

$$d((1-\lambda)x' + \lambda x'') \geq (1-\lambda)d(x') + \lambda d(x''), \quad \forall \lambda \in [0, 1].$$

Now use the fact that the function $\log t$ is increasing and concave [the graph lies above the chords] to prove inequality $(2f)$ for $v = -\log d$.

6.7. Let D be a connected domain in $\mathbf{R}^n$. A point $b \in \partial D$ is called a *boundary point of nonconvexity* for D if there is a straight line segment S throught b in $\overline{D}$ whose end points belong to D and which is the limit of a continuous family of line segments inside D. Suppose now that D is nonconvex. Prove that D has a boundary point of nonconvexity. [There must be points x', x'' in D such that the segment $[x', x'']$ does not belong to D. Connecting x' to x'' by a polygonal path in D, one may deduce that D must contain segments $[x^0, x^1]$ and $[x^0, x^2]$ such that $[x^1, x^2]$ does not belong to D. Now consider segments parallel to $[x^1, x^2]$.]

6.8. (Continuation) Let D be a connected domain in $\mathbf{R}^n$.
   (i) Suppose that the function $v(x) = \log 1/d(x)$ is convex on D.
   Deduce that D is convex.
   (ii) Prove that D is convex if and only if it has a convex exhaustion function.

6.9. Let $K \subset \mathbf{R}^n$ be compact. Characterize the points x of the convex hull $CH(K)$ by means of a family of inequalities involving real linear functions, cf. ($3b$).

6.10. Let $K \subset \Omega \subset \mathbf{C}^n$ be compact, $\hat{K} = \hat{K}_\Omega$ the holomorphically convex hull of K relative to Ω. Prove that
   (i) $(\hat{K})^\wedge = \hat{K}$;
   (ii) if $|z_j| \le r_j$ for all $z \in K$, then $|z_j| \le r_j$ for all $z \in \hat{K}$ [hence $\hat{K}$ is bounded].

6.11. Suppose $K \subset Z_f \subset \Omega$, where Z_f is the zero set of a function $f \in \mathcal{O}(\Omega)$. Prove that $\hat{K} \subset Z_f$.

6.12. Let $\Omega \subset \mathbf{C}$ be the annulus $A(0, \rho, R)$ and let K be the circle $C(0, r)$, where $\rho < r < R$. Determine $\hat{K}_\Omega$.

6.13. Let $K \subset \mathbf{C}$ be compact, $\mathbf{C} - K$ connected, $a \in \mathbf{C} - K$. Prove that there is a simple holomorphic function f on a neighbourhood Ω of $K \cup \{a\}$ with connected complement $\mathbf{C}_e - \Omega$ such that
$$|f(z)| \le 1 \quad \text{on} \quad K, \qquad |f(a)| > 1.$$
Next use Runge's theorem 1.75 to show that there is a polynomial $p(z)$ such that
$$|p(z)| \le 1 \quad \text{on} \quad K, \qquad |p(a)| > 1.$$
Finally, describe $\hat{K}_{\mathbf{C}}$.

6.14. Let K be an arbitrary compact set in $\mathbf{C}$ [$\mathbf{C} - K$ need not be connected]. Describe $\hat{K}_{\mathbf{C}}$.

6.15. Let $\overline{\Delta} \subset \Omega \subset \mathbf{C}^n$ be a flat analytic disc, $\overline{\Delta} = \{z \in \mathbf{C}^n : z = a + w\zeta, \quad |w| \le 1\}$, where $\zeta \in \mathbf{C}^n - \{0\}$. Determine $\hat{\Gamma}_\Omega$ where $\Gamma = \text{edge}\,\overline{\Delta}$.

6.16. (*Polynomially convex hull*) For $K \subset \mathbf{C}^n$ compact, the polynomially convex hull $\tilde{K}$ is defined by
$$\tilde{K} = \{z \in \mathbf{C}^n : |p(z)| \le \|p\|_K \quad \text{for all polynomials} \quad p\}.$$
Prove:

(i) For any domain Ω containing K, $\quad \hat{K}_\Omega \subset \tilde{K}$;
(ii) For any polydisc Δ containing K, $\quad \hat{K}_\Delta = \tilde{K}$;
(iii) $\tilde{K} = \hat{K}_{\mathbf{C}^n} \subset CH(K)$.

6.17. Let $K \subset \Omega \subset \mathbf{C}^n$ be compact and let E be the set of those points $z \in \Omega$, for which there is a constant M_z such that $|f(z)| \leq M_z \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$. Prove that $E = \hat{K}_\Omega$.

6.18. Prove directly from Definition 6.35 that the following domains are holomorphically convex:
(i) $\mathbf{C}^n$;
(ii) polydiscs in $\mathbf{C}^n$;
(iii) convex domains in $\mathbf{C}^n$.

6.19. Prove that the intersection of two domains of holomorphy is again a domain of holomorphy.

6.20. Let D_1 be a connected domain of holomorphy in $\mathbf{C}^n$ and let $D_2 \subset \mathbf{C}^n$ be analytically isomorphic to D_1. Prove that D_2 is also a domain of holomorphy.

6.21. Let Ω be a domain of holomorphy and let Ω^ε be the "ε-contraction" of Ω, that is, $\Omega^\varepsilon = \{z \in \Omega : d(z) > \varepsilon\}$. Prove that Ω^ε is also a domain of holomorphy. [For $K \subset \Omega^\varepsilon$, $\quad \hat{K}_{\Omega^\varepsilon} \subset \hat{K}_\Omega$.]

6.22. (*Another characterization of domains of holomorphy*) Prove that $\Omega \subset \mathbf{C}^n$ is a domain of holomorphy if and only if the following condition is satisfied:

"For every boundary point b and every sequence of points $\{\zeta_\nu\}$ in Ω that converges to b, there is a holomorphic function f on Ω which is unbounded on the sequence $\{\zeta_\nu\}$".

[Let Ω be a domain of holomorphy and $\{K_\nu\}$ an increasing exhaustion of Ω by compact subsets, determined in such a way that $K_{\mu+1} - \hat{K}_\mu$ contains a point θ_μ of the sequence $\{\zeta_\nu\}$. Define $f = \sum g_\mu$, where the functions $g_\mu \in \mathcal{O}(\Omega)$ are determined inductively such that

$$|g_\mu(\theta_\mu)| > \mu + \sum_{\nu-1}^{\mu-1} |g_\nu(\theta_\mu)|, \quad \|g_\mu\|_{K_\mu} < 2^{-\mu}.$$

What can you say about $|f(\theta_\mu)|$?]

6.23. Prove that $\Omega \subset \mathbf{C}^n$ is a domain of holomorphy if and only if the following condition is satisfied: For every compact $K \subset \Omega$ and every function $f \in \mathcal{O}(\Omega)$,

$$\sup_K \frac{|f(z)|}{d(z)} = \sup_{\hat{K}} \frac{|f(z)|}{d(z)}.$$

6.24. Let D be a connected domain of holomorphy, $K \subset D$ compact. Prove that there is an analytic polyhedron P such that $K \subset \overline{P} \subset D$. Deduce that D is the limit of an increasing sequence of analytic polyhedra $\{P_\nu\}$ such that $\overline{P}_\nu \subset P^0_{\nu+1}$. [Assuming

$K = \hat{K}$ (as we may), associate certain functions to the boundary points of an ε-neighbourhood of K.]

6.25. Use the mean value inequality (5a) for $v = \log 1/d$ on straight line segments to prove that every convex domain in $\mathbf{C}^n$ is pseudoconvex.

6.26. (*Continuity principle for analytic discs*, cf. Hartogs' continuity theorem 2.61) Let D be a connected domain in $\mathbf{C}^n$ and let $\{\overline{\Delta}_\nu\},\ \nu = 1, 2, \ldots$ be a sequence of analytic discs in D which converges to a set E in $\mathbf{C}^n$. Suppose that the edges Γ_ν of the discs $\overline{\Delta}_\nu$ all belong to a compact subset K of D. Setting $d(K) = \rho$ and taking any point $b \in E$, prove that (suitable restrictions of) the functions $f \in \mathcal{O}(D)$ can be continued analytically to the ball $B(b, \rho)$.

6.27 Let Ω be a domain of holomorphy. For $a \in \mathbf{C}^n \setminus \{0\}$ let

$$d_a(z) = d_a(z, \partial\Omega) = \sup\{r :\ (z + \lambda a) \in \Omega\ \forall |\lambda| < r\}.$$

Let Γ be the edge of an analytic disc $\bar{\Delta}$ in Ω. Show that

$$d_a(\Gamma) = d_a(\bar{\Delta}).$$

Next show that $-\log d_a(z)$ is psh on Ω and deduce a proof of step ii of Hartogs' singularity theorem (4.82), for all $n \geq 2$.

6.28. (*Prism Lemma*). Let H_x be a domain in $\mathbf{R}^n_x$ which contains two closed line segments $[x^0, x^1]$ and $[x^0, x^2]$ that do not belong to a straight line and let f be holomorphic on $H_x + i\mathbf{R}^n_y \subset \mathbf{C}^n$. Prove that f has an analytic continuation to a neighborhood of $T_x + i\mathbf{R}^n_y$, where T_x is the closed triangular region with vertices x^0, x^1, x^2. [Take $x^0 = (0, 0, 0, \ldots, 0)$, $x^1 = (1, 1, 0, \ldots, 0)$, $x^2 = (-1, 1, 0, \ldots, 0)$. By translation invariance, it is sufficient to prove that f has an analytic extension to a neighborhood of an arbitrary point $a = (a_1, a_2, \ldots, 0) \in T^0_x :\ |a_1| < a_2 < 1$. Determine c and d such that a lies on the parabola $x_2 = cx_1^2 + d$ through x^1 and x^2 and then consider the family of analytic discs $\overline{\Delta}_\lambda = \{z \in T + i\mathbf{R}^n_y :\ z_2 = cz_1^2 + \lambda, z_3 = \cdots = z_n = 0\}$, $0 \leq \lambda \leq d = 1 - c$. The boundaries Γ_λ belong to $\{[x^0, x^1] \cap [x^0, x^2]\} + i\mathbf{R}^n_y$. Verify that the projection of $\overline{\Delta}_\lambda$ onto the z_1-plane is given by the inequality $|x_1| \leq c(x_1^2 - y_1^2) + \lambda$.]

CHAPTER 7

The first Cousin problem, $\bar\partial$ and the Levi problem

Towards the end of the nineteenth century, prominent mathematicians solved the following problems:
– Construct a meromorphic function f on $\mathbf{C}$, or on a domain $\Omega \subset \mathbf{C}$, with poles at prescribed points and with a prescribed way of becoming infinite at the poles (results of Mittag-Leffler);
– Construct a holomorphic function f on $\mathbf{C}$, or on $\Omega \subset \mathbf{C}$, with zeros at prescribed points and with prescribed multiplicities of the zeros (results of Weierstrass).

The corresponding questions for $\mathbf{C}^n$ and for domains $\Omega \subset \mathbf{C}^n$ were also raised and led to the important first and second Cousin problem, respectively, see Section 1.10. However, solutions for domains other than simple product domains did not appear until Oka started to make his major contributions around 1936. It seemed then that the Cousin problems are best considered on domains of holomorphy. Still, that was not the whole story. Complete understanding came only with the application of sheaf cohomology [Cartan-Serre, early 1950's, cf. Chapter 12 and the monograph [Grauert-Remmert]]. More recently, there has been increased emphasis on the role of the $\bar\partial$ equation, especially after Hörmander found a direct analytic solution for the general $\bar\partial$ problem on pseudoconvex domains [cf. Chapter 11].

Let us elaborate. The Cousin problems require a technique, whereby local solutions may be patched together to obtain a global solution. Techniques for the first Cousin problem can be applied also to other problems, such as the holomorphic extension of functions, defined and analytic on the intersection of a domain with a complex hyperplane, and the patching together of local solutions of the $\bar\partial$ equation to a global solution. In this Chapter it will be shown that the reduced, so-called holomorphic Cousin-I problem is generally solvable on a domain $\Omega \subset \mathbf{C}^n$ if and only if the "first order" $\bar\partial$ equation $\bar\partial u = v$ is generally solvable on Ω. Indeed, *smooth* solutions of the holomorphic Cousin-I problem exist no matter what domain Ω one considers. Such smooth solutions can be modified to a *holomorphic* solution by the method of "subtraction of nonanalytic parts" if an only if one can solve a related $\bar\partial$ equation.

In Chapter 11 it will be shown analytically that every pseudoconvex domain is a $\bar\partial$ domain, that is, a domain on which all (first order) $\bar\partial$ equations are solvable. For $\mathbf{C}^2$ it will then follow that pseudoconvex domains, $\bar\partial$ domains, Cousin-I domains (domains on which all first Cousin problems are solvable) and domains of holomorphy are all the same, cf. Sections 7.2, 7.7. The situation in $\mathbf{C}^n$ with $n \geq 3$ is more complicated: see Sections 7.2, 7.5 and the discussion of the Levi problem in Section 7.7; cf. also Chapter 12.

7.1 Meromorphic and holomorphic Cousin-I. A meromorphic function f on an open set $\Omega \subset \mathbf{C}$ is a function which is holomorphic except for poles. The poles must be isolated: they can not have a limit point inside Ω, but there are no other restrictions. [A limit point of poles inside Ω would be a singular point of f but not an isolated singularity, hence certainly not a pole.] The classical existence theorem here is Mittag-Leffler's: For any open set $\Omega \subset \mathbf{C}$, any family of isolated points $\{a_\lambda\} \subset \Omega$ and any corresponding family of principal parts

$$f_\lambda(z) = \sum_{s=1}^{m_\lambda} c_{\lambda s}(z - a_\lambda)^{-s},$$

there is a meromorphic function f on Ω which has principal part f_λ at a_λ for each λ but no poles besides the points a_λ. That is, for each point a_λ there is a small neighbourhood $U_\lambda \subset \Omega$ such that

$$(1a) \qquad f = f_\lambda + h_\lambda \quad \text{on} \quad U_\lambda, \quad \text{with} \quad h_\lambda \in \mathcal{O}(U_\lambda)$$

[h_λ holomorphic on U_λ], while f is holomorphic on $\Omega - \cup_\lambda U_\lambda$. For a classical proof of Mittag-Leffler's theorem, cf. Exercise 7.1, for a proof in the spirit of this Chapter, cf. Exercise 7.3.

Since analytic functions in $\mathbf{C}^n$ with $n \geq 2$ have no isolated singularities [Chapters 2, 3], meromorphy in $\mathbf{C}^n$ must be defined in a different way:

DEFINITION 7.11. A MEROMORPHIC function f on $\Omega \subset \mathbf{C}^n$ [we write $f \in \mathcal{M}(\Omega)$] is a function which, in some (connected) neighbourhood U_a of each point $a \in \Omega$, can be represented as a quotient of holomorphic functions,

$$f = g_a/h_a \quad \textit{on} \quad U_a, \quad \textit{with} \quad h_a \not\equiv 0.$$

[The question of global quotient representations of meromorphic functions f will be considered in Theorem 12.6x.]

Observe that a meromorphic function need not be a function in the strict sense that it has a precise value everywhere: it is locally defined as an element of a quotient field. On U_a, the above f may be assigned a precise value (at least) wherever $h_a \neq 0$ or $g_a \neq 0$. A meromorphic function is determined by its finite values,
cf. Exercise 7.5.

Suppose now that for $\Omega \subset \mathbf{C}^n$ one is given a covering by *open subsets* U_λ, where λ runs over some index set Λ, and that on each set U_λ one is given a meromorphic function f_λ. One would like to know if there is a *global* meromorphic function f on Ω which on each set U_λ becomes singular just like f_λ, in other words, f should satisfy the conditions (1a). Of course, this question makes sense only if on each nonempty intersection

$$(1b) \qquad U_{\lambda\mu} = U_\lambda \cap U_\mu,$$

the functions f_λ and f_μ have the same singularities. One thus arrives at the following initial form of the first Cousin problem:

MEROMORPHIC FIRST COUSIN PROBLEM 7.12. Let $\{U_\lambda\}$, $\lambda \in \Lambda$ be a covering of $\Omega \subset \mathbf{C}^n$ by open subsets and let the meromorphic functions $f_\lambda \in \mathcal{M}(U_\lambda)$ satisfy the *compatibility conditions*

$$(1c) \qquad f_\lambda - f_\mu = h_{\lambda\mu} \quad \text{on} \quad U_{\lambda\mu} \quad \text{with} \quad h_{\lambda\mu} \in \mathcal{O}(U_{\lambda\mu}), \ \forall \lambda, \mu \in \Lambda.$$

The question is if there exists a meromorphic function $f \in \mathcal{M}(\Omega)$ such that

$$(1d) \qquad f = f_\lambda + h_\lambda \quad \text{on} \quad U_\lambda \quad \text{with} \quad h_\lambda \in \mathcal{O}(U_\lambda), \ \forall \lambda \in \Lambda.$$

In looking for f, one may consider the holomorphic functions h_λ as the unknowns. They must then be determined such that $h_\mu - h_\lambda = h_{\lambda\mu}$ on $U_{\lambda\mu}$, cf. (1d) and (1c). By (1c), the functions $h_{\lambda\mu}$ will have to satisfy certain requirements:

$$h_{\lambda\mu} = -h_{\mu\lambda} \quad \text{on} \quad U_{\lambda\mu}, \qquad h_{\lambda\mu} = h_{\lambda\nu} + h_{\nu\mu} \quad \text{on} \quad U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu,$$

etc. It turns out that the meromorphic Cousin problem can be reduced to a holomorphic problem involving the known functions $h_{\lambda\mu}$ and the unknown functions h_λ:

(HOLOMORPHIC) COUSIN-I PROBLEM or ADDITIVE COUSIN PROBLEM **7.13**. Let $\{U_\lambda\}$, $\lambda \in \Lambda$ be an open covering of $\Omega \subset \mathbf{C}^n$ and let $\{h_{\lambda\mu}\}$ be a family of holomorphic functions on the (nonempty) intersections $U_{\lambda\mu}$ that satisfy the COMPATIBILITY CONDITIONS

$$(1e) \qquad \begin{cases} h_{\lambda\mu} + h_{\mu\lambda} = 0 & \text{on} \quad U_{\lambda\mu}, \ \forall \lambda, \mu, \nu \in \Lambda \\ \\ h_{\lambda\mu} + h_{\mu\nu} + h_{\nu\lambda} = 0 & \text{on} \quad U_{\lambda\mu\nu}, \ \forall \lambda, \mu, \nu \in \Lambda. \end{cases}$$

The question is if there exist holomorphic functions $h_\lambda \in \mathcal{O}(U_\lambda)$ such that

$$(1f) \qquad h_\lambda - h_\mu = h_{\lambda\mu} \quad \textit{on} \quad U_{\lambda\mu}, \ \forall \lambda, \mu \in \Lambda.$$

A family $\{U_\lambda, f_\lambda\}$ with functions $f_\lambda \in \mathcal{M}(U_\lambda)$ that satisfy condition (1c) will be called a set of *meromorphic Cousin data* on Ω; a family $\{U_\lambda, h_{\lambda\mu}\}$ with holomorphic functions $h_{\lambda\mu} \in \mathcal{O}(U_{\lambda\mu})$ that satisfy the conditions (1e) will be called a set of (*holomorphic*) *Cousin-I data* on Ω. The above forms of the first Cousin problem are related in the following way:

Proposition 7.14. *The meromorphic first Cousin problem on Ω with data $\{U_\lambda, f_\lambda\}$ has a solution f (in the sense of* (1d) *if and only if there is a solution $\{h_\lambda\}$ to the holomorphic Cousin-I problem on Ω with the data $\{U_\lambda, h_{\lambda\mu}\}$ derived from* (1c). *In particular, if all holomorphic Cousin-I problems on Ω are solvable, then so are all meromorphic first Cousin problems on Ω.*

PROOF. If the family $\{h_\lambda\}$ solves the holomorphic problem corresponding to the functions $h_{\lambda\mu}$ coming from (1c), then in view of (1f):

$$f_\lambda + h_\lambda = f_\mu + h_\mu \quad \text{on} \quad U_{\lambda\mu}, \ \forall \lambda, \mu,$$

hence one may define a global meromorphic function f on Ω by setting

$$f \stackrel{\text{def}}{=} f_\lambda + h_\lambda \quad \text{on} \quad U_\lambda, \ \forall \lambda.$$

Conversely, if f solves the meromorphic problem, then the family $\{h_\lambda\}$ given by (1d) solves the corresponding holomorphic problem.

We will see in the sequel that the theory for the holomorphic Cousin-I problem has a number of applications besides the meromorphic problem.

DEFINITION 7.15. An open set Ω in $\mathbf{C}^n$ will be called a COUSIN-I DOMAIN if all HOLOMORPHIC first Cousin problems on Ω are solvable. By Proposition 7.14, all meromorphic first Cousin problems on such a domain are also solvable. If it is only known that all meromorphic first Cousin problems on Ω are solvable, one may speak of a MEROMORPHIC Cousin-I domain.

Are all meromorphic Cousin-I domains in $\mathbf{C}^n$ also holomorphic Cousin-I domains? The answer is known to be yes for $n = 2$, cf. Exercise 7.23. For $n = 1$ the answer is yes for a trivial reason: by the theory below, all domains Ω in $\mathbf{C}$ are Cousin-I domains!

EXAMPLE 7.16. Let U_1 and U_2 be domains in $\mathbf{C}$ with nonempty intersection U_{12}. Let h_{12} be any holomorphic function on U_{12}. All boundary points of U_{12} may be bad singularities for h_{12}! Nevertheless, by the Cousin-I theory h_{12} can be represented as $h_2 - h_1$ with $h_j \in \mathcal{O}(U_j)$. It does not seem easy to prove this directly! Cf. also Exercise 7.14.

Not all domains in $\mathbf{C}^2$ are Cousin-I domains:

EXAMPLE 7.17. Take $\Omega = \mathbf{C}^2 - \{0\}$,

$$U_j = \{z \in \mathbf{C}^2 : z_j \neq 0\}, \quad j = 1, 2; \quad h_{12} = -h_{21} = \frac{1}{z_1 z_2}, \quad h_{11} = h_{22} = 0.$$

Question: Can one write h_{12} as $h_2 - h_1$ with $h_j \in \mathcal{O}(U_j)$?

Observe that U_1 is a multicircular domain and that $U_1 = [\mathbf{C} - \{0\}] \times \mathbf{C}$. Every holomorphic function h_1 on U_1 is the sum of a (unique) absolutely convergent Laurent series

$$\sum_{p,q} a_{pq} z_1^p z_2^q,$$

at least where $z_1 z_2 \neq 0$, cf. Section 2.7. Here $a_{pq} = 0$ whenever $q < 0$: indeed, for fixed $z_1 \neq 0$,

$$h_1(z_1, z_2) = \sum_q \left(\sum_q a_{pq} z_1^p \right) z_2^q$$

will be an entire function of z_2, hence $\sum_p a_{pq} z_1^p = 0$ for every $q < 0$ and all $z_1 \neq 0$. Another application of the uniqueness theorem for Laurent series in one variable completes the proof that $a_{pq} = 0$ for all (p, q) with $q < 0$. Similarly every holomorphic function h_2 on U_2 is the sum of a Laurent series

$$\sum_{p,q} b_{pq} z_1^p z_2^q$$

with $b_{pq} = 0$ whenever $p < 0$. It follows in particular that

$$a_{-1,-1} = b_{-1,-1} = 0.$$

Thus a difference $h_2 - h_1$ with $h_j \in \mathcal{O}(U_j)$ can not possibly be equal to the prescribed function h_{12} on $U_1 \cap U_2$: the latter has Laurent series $\sum c_{pq} z_1^p z_2^q = z_1^{-1} z_2^{-1}$ hence $1 = c_{-1,-1} \neq b_{-1,-1} - a_{-1,-1}$. The present Cousin-I problem is not solvable.
[A shorter but less informative proof is suggested in Exercise 7.6.]

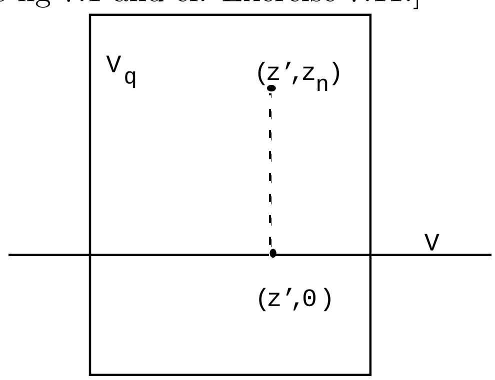
The domain $\Omega = \mathbf{C}^2 - \{0\}$ is not a domain of holomorphy. [All holomorphic functions on Ω have an analytic continuation to $\mathbf{C}^2$, cf. Sections 2.6, 3.4.] This is not a coincidence: on a domain of holomorphy, all Cousin-I problems will be solvable [see Theorem 7.71].

7.2 Holomorphic extension of analytic functions defined on a hyperplane section. Certain holomorphic extensions may be obtained by solving Cousin problems:

Theorem 7.21. *Let $\Omega \subset \mathbf{C}^n$ be a [meromorphic] Cousin-I domain and let Ω' be the nonempty intersection of Ω with some (affine) complex hyperplane $V \subset \mathbf{C}^n$. Then every holomorphic function h on Ω' [interpreted as a subset of $\mathbf{C}^{n-1}$] has a holomorphic extension g to Ω.*

PROOF. By suitable choice of coordinates it may be assumed that V is the hyperplane $\{z_n = 0\}$, so that the $(n-1)$-tuples $(z_1, \ldots, z_{n-1}) = z'$ can serve as coordinates in $\Omega' = V \cap \Omega$. The given holomorphic function $h(z')$ on Ω' can, of course, be extended to a holomorphic function [independent of z_n] on the cylinder $\Omega' \times \mathbf{C}$ by setting $\tilde{h}(z', z_n) = \tilde{h}(z', 0) = h(z')$. This observation solves the extension problem if $\Omega \subset \Omega' \times \mathbf{C}$; the general case will be handled via a meromorphic Cousin problem.

We introduce a covering of Ω by a family of polydiscs contained in Ω; the polydiscs which contain no point z with $z_n = 0$ will be called U_p's, those containing some point of V will be called V_q's. Observe that if $(z', z_n) \in V_q$, then also $(z', 0) \in V_q \subset \Omega$, hence $z' \in \Omega'$ [see fig 7.1 and cf. Exercise 7.11.]



One associates meromorphic Cousin data to the above covering that depend on the given function h:

$$(2a) \qquad f_p = 0 \quad \text{on each} \quad U_p, \qquad f_q(z', z_n) = \frac{h(z')}{z_n} \quad \text{on each} \quad V_q.$$

Since an intersection $U_p \cap V_q$ contains no points z with $z_n = 0$, the corresponding difference $f_p - f_q = h_{pq}$ is holomorphic on that intersection.

By the hypothesis, our meromorphic first Cousin problem is solvable. Let f be a meromorphic solution on Ω:

$$(2b) \qquad f = \begin{cases} f_p + h_p = h_p & \text{on} \quad U_p, \qquad h_p \in \mathcal{O}(U_p), \ \forall p, \\ \\ f_q + h_q = h(z')/z_n + h_q & \text{on} \quad V_q, \qquad h_q \in \mathcal{O}(V_q), \ \forall q. \end{cases}$$

We now define

$$(2c)\qquad g = z_n f = \begin{cases} z_n h_p & \text{on the polydisc } U_p, \ \forall p. \\ h(z') + z_n h_q & \text{at each point } (z', z_n) \\ & \text{in the polydisc } V_q, \ \forall q. \end{cases}$$

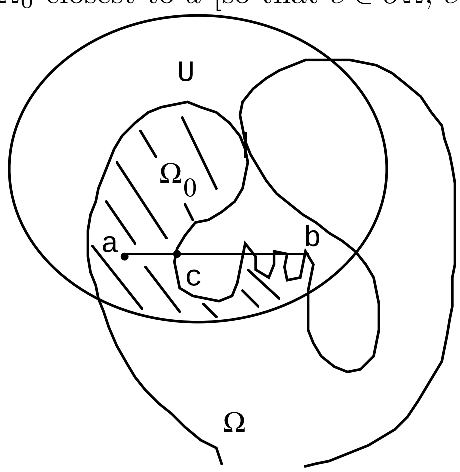
The function g is clearly holomorphic on Ω. It is equal to h on Ω': the points of Ω with $z_n = 0$ belong to polydiscs V_q, hence $g(z', 0) = h(z')$.

Appropriate choice of h will provide the following important step in an inductive solution of the *Levi problem*:

Theorem 7.22. *Let $\Omega \subset \mathbf{C}^n$ be a [meromorphic] Cousin-I domain. Suppose that the (nonempty) intersections of Ω with the complex hyperplanes in $\mathbf{C}^n$ are domains of holomorphy when considered as subsets of $\mathbf{C}^{n-1}$. Then Ω is a domain of holomorphy.*

PROOF. Choose any (small) connected domain U that intersects the boundary of Ω and any component Ω_0 of $U \cap \Omega$. We will construct a function $g \in \mathcal{O}(\Omega)$ whose restriction $g \mid \Omega_0$ can not be continued analytically to U [cf. Definition 6.11].

Take a point $b \in U \cap \partial\Omega_0$ and a point $a \in \Omega_0$ such that the segment $[a, b]$ belongs to U; let c be the point of $[a, b] \cap \partial\Omega_0$ closest to a [so that $c \in \partial\Omega$; c may coincide with b].



We next select a complex hyperplane V which contains $[a, b]$. Since by hypothesis the intersection $\Omega' = V \cap \Omega$ is a domain of holomorphy, there is a holomorphic function h on Ω' which becomes singular at c for approach along $[a, c)$. [One may take a function h that is unbounded on $[a, c)$, cf. Exercise 6.22. For $n = 2$, cf. also Exercise 7.12.] Let g, finally, be a holomorphic extension of h to the Cousin-I domain Ω. Then the restriction of g to Ω_0 has no analytic continuation to U: g must also become singular at c for approach along $[a, c)$.

COROLLARY 7.23. Let $\Omega \subset \mathbf{C}^n$ be a [meromorphic] Cousin-I domain and suppose that the same is true for the intersection of Ω with any affine complex subspace of $\mathbf{C}^n$ of any dimension k between 1 and n. Then Ω is a domain of holomorphy.

[Use induction on n; the intersections of Ω with complex lines are planar open sets, hence domains of holomorphy.]

Thus in $\mathbf{C}^2$, every [meromorphic] Cousin-I domain is a domain of holomorphy. This is no longer true in $\mathbf{C}^n$ with $n \geq 3$. For example, it was shown by Cartan that $\Omega = \mathbf{C}^n \setminus \mathbf{C}^m \times \{(0, \ldots, 0)\}$, $n \geq m + 3$ is a Cousin-I domain, cf. Exercise 7.10.

The method of proof of Theorem 7.22 gives another interesting criterion for a domain of holomorphy, see Exercise 7.24.

7.3 Refinement of coverings and partitions of unity. It is sometimes desirable to refine a given covering of Ω by open subsets. A covering $\{V_j\}$, $j \in J$ of Ω is called a *refinement* of the covering $\{U_\lambda\}$, $\lambda \in \Lambda$ if each set V_j is contained in some set U_λ. In order to transform given Cousin data for the covering $\{U_\lambda\}$ into Cousin data for the covering $\{V_j\}$, we introduce a *refinement map*, that is, a map $\sigma : J \to \Lambda$ such that

$$V_j \subset U_{\sigma(j)} \quad \text{for each} \quad j \in J. \tag{3a}$$

[There may be several possible choices for $U_\sigma(j)$: we make one for each j.]

REFINEMENT OF COUSIN DATA 7.31. Let $\{V_j\}$ be an open covering of Ω, $\varphi_{jk} \in \mathcal{O}(V_{jk})$. The data $\{V_j, \varphi_{jk}\}$ are called a refinement of given Cousin-I data $\{U_\lambda, h_{\lambda\mu}\}$ on Ω if the covering $\{V_j\}$ is a refinement of $\{U_\lambda\}$, and if the functions φ_{jk} are obtained from the functions $h_{\lambda\mu}$ via a refinement map σ, combined with restriction:

$$\varphi_{jk} = h_{\sigma(j)\sigma(k)} \,|_{V_{jk}}, \quad \forall j,\ k \in J \quad [V_{jk} \subset U_{\sigma(j)\sigma(k)}]. \tag{3b}$$

Let $\{U_\lambda, h_{\lambda\mu}\}$ be given holomorphic Cousin-I data for Ω and let $\{V_j, \varphi_{jk}\}$ be a refinement. It is clear that the functions φ_{jk} (3b) will then satisfy the compatibility conditions for the covering $\{V_j\}$, cf. (1e), hence the data $\{V_j, \varphi_{jk}\}$ are also Cousin-I data for Ω.

Proposition 7.32. *The original Cousin-I problem $\{U_\lambda, h_{\lambda\mu}\}$ on Ω is (holomorphically) solvable if and only if the refined problem $\{V_j, \varphi_{jk}\}$ is.*

PROOF. Suppose we have a solution $\{\varphi_j\}$ of the refined problem:

$$\varphi_k - \varphi_j = \varphi_{jk} = h_{\sigma(j)\sigma(k)} \quad \text{on} \quad V_{jk} \subset U_{\sigma(j)\sigma(k)}, \ \forall j,\ k \in J.$$

We want to construct appropriate functions h_λ on the sets U_λ from the functions φ_j and $h_{\lambda\mu}$. By the compatibility conditions (1e),

$$h_{\sigma(j)\sigma(k)} + h_{\sigma(k)\lambda} - h_{\sigma(h)\lambda} = 0 \quad \text{on} \quad U_{\sigma(j)\sigma(k)} \cap U_\lambda.$$

Combination of the two formulas shows that

$$\varphi_k + h_{\sigma(k)\lambda} = \varphi_j + h_{\sigma(j)\sigma(k)} + h_{\sigma(k)\lambda} = \varphi_j + h_{\sigma(j)\lambda} \quad \text{on} \quad V_{jk} \cap U_\lambda.$$

For each $\lambda \in \Lambda$ we may therefore define a function h_λ in a consistent manner throughout U_λ by setting

$$(3c) \qquad h_\lambda \stackrel{\text{def}}{=} \varphi_j + h_{\sigma(j)\lambda} \quad \text{on} \quad U_\lambda \cap V_j, \quad \forall j.$$

[Each point of U_λ belongs to some set V_j.] The (holomorphic) functions $h_\lambda,\ h_\mu$ will then satisfy the relation

$$h_\mu - h_\lambda = \varphi_j + h_{\sigma(j)\mu} - \varphi_j - h_{\sigma(j)\lambda} = h_{\lambda\mu} \quad \text{on} \quad U_{\lambda\mu} \cap V_j$$

for each j, hence $h_\mu - h_\lambda = h_{\lambda\mu}$ throughout $U_{\lambda\mu}$. Thus the family $\{h_\lambda\}$ will solve the original Cousin-I problem.

The proof in the other direction is immediate: if $\{h_\lambda\}$ solves the original problem, the family obtained via the map σ, combined with restriction, will solve the refined problem. Indeed, if

$$\varphi_j \stackrel{\text{def}}{=} h_{\sigma(j)} \mid V_j, \quad \forall j \in J,$$

then

$$\varphi_k - \varphi_j = h_{\sigma(k)} - h_{\sigma(j)} = h_{\sigma(j)\sigma(k)} = \varphi_{jk} \quad \text{on} \quad V_{jk}.$$

SPECIAL OPEN COVERINGS 7.33. It is convenient to consider open coverings $\{V_j\}$ of Ω that have the following properties:

(3d) $\{V_j\}$ is *locally finite*, that is, every compact subset of Ω intersects only finitely many sets V_j;

(3e) V_j has *compact closure* on Ω for each j.

Every special open covering $\{V_j\}$ as above will be countably infinite: by (3e) it must be infinite, and by (3d) it is countable [cf. the proof below].

Lemma 7.34. *Every open covering $\{U_\lambda\}$ of Ω has a special refinement $\{V_j\}$ – one that satisfies the conditions* (3d, e).

PROOF. One may obtain such a refinement $\{V_j\}$ of $\{U_\lambda\}$ with the aid of the standard exhaustion of Ω by the compact subsets

$$E_\nu = \{z \in \Omega : d(z, \partial\Omega) \geq 1/\nu, \quad |z| \leq \nu\}, \quad \nu = 1, 2, \ldots\ .$$

Assuming E_2 nonempty (as we may by changing the scale if necessary), one picks out finitely many sets U_λ that jointly cover E_2. The corresponding subsets $U_\lambda \cap E_3^0$ [E^0=interior of E] will provide the first sets V_j; together, they cover E_2. One next covers $E_3 - E_2^0$ by finitely many sets U_λ and uses the corresponding subsets $U_\lambda \cap (E_4^0 - E_1)$ as the next sets V_j; jointly they cover $E_3 - E_2^0$. In the next step one covers $E_4 - E_3^0$ by infinitely many sets $U_\lambda \cap (E_5^0 - E_2)$, etc.

DEFINITION 7.35. A C^∞ PARTITION OF UNITY on Ω subordinate to an open covering $\{U_\lambda\}$ is a family of nonnegative C^∞ functions $\{\beta_\lambda\}$ on Ω such that

$$\sum_\lambda \beta_\lambda \equiv 1 \quad on \quad \Omega \quad and \quad \operatorname{supp} \beta_\lambda \subset U_\lambda, \ \forall \lambda,$$

Here $\operatorname{supp} \beta_\lambda$ is the support relative to Ω, that is, the smallest relatively closed subset of Ω outside of which β_λ is equal to 0.

Proposition 7.36. *For every special covering $\{V_j\}$, $j = 1, 2, ...$ of Ω satisfying the conditions $(3d, e)$ there exists a C^∞ partition of unity $\{\beta_j\}$ on Ω with $\beta_j \in C_0^\infty(V_j)$, that is,* $\operatorname{supp} \beta_j$ *is a compact subset of V_j, $\forall j$.*

[Actually, there exist C^∞ partitions of unity subordinate to any open covering $\{U_\lambda\}$; they may be obtained from those for special coverings by a simple device, cf. Exercise 7.15.]

PROOF of the Proposition. We begin by constructing a family of nonnegative C^∞ functions $\{\alpha_j\}$ on Ω such that $\operatorname{supp} \alpha_j$ is a compact subset of V_j while $\alpha = \sum_j \alpha_j$ is a strictly positive C^∞ function on Ω. For appropriate $\varepsilon_j > 0$ with $4\varepsilon_j < \operatorname{diam} V_j$, let W_j denote the set of all points in V_j whose distance to the boundary ∂V_j is greater than $2\varepsilon_j$. It may and will be assumed that the numbers ε_j have been chosen in such a way that the family $\{W_j\}$ is still a covering of Ω. [One may first choose ε_1 so small that the family $W_1, V_2, V_3, \ldots$ is still a covering, then choose ε_2 so small that the family $W_1, W_2, V_3, \ldots$ is still a covering, etc.]

For each j we now determine a nonnegative C^∞ function α_j on Ω which is strictly positive on W_j and has compact support in V_j. [One may obtain α_j by smoothing of the characteristic function of W_j through convolution with a nonnegative C^∞ approximation to the identity ρ_ε, $\varepsilon = \varepsilon_j$, whose support is the ball $\overline{B}(0, \varepsilon_j)$, cf. Section 3.3.] Observe that at any given point $a \in \Omega$, at least one function α_j will be > 0.

Since the covering $\{V_j\}$ is locally finite, a closed ball $\overline{B} \subset \Omega$ intersects only finitely many sets V_j. Hence all but a finite number of functions α_j are identically zero on B. It follows that the sum $\sum_j \alpha_j$ defines a C^∞ function α on B, and hence on Ω. By the preceding, the sum function α will be > 0 throughout Ω.

The proof is completed by setting

$$\beta_j \stackrel{\text{def}}{=} \alpha_j / \alpha, \quad \forall j.$$

7.4 Analysis of Cousin-I. Existence of smooth solutions. Suppose that the holomorphic Cousin-I problem with data $\{U_\lambda, h_{\lambda\mu}\}$ on Ω has a holomorphic or *smooth* solution $\{h_\lambda\}: \quad h_\lambda \in \mathcal{O}(U_\lambda)$ or $h_\lambda \in C^\infty(U_\lambda)$ and $h_\mu - h_\lambda = h_{\lambda\mu}$ on $U_{\lambda\mu}$. By refinement of the data we may assume that the covering $\{U_\lambda\}$ is *locally finite* and that we have been able to construct a C^∞ *partition of unity* $\{\beta_\lambda\}$ on Ω, subordinate to the covering $\{U_\lambda\}$, cf. Section 7.3.

We wish to analyze the function h_λ and *focus on a point* a in U_λ. At such a point a we will have

$$h_\lambda = h_\nu + h_{\nu\lambda} \tag{4a}$$

for all indices ν such that $a \in U_\nu$. There are only finitely many such indices ν! We multiply (4a) by β_ν and initially sum over precisely those indices ν for which $a \in U_\nu$ (symbol $\sum_\nu^a$):

$$\Big(\sum_\nu{}^a \beta_\nu\Big) h_\lambda = \sum_\nu{}^a \beta_\nu h_\nu + \sum_\nu{}^a \beta_\nu h_{\nu\lambda} \qquad \text{(at the point} \quad a \in U_\lambda). \tag{4b}$$

The value of the *first sum* will not change if we add the terms β_ν (equal to 0!) which correspond to the indices ν for which $a \notin U_\nu$. The sum over all ν's is equal to 1 and this will hold at every point $a \in U_\lambda$:

$$\sum_\nu{}^a \beta_\nu = \sum_{\nu \in \Lambda} \beta_\nu = 1 \quad \text{(at} \quad a \in U_\lambda). \tag{4c}$$

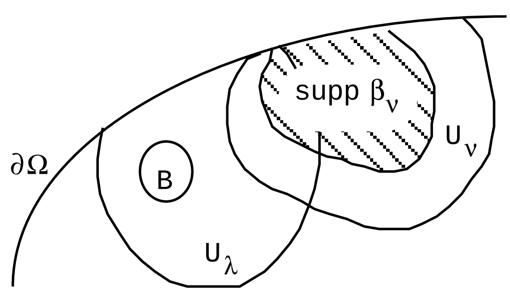
Products $\beta_\nu h_\nu$, whether they occur in the *second sum* (4b) or not, may be extended to C^∞ functions on Ω by defining $\beta_\nu h_\nu = 0$ on $\Omega - U_\nu$; indeed, $\beta_\nu h_\nu = 0$ outside a closed subset of U_ν anyway [closed relative to Ω]. The value of the second sum will not change if we add the terms zero corresponding to those ν's, for which $a \notin U_\nu$:

$$\sum_\nu{}^a \beta_\nu h_\nu = \sum_{\nu \in \Lambda} \beta_\nu h_\nu \quad \text{(at} \quad a \in U_\lambda). \tag{4d}$$

What can we say about the last sum (4d) on U_λ or elsewhere?

On a closed ball $\overline{B} \subset \Omega$, only finitely many terms in the full sum $\sum \beta_\nu h_\nu$ are ever $\neq 0$. Thus the full sum defines a C^∞ function on B and hence on Ω:

$$\sum_{\nu \in \Lambda} \beta_\nu h_\nu \overset{\text{def}}{=} u \in C^\infty(\Omega). \tag{4e}$$



We now turn to the *third sum* in (4b), but there we will not go outside U_λ. Products $\beta_\nu h_{\nu\lambda}$ are defined only on $U_{\lambda\nu}$. Such products can be extended to C^∞ *functions* on U_λ by setting them equal to 0 on $U_\lambda - U_\nu$: they vanish at the points of U_ν close to $U_\lambda - U_\nu$ anyway (fig 7.3). For indices ν such that U_ν does not meet U_λ, we may simply define

$\beta_\nu h_{\nu\lambda}$ as 0 throughout U_λ. We are again going to sum over all ν's; at $a \in U_\lambda$ this only means that we add a number of zero terms to the original sum:

$$\sum_\nu{}^a \beta_\nu h_{\nu\lambda} = \sum_{\nu\in\Lambda} \beta_\nu h_{\nu\lambda} \quad (\text{at} \quad a \in U_\lambda). \tag{4f}$$

What can we say about the last sum $(4f)$?

On a closed ball $\overline{B}$ in U_λ, only finitely many terms in the full third sum $\sum \beta_\nu h_{\nu\lambda}$ are ever $\neq 0$. Thus that sum defines a C^∞ function on B and hence on U_λ:

$$\sum_{\nu\in\Lambda} \beta_\nu h_{\nu\lambda} \stackrel{\text{def}}{=} g_\lambda \in C^\infty(U_\lambda). \tag{4g}$$

Conclusion. Combining $(4b-g)$, we see that at any point $a \in U_\lambda$:

$$h_\lambda = \Big(\sum_\nu{}^a \beta_\nu\Big) h_\lambda = \sum_{\nu\in\Lambda} \beta_\nu h_\nu + \sum_{\nu\in\Lambda} \beta_\nu h_{\nu\lambda} = u + g_\lambda.$$

Thus any holomorphic or C^∞ soluation of the Cousin-I problem under consideration can be represented in the form

$$h_\lambda = u + g_\lambda \quad \text{on} \quad U_\lambda, \qquad \forall \lambda \in \Lambda, \tag{4h}$$

where $g_\lambda \in C^\infty(U_\lambda)$ is given by $(4g)$ and $u \in C^\infty(\Omega)$.

Conversely, if we *define* functions h_λ by $(4h, g)$, they will always form at least a *smooth* solution of the Cousin-I problem, no matter what open set Ω we have
[see below]:

Theorem 7.41. *Let $\{U_\lambda, h_{\lambda\mu}\}$ be any family of holomorphic Cousin-I data on $\Omega \subset \mathbf{C}^n$ which has been refined so that the covering $\{U_\lambda\}$ is locally finite and there is a C^∞ partition of unity $\{\beta_\lambda\}$ on Ω subordinate to $\{U_\lambda\}$. Then the functions h_λ defined by $(4h, g)$, with an arbitrary C^∞ function u on Ω, constitute a C^∞ solution of the Cousin-I problem with the given data, and every C^∞ solution of the problem is of that form.*

PROOF. For functions h_λ as in $(4h, g)$ one has

$$\begin{aligned} h_\mu - h_\lambda = g_\mu - g_\lambda &= \sum_{\nu\in\Lambda}\Big(\beta_\nu h_{\nu\mu} - \beta_\nu h_{\nu\lambda}\Big) \\ &= \Big(\sum_{\nu\in\Lambda} \beta_\nu\Big) h_{\lambda\mu} = h_{\lambda\mu} \quad \text{on} \quad U_{\lambda\mu}, \ \forall \lambda, \mu \end{aligned} \tag{4i}$$

because of the compatibility conditions $(1e)$. Thus the functions h_λ form a C^∞ solution of the Cousin-I problem, cf. $(1f)$. That all C^∞ solutions of the problem have the form $(4h, g)$ follows from the earlier analysis.

Remark. That every holomorphic (or C^∞!) Cousin-I problem for arbitrary open Ω is C^∞ solvable can also be proved without refinement of the Cousin data – it suffices to refine the covering (if necessary), cf. Exercise 7.20.

7.5 Holomorphic solutions of Cousin-I via $\overline{\partial}$. In Section 7.4 we have determined all C^∞ solutions $\{h_\lambda\}$ of a given (suitably refined) holomorphic Cousin-I problem $\{U_\lambda, h_{\lambda\mu}\}$ on an open set Ω. For a fixed C^∞ partition of unity $\{\beta_\lambda\}$ subordinate to $\{U_\lambda\}$, they have the form $(4h)$:

$$h_\lambda = u + g_\lambda \quad \text{on} \quad U_\lambda,$$

with g_λ as in $(4g)$ and an arbitrary C^∞ function u on Ω.

QUESTION 7.51. Will there be a *holomorphic* solution among all the C^∞ solutions $\{h_\lambda\}$?

We still have the function u at our disposal. For holomorphy of the smooth functions h_λ it is *necessary and sufficient* that

$$0 = \overline{\partial} h_\lambda = \overline{\partial} u + \overline{\partial} g_\lambda,$$

or

$$(5a) \qquad \overline{\partial} u = -\overline{\partial} g_\lambda \quad \text{on} \quad U_\lambda, \ \forall \lambda \in \Lambda.$$

The second members in $(5a)$ can be used to define a global differential form v on Ω. Indeed, on an intersection $U_\lambda \cap U_\mu$, $(4i)$ gives

$$(5b) \qquad \overline{\partial}(g_\mu - g_\lambda) = \overline{\partial}(h_\mu - h_\lambda) = \overline{\partial} h_{\lambda\mu} = 0$$

since $h_{\lambda\mu} \in \mathcal{O}(U_{\lambda\mu})$! Thus we obtain a C^∞ form v on Ω by setting

$$(5c) \qquad v = \sum_{j=1}^n v_j d\overline{z}_j \overset{\text{def}}{=} -\overline{\partial} g_\lambda = -\sum_{j=1}^n \frac{\partial g_\lambda}{\partial \overline{z}_j} d\overline{z}_j \quad \text{on} \quad U_\lambda, \ \forall \lambda \in \Lambda.$$

The conditions $(5a)$ on u can now be summarized by the single equation $\overline{\partial} u = v$ on Ω. If u satisfies this condition, then $\overline{\partial} h_\lambda = 0$, so that h_λ is a holomorphic function for each λ. We have thus proved:

Proposition 7.52. *The (suitably refined) holomorphic Cousin-I problem 7.13 on Ω has a holomorphic solution $\{h_\lambda\}$ if and only if the associated $\overline{\partial}$ equation*

$$(5d) \qquad \overline{\partial} u = v \quad \textit{on} \quad \Omega,$$

with v given by $(5c)$ *and* $(4g)$*, has a C^∞ solution u on Ω.*

Incidentally, it is clear from $(5c)$ that v satisfies the integrability conditions $\partial v_k / \partial \overline{z}_j = \partial v_j / \partial \overline{z}_k$. It will be convenient to introduce the following terminology:

DEFINITION 7.53. An open set $\Omega \subset C^n$ will be called a $\overline{\partial}$ DOMAIN if all equations $\overline{\partial} u = v$ on Ω, with $(0,1)$-forms v of class C^∞ that satisfy the integrability conditions, are C^∞ solvable on Ω.

We will now prove the following important

Theorem 7.54. *Every $\overline{\partial}$ domain Ω in $\mathbf{C}^n$ is a Cousin-I domain 7.15, and conversely.*

PROOF of the direct part. Let Ω be a $\overline{\partial}$ domain. Then equation $(5d)$ is C^∞ solvable, hence by Proposition 7.52, every suitable refined (holomorphic) Cousin-I problem on Ω is holomorphically solvable. Proposition 7.32 on refinements now tells us that *every* Cousin-I problem on Ω is holomorphically solvable, hence Ω is a Cousin-I domain.

For the converse we need an auxiliary result on local solvability of the $\overline{\partial}$ equation that will be proved in Section 7.6:

Proposition 7.55. *Let v be a differential form $\sum_1^n v_j d\overline{z}_j$ of class C^p $(1 \le p \le \infty)$ on the polydisc $\Delta(a,r) \subset \mathbf{C}^n$ that satisfies the local integrability conditions $\partial v_k/\partial\overline{z}_j = \partial v_j/\partial\overline{z}_k$. Then the equation $\overline{\partial} u = v$ has a C^p solution $f = f_s$ on every polydisc $\Delta(a,s)$ with $s < r$.* [If a certain differential $d\overline{z}_k$ is absent from v (that is, if the coefficient v_k is identically 0), one may take the corresponding number s_k equal to r_k. The solution constructed in Section 7.6 will actually be of class $C^{p+\alpha}$, $\forall \alpha \in (0,1)$.]

PROOF of Theorem 7.54, converse part. Let Ω be a Cousin-I domain and let v be any C^∞ differential form $\sum_1^n v_j d\overline{z}_j$ on Ω that satisfies the integrability conditions. We cover Ω by a family of "good" polydiscs U_λ, $\lambda \in \Lambda$, that is, polydiscs $U_\lambda \subset \Omega$ on which there exists a C^∞ solution f_λ of the equation $\overline{\partial} u = v$. Then on the intersections $U_{\lambda\mu}$:

$$\overline{\partial}(f_\lambda - f_\mu) = v - v = 0,$$

hence

$$h_{\lambda\mu} \stackrel{\text{def}}{=} f_\lambda - f_\mu \in \mathcal{O}(U_{\lambda\mu}), \quad \forall \lambda,\ \mu.$$

Just as in Section 7.1 the differences $h_{\lambda\mu}$ will satisfy the compatibility conditions $(1e)$, hence $\{U_\lambda, h_{\lambda\mu}\}$ is a family of holomorphic Cousin-I data for Ω. Since by the hypothesis all Cousin-I problems on Ω are (holomorphically) solvable, there is a family of functions $h_\lambda \in \mathcal{O}(U_\lambda)$ such that

$$h_{\lambda\mu} = h_\mu - h_\lambda$$

on each nonempty intersection $U_{\lambda\mu}$. We now set

$$u \stackrel{\text{def}}{=} f_\lambda + h_\lambda \quad \text{on} \quad U_\lambda,\ \forall \lambda.$$

This formula will furnish a global C^∞ solution of the equation $\overline{\partial} u = v$ on $\Omega:$ $f_\lambda + h_\lambda = f_\mu + h_\mu$ on $U_{\lambda\mu}$ and

$$\overline{\partial} u = \overline{\partial} f_\lambda + \overline{\partial} h_\lambda = v + 0 = v \quad \text{on} \quad U_\lambda,\ \forall \lambda.$$

Conclusion: Ω is a $\overline{\partial}$ domain.

7.6 Solution of $\overline{\partial}$ on polydiscs. We first prove the important local solvability of the $\overline{\partial}$ equation asserted in Proposition 7.55. Next we will show that polydiscs (and in particular $\mathbf{C}^n$ itself) are $\overline{\partial}$ domains ["Dolbeault's lemma"]. The latter result is not needed for the sequel, but we derive it to illustrate the approximation technique that may be used to prove the general solvability of $\overline{\partial}$ on domains permitting appropriate polynomial approximation, cf. [Hörmander 1], [Range].

PROOF of Proposition 7.55. It is convenient to take $a = 0$. For $n = 1$ the proof is very simple. Just let ω be a C^∞ cutoff function which is equal to 1 on $\Delta(0, s)$ and has support in $\Delta(0, r)$. Then ωv can be considered as a C^p form on $\mathbf{C}$ which vanishes outside $\Delta(0, r)$. Hence the Cauchy-Green transform will provide a C^p solution of the equation $\overline{\partial} u = \omega v$ on $\mathbf{C}$ and thus of the equation $\overline{\partial} u = v$ on $\Delta(0, s)$, cf. Theorem 3.13.

For $n \geq 2$ we try to imitate the procedure used in Section 3.2 for the case where v has compact support, but now there will be difficulties. These are due to the fact that we can not multiply v by a nonzero C^∞ function of compact support in $\Delta(0, r)$ and still preserve the integrability conditions. To get around that problem one may use induction on the number of differentials $d\overline{z}_j$ that there actually present in v.

If v contains no differentials $d\overline{z}_j$ at all, the equation is $\overline{\partial} u = 0$ and every holomorphic function on $\Delta(0, r)$ is a C^p solution on the whole polydisc. Suppose now that *precisely q differentials $d\overline{z}_j$ are present* in v, among them $d\overline{z}_n$, and that the Proposition has been established already for the case in which only $q - 1$ differentials $d\overline{z}_j$ are present. As usual we write $z = (z', z_n)$ and we set

$$\Delta(0, r) = \Delta_{n-1}(0, r') \times \Delta_1(0, r_n) = \Delta_{n-1} \times \Delta_1(0, r_n).$$

Choosing s as in the Proposition, so that in particular $s_n < r_n$, we let $\omega = \omega(z_n)$ be a cutoff function of class C_0^∞ on $\Delta_1(0, r_n) \subset \mathbf{C}$ which is equal to 1 on $\Delta_1(0, s_n)$. Defining $\omega v_j = 0$ for $|z_n| \geq r_n$, the product

$$\omega(z_n) v(z', z_n) = \omega v_1 d\overline{z}_1 + \ldots + \omega v_n d\overline{z}_n$$

represents a C^p form on $\Delta_{n-1} \times \mathbf{C}$ which for fixed $z' \in \Delta_{n-1}$ vanishes when $|z_n| \geq R_n$. Thus the Cauchy-Green transform of ωv_n relative to z_n provides a solution φ to the equation $\partial u / \partial \overline{z}_n = \omega v_n$ on $\Delta_{n-1} \times \mathbf{C}$, cf. Theorem 3.13:

$$\varphi(z) = \varphi(z', z_n) = -\frac{1}{\pi} \int_{\mathbf{C}} \frac{\omega(\zeta) v_n(z', \zeta)}{\zeta - z_n} d\xi d\eta. \tag{6a}$$

Observe that the function φ is of class C^p on $\Delta(0, r)$ and that the same holds for $\partial \varphi / \partial \overline{z}_n = \omega v_n$.

We will determine the derivatives $\partial \varphi / \partial \overline{z}_j$ with $j < n$ by differentiation under the integral sign, noting that by the integrability conditions,

$$\frac{\partial}{\partial \overline{z}_j} \{\omega(\zeta) v_n(z', \zeta)\} = \omega(\zeta) \frac{\partial v_j}{\partial \overline{z}_n}(z', \zeta) = \frac{\partial}{\partial \overline{\zeta}} \{\omega(\zeta) v_j(z', \zeta)\} - v_j(z', \zeta) \frac{\partial \omega}{\partial \overline{\zeta}}.$$

Thus, referring to the representation for compactly supported functions of Corollary 3.12 for the second step,

$$(6b)\qquad \begin{aligned}\frac{\partial\varphi}{\partial\overline{z}_j} &= -\frac{1}{\pi}\int_{\mathbf{C}}\frac{\partial}{\partial\overline{\zeta}}\{\omega(\zeta)v_j(z',\zeta)\}\frac{d\xi d\eta}{\zeta-z_n}+\frac{1}{\pi}\int_{\mathbf{C}}\frac{v_j(z',\zeta)}{\zeta-z_n}\frac{\partial\omega}{\partial\overline{\zeta}}(\zeta)d\xi d\eta\\ &= \omega(z_n)v_j(z',z_n)+I(v_j,\omega),\quad 1\le j<n,\end{aligned}$$

say. By inspection, the derivatives $\partial\varphi/\partial\overline{z}_j$ are of class C^p on $\Delta(0,r)$.

We now introduce the differential form

$$(6c)\qquad w = v-\overline{\partial}\varphi.$$

This new form is also of class C^p on $\Delta(0,r)$ and it satisfies the integrability conditions [forms $\overline{\partial}\varphi$ always do]. Moreover, if we restrict the form to $\Delta_{n-1}(0,r')\times\Delta_1(0,s_n)$, then it may be written as

$$w = w_1 d\overline{z}_1+\ldots+w_{n-1}d\overline{z}_{n-1},$$

since $\partial\varphi/\partial\overline{z}_n = \omega v_n = v_n$ on that polydisc. Finally, if $d\overline{z}_k$ was absent from v, that is, $v_k=0$ and $k>n$, then by (6b) also $\partial\varphi/\partial\overline{z}_k=0$, so that $w_k=0$. Thus the new form w restricted to $\Delta_{n-1}(0,r')\times\Delta_1(0,s_n)$ contains at most $q-1$ differentials $d\overline{z}_j$. Hence by the induction hypothesis, there is a C^p function ψ on the polydisc $\Delta_{n-1}(0,s')\times\Delta_1(0,s_n)$ such that $w=\overline{\partial}\psi$ there. Conclusion:

$$(6d)\qquad v=\overline{\partial}\varphi+w=\overline{\partial}(\varphi+\psi)\quad\text{on}\quad\Delta(0,s),$$

with $\varphi+\psi\in C^p$.

[Actually, the function φ in (6a) will be of class $C^{p+\alpha}$, $\forall\alpha\in(0,1)$, cf. Remarks 3.14 and Exercises 3.6, 3.9. Likewise, by induction, $\psi\in C^{p+\alpha}$.]

Theorem 7.61. *Let $v=\sum_1^n v_j d\overline{z}_j$ be a differential form of class C^p $(1\le p\le\infty)$ on the polydisc $\Delta(a,r)\subset\mathbf{C}^n$ that satisfies the integrability conditions. Then the equation $\overline{\partial}u=v$ has a C^p solution on $\Delta(a,r)$.*

PROOF. We take $a=0$ and introduce a strictly increasing sequence of polydiscs Δ_k [more precisely, $\overline{\Delta}_k\subset\Delta_{k+1}$, $k=0,1,2,\ldots$] with center 0 and union $\Delta=\Delta(0,r)$. By Proposition 7.55 there are C^p functions f_k on Δ_{k+1}, $k=1,2,\ldots$ such that $\overline{\partial}f_k=v$ on Δ_k. Starting with such functions f_k, we will inductively determine C^p functions u_k, $k=1,2,\ldots$ on Δ such that

(i) $\operatorname{supp} u_k\subset\Delta_{k+1}$,
(ii) $\overline{\partial}u_k=v$ on Δ_k,
(iii) $|u_k-u_{k-1}|<2^{-k}$ on Δ_{k-2}, $k\ge 2$.

Let $\{\omega_k\}$ be a sequence of C^p cutoff functions on $\mathbf{C}^n$ such that $\omega_k=1$ on Δ_k and $\operatorname{supp}\omega_k\subset\Delta_{k+1}$. We define $u_1=\omega_1 f_1$ on Δ_2, $u_1=0$ on $\Delta-\Delta_2$ so that (i) and (ii) hold for $k=1$. Now suppose that $u_1,\ldots,u_k$ have been determined in accordance with conditions (i)-(iii). Since $\overline{\partial}(f_{k+1} - u_k) = 0$ on Δ_k, the difference $f_{k+1} - u_k$ is holomorphic on that polydisc, hence equal to the sum of a power series around 0 which is uniformly convergent on Δ_{k-1}. Thus one can find a polynomial p_k such that

$$|f_{k+1} - u_k - p_k| < 2^{-k-1} \quad \text{on} \quad \Delta_{k-1}.$$

We may now define

$$u_{k+1} = \begin{cases} (f_{k+1} - p_k)\omega_{k+1} & \text{on} \quad \Delta_{k+2}, \\ \\ 0 & \text{on} \quad \Delta - \Delta_{k+2} \end{cases}$$

to obtain (i)-(iii) with $k+1$ instead of k.

By (iii) we may define a function u on Δ by

$$u = u_1 + \sum_2^\infty (u_k - u_{k-1});$$

the series will be uniformly convergent on every compact subset of Δ. Condition (ii) shows that the terms $u_k - u_{k-1}$ with $k > j$ are holomorphic on Δ_j, hence $\varphi_j = \sum_{k>j}(u_k - u_{k-1})$ is holomorphic on Δ_j. It follows that $u = u_j + \varphi_j$ is of class C^p on Δ_j. Moreover, on Δ_j

$$\overline{\partial}u = \overline{\partial}u_j + \overline{\partial}\varphi_j = v + 0 = v.$$

Since these results hold for each $j = 1, 2, \ldots$ we are done.

The method of the previous theorem may be extended to show that products of planar domains are Cousin-I domains, cf. exercise 7.21. It seems difficult to determine if a given domain is Cousin-I. The following theorem is sometimes useful.

Theorem 7.62. *Let $\{U_\lambda\}$ be some open cover of the domain Ω consisting of Cousin-I domains. Ω is a Cousin-I domain if and only if for all Cousin-I data of the form $\{U_\lambda, h^{\lambda\mu}\}$ the Cousin-I problem is solvable.*

PROOF. The only if part is clear. For the if part, suppose that we are given Cousin-I data $\{V_i, h_{ij}\}$. Then $\{V_i \cap U_\lambda, h_{ij}\}$ are Cousin-I data on U_λ (here and in the sequel we denote the restriction of a function to a smaller domain and the function itself with the same symbol), hence there exist $h_i^\lambda \in \mathcal{O}(V_i \cap U_\lambda)$ with

$$h_i^\lambda - h_j^\lambda = h_{ij} \text{ on } U_\lambda \cap V_{ij}.$$

On $U_{\lambda\mu} \cap V_{ij}$ we find $h_i^\lambda - h_j^\lambda = h_{ij} = h_i^\mu - h_j^\mu$, therefore

$$h_i^\lambda - h_j^\lambda = h_i^\mu - h_j^\mu.$$

Thus $h_i^\lambda - h_i^\mu = h_j^\lambda - h_j^\mu$ for all i, j on $U_{\lambda\mu}$ and we may define $h^{\lambda\mu} \in \mathcal{O}(U_{\lambda\mu})$ by

$$h^{\lambda\mu} = h_i^\lambda - h_i^\mu \text{ on } V_i \cap U_{\lambda\mu}.$$

Are these consistent Cousin-I data? Yes: on $V_i \cap U_{\lambda\mu\nu}$ we have

$$h^{\lambda\mu} + h^{\mu\nu} + h^{\nu\lambda} = h_i^\lambda - h_i^\mu + h_i^\mu - h_i^\nu + h_i^\nu - h_i^\lambda = 0.$$

We can solve this Cousin-I problem with functions $h^\lambda \in \mathcal{O}(U_\lambda)$. Now for all i we find on $V_i \cap U_{\lambda\mu}$

$$h_i^\lambda - h_i^\mu = h^{\lambda\mu} = h^\lambda - h^\mu.$$

Hence $h_i^\lambda - h^\lambda = h_i^\mu - h^\mu$ on $V_i \cap U_{\lambda\mu}$ and we may conclude that $h_i^\lambda - h^\lambda$ extends analytically to a function $h_i \in \mathcal{O}(V_i)$. We claim that the h_i provide a solution. Indeed, for all λ we have on $U_\lambda \cap V_{ij}$

$$h_i - h_j = h_i^\lambda - h^\lambda - h_j^\lambda + h^\lambda = h_{ij}.$$

7.7 The Levi problem. It will be shown in Chapter 11 that every domain with a plurisubharmonic exhaustion function, or pseudoconvex domain, is a $\overline{\partial}$ domain. Once that fundamental result has been established, we can use Theorem 7.54 to conclude:

Theorem 7.71. *Every pseudoconvex domain, and hence every domain of holomorphy, is a Cousin-I domain.*

More important, the result of Chapter 11 will enable us to complete the solution of the Levi problem begun in Section 7.2:

Theorem 7.72. *Every domain Ω in $\mathbf{C}^n$ with a plurisubharmonic exhaustion function, or pseudoconvex domain, is a domain of holomorphy.*

PROOF. We use induction on the dimension. Suppose then that the result has been established for dimension $n-1$; dimension 1 is no problem since every domain in $\mathbf{C}$ is a domain of holomorphy. Now let Ω be a *psh* exhaustible domain in $\mathbf{C}^n$, $n \geq 2$. By the fundamental result to be proved in Chapter 11, Ω is a $\overline{\partial}$ domain and hence a Cousin-I domain [Theorem 7.54]. On the other hand, the intersections Ω' of Ω with (affine) complex hyperplanes are also *psh* exhaustible [Proposition 6.56]. Hence by the induction hypothesis, they are domains of holomorphy when considered as open subsets of $\mathbf{C}^{n-1}$. Thus by Theorem 7.22, Ω is a domain of holomorphy.

Remarks. For $n = 2$ the Levi problem was settled by Oka in 1942, while solutions for $n \geq 3$ were obtained almost simultaneously by Bremermann, Norguet and Oka in the years 1953-1954. After Dolbeault's work on cohomology (1953-1956, cf. Chapter 12), it became clear that a solution of the Levi problem could also be based on an analytic solution of $\overline{\partial}$, but such a solution did not exist at the time!

Exercises

7.1. Prove the *Mittag-Leffler theorem* for $\Omega = \mathbf{C}$ (Section 7.1) along the following classical lines:

(i) Write down a rational function g_k with the prescribed poles and principal parts on the disc $\Delta(0,k)$, $k = 1, 2, \ldots$;

(ii) Does the series $g_1 + \sum_2^\infty (g_k - g_{k-1})$ converge on Ω? If not, how can it be modified to ensure convergence, taking into account that $g_k - g_{k-1}$ is holomorphic on $\Delta(0, k-1)$?

7.2. (*Related treatment of* $\overline{\partial}$ *on* $\mathbf{C}$) Let v be a C^p function on $\Omega = \mathbf{C}$, $1 \le p \le \infty$.

(i) Use a suitable Cauchy-Green transform (Section 3.1-(1f)) to obtain a C^p function u_k on $\mathbf{C}$ such that $\partial u_k/\partial \overline{z} = v$ on $\Delta(0,k)$;

(ii) Determine a C^p solution u of the equation $\partial u/\partial \overline{z} = v$ on Ω by using a suitable modification of the series $u_1 + \sum_2^\infty (u_k - u_{k-1})$.

7.3. Describe how one can solve the meromorphic first Cousin problem for $\mathbf{C}$ (Section 7.1) directly with the aid of a $\overline{\partial}$ problem. [Using nonoverlapping discs $\Delta(a_\lambda, r_\lambda)$, let ω_λ be a C^∞ function on $\mathbf{C}$ with support in $\Delta(a_\lambda, r_\lambda)$ and equal to 1 on $\Delta(a_\lambda, \frac{1}{2}r_\lambda)$. Then $u = f - \sum \omega_\lambda f_\lambda$ must be a C^∞ function on $\mathbf{C}$. What conditions does $\partial u/\partial \overline{z}$ have to satisfy?]

7.4. Extend the constructions in Exercises 7.1, 7.2 to the case where Ω is:

(i) the unit disc $\Delta(0,1)$;

(ii) the annulus $A(0;1,2)$.

7.5. Let U be a connected domain in $\mathbf{C}^n$, $g,\ h,\ \tilde{g},\ \tilde{h} \in \mathcal{O}(U)$, $h \not\equiv 0$, $\tilde{h} \not\equiv 0$. Suppose that $g/h = \tilde{g}/\tilde{h}$ outside $Z(h) \cup Z(\tilde{h})$. Prove that $g\tilde{h} = h\tilde{g}$ on U, so that $[g/h] = [\tilde{g}/\tilde{h}]$ in the quotient field for $\mathcal{O}(U)$.

7.6. Prove directly [without Laurent series] that the following meromorphic first Cousin problem on $\Omega = \mathbf{C}^2 - \{0\}$ must be unsolvable:

$$f_1 = \frac{1}{z_1 z_2} \quad \text{on} \quad U_1 = \{z_1 \neq 0\}, \qquad f_2 = 0 \quad \text{on} \quad U_2 = \{z_2 \neq 0\}.$$

[Cf. formula (2c).]

7.7. Let Ω_1 be a Cousin-I domain in $\mathbf{C}^n$ and let Ω_2 be analytically isomorphic to Ω_1. Prove that Ω_2 is also a Cousin-I domain.

7.8. Which holomorphic Cousin-I problems for $\Omega = \mathbf{C}^2 - \{0\}$ and $U_j = \{z \in \mathbf{C}^2 : z_j \neq 0\}$, $j = 1, 2$ are solvable and which are not?

7.9. Let Ω be the multicircular domain in $\mathbf{C}^n$ $(n \ge 2)$ given by

$$\{|z'| < 1,\ |z_n| < 3\} \cup \{|z'| < 3, \quad 1 < |z_n| < 3\}, \qquad z' = (z_1, \ldots, z_{n-1}).$$

Prove that Ω is not a Cousin-I domain by indicating a holomorphic function on $\Omega \cap \{z_1 = 2\}$ that has no analytic extension to Ω.

7.10. Prove by computation that all Cousin-I problems for $\Omega = \mathbf{C}^3 - \{0\}$, $U_j = \{z \in \mathbf{C}^3 : z_j \neq 0\}$, $\quad j = 1, 2, 3$ are solvable:

(i) Show that it is sufficient to consider the case where

$$h_{23} = a\, z_1^p z_2^q z_3^r, \quad h_{31} = b\, z_1^p z_2^q z_3^r, \quad h_{12} = c\, z_1^p z_2^q z_3^r.$$

(ii) Suppose $p < 0$. Show that then $a = 0,\ c = -b$ and (assuming $b \neq 0$) $q \geq 0,\ r \geq 0$. Solve.

(iii) Finally deal with the case $p \geq 0,\ q \geq 0,\ r \geq 0$.

(iv) Can you extend this to $\mathbf{C}^n \setminus \mathbf{C}^m$, $n - m \geq 3$?

7.11. Suppose that the polydisc $\Delta(a, r)$ in $\mathbf{C}^n$ meets the hyperplane $\{z_n = 0\}$. What does this mean for a_n and r_n? Prove that the intersection is precisely the projection of $\Delta(a, r)$ onto the hyperplane.

7.12. Prove more directly than in the text that a Cousin-I domain Ω in $\mathbf{C}^2$ is a domain of holomorphy, using the following idea. Starting out as in the proof of Theorem 7.22, take $c = 0$ and let the complex line $z_2 = 0$ pass through $[a, b]$. Cover Ω by polydiscs U_p (not containing points with $z_2 = 0$) and V_q (containing points with $z_2 = 0$). Now solve the associated meromorphic Cousin problem with $f_p = 0$ on each U_p, $f_q = 1/z_1 z_2$ on each V_q, etc.

7.13. C^∞ partitions of unity subordinate to a given open covering $\{U_\lambda\}$ are relatively easy to construct for the case of open sets Ω in $\mathbf{R}$. Verify the following steps:

(i) It may be assumed that Ω is an open interval I and that $\{U_\lambda\}$ is a locally finite covering by open intervals I_λ;

(ii) There are subintervals $J_\lambda \subset I_\lambda$ which are relatively closed in I and jointly cover I, and for $J_\lambda \subset I_\lambda$ there is a C^∞ function $\alpha_\lambda \geq 0$ on I such that $\alpha_\lambda > 0$ on J_λ and $\alpha_\lambda = 0$ on a neighbourhood of $I - I_\lambda$ in I;

(iii) The functions $\beta_\lambda = \alpha_\lambda / \sum \alpha_\nu$ form a C^∞ partition of unity on I, subordinate to the covering $\{I_\lambda\}$.

7.14. Let f be holomorphic on $D = D_1 \cap D_2$ in $\mathbf{C}$. Which $\overline{\partial}$ problem do you have to solve in order to represent f in the form $f_1 + f_2$ with $f_j \in \mathcal{O}(D_j)$?

7.15. Let $\Omega \subset \mathbf{R}^n$ be open and let $\{U_\lambda\},\ \lambda \in \Lambda$ be an arbitrary covering of Ω by open subsets. Construct a C^∞ partition of unity $\{\beta_\lambda\},\ \lambda \in \Lambda$ on Ω subordinate to the covering $\{U_\lambda\}$. [Start out with a special refinement $\{V_j\}$ as in 7.33 and an associated partition of unity $\{\alpha_j\}$. Try to define β_λ in terms of functions α_j.]

7.16. Determine a smooth solution of the Cousin-I problem in Example 7.17 with the aid of the pseudo-partition of unity

$$\beta_j = z_j \overline{z}_j / |z|^2, \quad j = 1, 2$$

associated with the covering $\{U_j\}$.

7.17. Determine a smooth form $v = v_1 d\overline{z}_1 + v_2 d\overline{z}_2$ on $\Omega = \mathbf{C}^2 - \{0\}$, with $\partial v_2 / \partial \overline{z}_1 = \partial v_1 / \partial \overline{z}_2$, for which the equation $\overline{\partial} u = v$ can not be solvable on Ω.

7.18. Let $\{U_\lambda, h_{\lambda\mu}\}$ be an arbitrary family of Cousin-I data on Ω of class C^p $(1 \le p \le \infty)$, that is, the functions $h_{\lambda\mu}$ are of class C^p and they satisfy the compatibility conditions ($1e$). Prove that the corresponding Cousin problem is C^p solvable.

7.19. Prove that $\Omega \subset \mathbf{C}^n$ is a Cousin-I domain if and only if for some p $(1 \le p \le \infty)$, the equation $\overline{\partial} u = v$ is C^p solvable on Ω for every $(0,1)$-form v of class C^p that satisfies the integrability conditions.

7.20. Let $\{U_\lambda, h_{\lambda\mu}\}$ be an arbitrary family of holomorphic Cousin-I data on Ω. Use a special refinement $\{V_j\}$ of the covering $\{U_\lambda\}$ with associated C^∞ partition of unity $\{\beta_j\}$ and with refinement map σ to prove the following. Every C^∞ solution of the Cousin-I problem with the original data can be represented in the form

$$h_\lambda = u + g_\lambda = u + \sum_j \beta_j h_{\sigma(j)\lambda} \quad \text{on} \quad U_\lambda, \qquad u \in C^\infty(\Omega),$$

and every family of functions $\{h_\lambda\}$ of this form is a C^∞ solution.

7.21. Formulate and prove a generalization of Proposition (7.55) to products of arbitrary planar domains.

7.22. Formulate and prove a generalization of Proposition (7.55) to products of arbitrary planar domains. [You may have to use Runge's theorem on rational approximation, which states that a holomorphic function defined in a neighborhood of a compact set K can be uniformly approximated with rational functions with poles atmost in one point of each component of the complement of K in the extended complex plane.]

(7.23) Using exercise 7.10, show that $\mathbf{C}^n \setminus \mathbf{C}^m$ is a Cousin-I domain if $n - m \ge 3$.

(7.24) (*Hefer's lemma*) Let $\Omega \subset \mathbf{C}^n$ be a domain of holomorphy, so that Ω and the intersections of Ω with affine complex subspaces are $\overline{\partial}$ domains (Chapter 11) and hence Cousin-I domains. Suppose that Ω meets the subspace $W_k = \{z_1 = z_2 = \ldots = z_k = 0\}$ of $\mathbf{C}^n$ and that $f \in \mathcal{O}(\Omega)$ vanishes on $\Omega \cap W_k$. Prove that there are holomorphic functions g_j on Ω such that

$$f(z) = \sum_1^k z_j g_j(z).$$

[Use induction on k.]

7.25. (*Hefer's theorem*) Let $\Omega \subset \mathbf{C}^n$ be a domain of holomorphy and let F be holomorphic on Ω. Prove that there are holomorphic functions $P_j(z,w)$ on $\Omega \times \Omega$ such that

$$F(z) - F(w) = \sum_1^n (z_j - w_j) P_j(z,w), \quad \forall z,\ w \in \Omega.$$

[Use $z_j - w_j = \zeta_j$ and $z_j = \zeta_{n+j},\ j = 1, \ldots, n$ as new coordinates on $\Omega \times \Omega$.]

7.26. Let Ω be a domain in $\mathbf{C}^2$ for which every meromorphic first Cousin problem is solvable. Prove that in this case, also every holomorphic Cousin-I problem on Ω is solvable. [If a more direct approach does not work, one can always use the general solvability of $\overline{\partial}$ on a pseudoconvex domain which is established in Chapter 11. It does not seem to be known if every meromorphic Cousin-I domain $\Omega \subset \mathbf{C}^n,\ n \geq 3$ is a (holomorphic) Cousin-I domain.]

7.27. (*Another characterization of domains of holomorphy*) Anticipating the general solvability of (first order) $\overline{\partial}$ on plurisubharmonically exhaustible domains (Chapter 11), one is asked to prove the following result:

> "$\Omega \subset \mathbf{C}^n$ is a domain of holomorphy if and only if for every complex line L that meets Ω and for every holomorphic function h_1 on $\Omega_1 = \Omega \cap L$, there is a holomorphic extension of h_1 to Ω".

CHAPTER 8

Subharmonic functions, plurisubharmonic functions and related aspects of potential theory

Subharmonic functions on a domain Ω in $\mathbf{C}$ or $\mathbf{R}^n$ are characterized by the local sub mean value property. Their name comes from the fact that they are majorized by harmonic functions with the same boundary values on subdomains of Ω.

Subharmonic functions in $\mathbf{C}$ play an important role in estimating the growth of holomorphic functions. The reason is that for holomorphic f, the functions $v = \log|f|$ is subharmonic. In the case of holomorphic f in $\mathbf{C}^n$, $\log|f|$ is even more special, namely, plurisubharmonic. In this chapter we will study subharmonic and plurisubharmonic (psh) functions in some detail. Because it serves as a model, the special case of $\mathbf{C}$ will receive a good deal of attention. Readers who are familiar with this case may wish to skip part of Sections 8.1–8.3.

Many properties of subharmonic and plurisubharmonic functions can be derived by means of approximation by smooth functions of the same class. Smooth subharmonic functions are characterized by nonnegative Laplacian and this property makes them easier to investigate. There is a related characterization of smooth psh functions. For arbitrary subharmonic and psh functions the desired C^∞ approximants are obtained by convolution with suitable approximate identities. The results on psh functions are used in Chapter 9 to construct smooth psh exhaustion functions of rapid growth on pseudoconvex domains. Such functions are essential for the solution of the $\bar\partial$ problem in Chapter 11.

Classical potential theory in $\mathbf{R}^n$ involves subharmonic functions [as well as their negatives, the superharmonic functions]. For applications to holomorphic functions in $\mathbf{C}^n$ one needs special $\mathbf{C}^n$ potential theory which involves plurisubharmonic functions. We will study aspects of that recent theory and discuss some applications, among them the useful lemma on the estimation of partial derivatives in terms of directional derivatives of the same order.

8.1 Harmonic and subharmonic functions. For these functions the theory is much the same in all spaces $\mathbf{R}^n \quad (n \geq 2)$. However, we will play special attention to the case $n = 2$. The theory is simpler there, due to the close relation between harmonic functions in $\mathbf{R}^2$ and holomorphic functions in $\mathbf{C}$. Moreover, the theory of plurisubharmonic functions in $\mathbf{C}^n \quad (n \geq 2)$ is in many ways closer to the theory of subharmonic functions in $\mathbf{C}$ than to the theory of such functions in $\mathbf{R}^{2n}$.

Accordingly, let Ω be an open set in $\mathbf{R}^2$ or $\mathbf{C}$. A function u on Ω is called *harmonic* if it is real valued of class C^2 and its Laplacian is identically zero:

$$\begin{aligned}\Delta u \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{r}\frac{\partial}{\partial r}\big(r\,\frac{\partial u}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= 4\,\frac{\partial^2 u}{\partial z \partial \bar z} = 0 \quad \text{on} \quad \Omega \qquad [x + iy = z = re^{i\theta}].\end{aligned}$$

[In the case of $\mathbf{R}^n$ one will use the n-dimensional Laplacian.] A function is called harmonic on an arbitrary set E if it has a harmonic extension to some open set containing E. Unless the contrary is explicitly stated, our harmonic functions will be *real-valued*.

For holomorphic f on $\Omega \subset \mathbf{C}$ both $u = \mathrm{Re}\, f$ and $v = \mathrm{Im} f$ are harmonic: the Cauchy-Riemann condition $\partial f/\partial \overline{z} = 0$ implies that $\Delta u + i\Delta v = \Delta f = 0$.

Conversely, let u be any harmonic function on Ω. Then u is locally the real part of a holomorphic function f. Indeed, by Laplace's equation, the derivative $\partial u/\partial z$ will be holomorphic. [It is of class C^1 and has $\partial/\partial \overline{z}$ equal to zero.] Suppose for a moment that $u = \mathrm{Re}\, f = \frac{1}{2}(f + \overline{f})$ for some holomorphic f. Then $\partial u/\partial z$ must equal $\frac{1}{2}\partial f/\partial z + \frac{1}{2}\partial \overline{f}/\partial z = \frac{1}{2}f'$ [since $\partial \overline{f}/\partial z = 0$], hence $\frac{1}{2}f$ must be a primitive of $\partial u/\partial z$. Starting then with our harmonic u, let $\frac{1}{2}g$ be any holomorphic primitive of $\partial u/\partial z$ on some disc B in Ω and set $u - \frac{1}{2}(g + \overline{g}) = v$. Then $\partial v/\partial z = \partial u/\partial z - \frac{1}{2}\partial g/\partial z = 0$, hence since v is real, $\partial v/\partial x = \partial v/\partial y = 0$. Thus v is equal to a real constant c and $u = \mathrm{Re}\,(g + c)$ on B.

As a corollary, the *composition* $u \circ h$ of a harmonic function u and a holomorphic function h is harmonic on any domain where it is well-defined.

Holomorphic functions f on Ω have the circular mean value property: by Cauchy's formula,

$$f(a) = \frac{1}{2\pi i} \int\limits_{C(a,r)} \frac{f(\zeta)}{\zeta - a}\, d\zeta = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(a + re^{it})dt$$

whenever the closed disc $\overline{B}(a, r)$ belongs to Ω. It follows that *harmonic functions u on Ω have the same mean value property*: representing u as $\mathrm{Re}\, f$ on discs, with f holomorphic, we find

$$(1a) \qquad u(a) = \overline{u}(a; r) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u(a + re^{it})dt, \qquad 0 \le r < d(a) = d(a, \partial\Omega).$$

One may use the analytic automorphisms of the unit disc and the mean value property at 0 to derive the *Poisson integral representation* for harmonic functions u on $\overline{B}(0, 1)$:

$$(1b) \qquad \begin{aligned} u(z) = P[u|_C](z) &= \frac{1}{2\pi} \int\limits_{C(0,1)} \frac{1 - |z|^2}{|\zeta - z|^2}\, u(\zeta)ds(\zeta) \\ &= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\, u(e^{it})dt, \quad z = re^{i\theta}, \quad 0 \le r < 1, \end{aligned}$$

cf. exercise 8.2. There is a corresponding Poisson integral formula for harmonic functions u on the unit ball $\overline{B} = \overline{B}(0, 1) \subset R^n \quad (n \ge 3)$:

$$(1b') \qquad u(x) = P[u|_S](x) \stackrel{\text{def}}{=} \frac{1}{\sigma_n} \int\limits_{S(0,1)} \frac{1 - |x|^2}{|\xi - x|^n}\, u(\xi)ds(\xi), \qquad S = \partial B,$$

$\sigma_n = \text{area } S(0,1) = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$, cf. exercises 8.3 and 8.51.

For any continuous function g on ∂B, the Poisson integral $u = P[g]$ provides a harmonic function on B with boundary function g: it solves the *Dirichlet problem* for the Laplace operator on B, cf. exercise 8.3. In general a Dirichlet problem for a partial differential operator L of order 2 on a domain D is to find for given functions u on D and g on the boundary of D a function F that satisfies

$$LF(x) = u(x), \quad x \in D, \qquad F|_{\partial D} = g.$$

Subharmonic functions v on $\Omega \subset \mathbf{C}$ are always real-valued; in addition, the value $-\infty$ is allowed (not $+\infty$). The essential requirement is that v have the *sub mean value property* [cf. Definition 6.52]: for every point $a \in \Omega$ there must exist some number $\delta(a) > 0$ such that

$$(1c) \qquad v(a) \le \overline{v}(a;r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(a + re^{it})dt, \qquad 0 \le r < \delta(a).$$

The local inequality (1c) on Ω will imply that $\overline{v}(a;r)$ is a nondecreasing function of r, see Corollaries 8.23. Thus in the final analysis, inequality (1c) will hold for *all* r such that $\overline{B}(a,r) \subset \Omega$. However, it is advantageous not to demand $\delta(a) = d(a)$ from the beginning.

To ensure the existence of the mean values $\overline{v}(a;r)$ one requires that subharmonic functions satisfy an appropriate continuity condition. In many applications we will have ordinary continuity, but in some situations one can not expect more than upper semi-continuity, cf. exercise 8.5. A function v on E in $\mathbf{C}$ [or $\mathbf{R}^n$] to $\mathbf{R} \cup \{-\infty\}$ is called *upper semi-continuous* (usc) if, for every point $a \in E$,

$$(1d) \qquad \limsup_{z \in E,\, z \to a} v(z) \le v(a).$$

In other words, whenever $A > v(a)$, then $A > v(z)$ on some neighborhood of a in E. There is an equivalent condition which is very useful in applications and perhaps easier to remember: A function v on E is upper semi-continuous if and only if, on every compact subset, it can be represented as the limit of a *decreasing* sequence of finite continuous functions $\{v_k\}$, cf. exercises 8.6, 8.7.

Similarly, a function v on E to $\mathbf{R} \cap \{\infty\}$ is called *lower semi-continuous* (lsc) if $-v$ is usc, or equivalently

$$(1d') \qquad \liminf_{z \in E,\, z \to a} v(z) \ge v(a).$$

One also has a description in terms of limit of an increasing sequence of finite continuous functions.

We will most often meet usc functions. Let v be usc. In terms of a sequence of continuous $\{v_k \downarrow v\}$ on the circle $C(a,r)$ in Ω, the mean value $\overline{v}(a;r)$ may be defined unambiguously as $\lim \overline{v}_k(a;r)$ [monotone convergence theorem]. It may happen that $\overline{v}(a;r) = -\infty$, but for a subharmonic function v on a connected domain Ω containing $\overline{B}(a,r)$, this will occur only if $v \equiv -\infty$, cf. Corollaries 8.23.

DEFINITION 8.11. *Subharmonic functions* on Ω in $\mathbf{C}$ or $\mathbf{R}^2$ are upper semi-continuous functions $v : \Omega \to \mathbf{R} \cup \{-\infty\}$ which have the sub mean value property: inequality (1c) must hold at every point $a \in \Omega$ for some $\delta(a) > 0$. There is a corresponding definition for the case of $\mathbf{R}^n$, with $\overline{v}(a;r)$ denoting the mean value of v over the sphere $S(a,r)$. We say that v is subharmonic on an arbitrary set E in $\mathbf{R}^n$ if v has a subharmonic extension to some open set containing E.

One easily deduces the following simple

PROPERTIES 8.12. For subharmonic functions v_1 and v_2 on Ω, the *sum* $v_1 + v_2$ and the *supremum* or least common majorant,

$$v(z) \stackrel{\text{def}}{=} \sup\{v_1(z), v_2(z)\}, \qquad z \in \Omega$$

are also subharmonic; in the latter case,

$$v_j(a) \le \overline{v}_j(a;r) \le \overline{v}(a;r), \quad j = 1,2 \Rightarrow v(a) \le \overline{v}(a;r).$$

In order to obtain extensions to infinite families of subharmonic functions on Ω, one has to know already that $\delta(a)$ in (1c) can be taken the same for all members of the family, cf. Corollaries 8.23. Assuming that much, it follows that the supremum or *upper envelope* of an infinite family of subharmonic functions on Ω [when $< +\infty$] also has the sub mean value property, hence it is subharmonic provided it is upper semi-continuous. The limit function of a *decreasing family* of subharmonic functions on Ω is always subharmonic: such a function is automatically usc. [But the infimum of an arbitrary family of subharmonic functions need not be subharmonic, cf. exercise 8.10!] We finally observe the following. If v is subharmonic on Ω then by upper semi-continuity (1d) and the sub mean value property (1c),

$$v(a) = \lim_{r \downarrow o} \overline{v}(a;r) = \limsup_{z \to a} v(z), \qquad \forall a \in \Omega. \tag{1e}$$

EXAMPLES 8.13. For holomorphic f on $\Omega \subset \mathbf{C}$ the functions $|f|$ and $\log|f|$ are subharmonic. For $|f|$ this follows immediately from the mean value property of f:

$$|f(a)| = \frac{1}{2\pi} \Big| \int_{-\pi}^{\pi} f(a + re^{it}) dt \Big| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{it})| dt.$$

For $\log|f|$ one distinguishes the cases $f(a) = 0$ [nothing to prove] and $f(a) \ne 0$ [then there is a holomorphic branch of $\log f$ around a, so that $\log|f|$ is harmonic around a]. In problems where one has to estimate the growth of $|f|$, it is usually best to work with $\log|f|$. An important subharmonic function on $\mathbf{C}$ is

$$v(z) = \log|z - a|.$$

Harmonic functions u on a connected domain $D \subset \mathbf{C}$ which depend only on $x = \operatorname{Re} z$ are linear in x, that is, of the form $u = c_1 x + c_2$. Subharmonic functions v which depend *only on* x will be sublinear on line segments or *convex*, cf. Example 8.35. Convex functions v are always subharmonic: the linear mean value inequality (6.5a) for all small complex ξ or ζ implies the circular mean value inequality (6.5b).

The negative of a subharmonic function is called *superharmonic*. Example: the logarithmic potential $\log 1/|z-a|$ on $\mathbf{C}$ of a unit mass at a. More generally, it can be shown that all *logarithmic potentials*

$$(1f) \qquad U^\mu(z) \stackrel{\text{def}}{=} \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta), \qquad z \in \mathbf{C}$$

with K compact, μ a finite positive measure, are superharmonic, cf. exercises 8.12, 8.22. Such potentials are harmonic on the complement of K. On K itself they need not be continuous (cf. exercise 8.5), even if μ is absolutely continuous so that $d\mu(\zeta) = \varphi(\zeta) d\xi d\eta$ with integrable density φ. [But for smooth φ, cf. Examples 8.33.]

8.2 Maximum principle and consequences. Although we restrict ourselves to $\mathbf{C}$ here, the main results and the proofs readily extend to $\mathbf{R}^n$. The following maximum principle is characteristic for subharmonic functions, cf. also exercise 8.13.

Theorem 8.21. *Let $D \subset \mathbf{C}$ be a bounded or unbounded connected domain, let v be subharmonic on D and u harmonic. Suppose that v is "majorized by u on the extended boundary of D". The precise meaning of this hypothesis is that*

$$\limsup_{z \to \zeta, z \in D} \{v(z) - u(z)\} \le 0, \qquad \forall \zeta \quad \text{in} \quad \partial_e D,$$

where $\partial_e D$ is the boundary of D in $\mathbf{C}_e = \mathbf{C} \cup \{\infty\}$. [Thus $\partial_e D$ includes the point at ∞ if D is unbounded.] Then v is majorized by u throughout D:

$$v(z) \le u(z), \qquad \forall z \in D.$$

PROOF. Since $v - u$ will be subharmonic we may as well replace $v - u$ by v or equivalently, set $u \equiv 0$. Put $M = \sup_D v \quad (\le +\infty)$; we have to prove that $M \le 0$.

Suppose on the contrary that $M > 0$. There will be a sequence of points $\{z_k\} \subset D$ such that $v(z_k) \to M$; taking a subsequence, we may assume that $z_k \to a$ in $\operatorname{clos}_e D$, the closure of D in $\mathbf{C}_e$. Because of the boundary condition $\limsup v(z) \le 0$ for $z \to \zeta \in \partial_e D$ and by the assumption $M > 0$, our point a must be inside D. Hence by upper semi-continuity (1d),

$$M = \lim v(z_k) \le \limsup_{z \to a} v(z) \le v(a) < +\infty;$$

since $M \ge v(a)$ we must have $M = v(a)$. Thus by the sub mean value property (1c),

$$(2a) \qquad \int_{-\pi}^{\pi} \{v(a + re^{it}) - M\} dt \ge 0 \quad \text{whenever} \quad 0 \le r < \delta(a).$$

Here the integrand is non-positive; being upper semi-continuous, it must vanish everywhere on $[-\pi, \pi]$. Indeed, if it would be negative at some point $t = c$, it would be negative on an interval around c (1d), contradicting (2a). Hence $v(z) = M$ on $C(a, r)$ and thus, varying r, $v(z) = M$ throughout the disc $B(a, \delta(a))$.

Let E be the subset of D where $v(z) = M$. Under the assumption $M > 0$ the set E is nonempty and open. By upper semi-continuity it will also be closed in D, hence $E = D$ so that $v \equiv M$. The boundary condition now shows that $M > 0$ is impossible, so that $v \leq 0$ everywhere on D.

In the general case we conclude that $v \leq u$ throughout D. If $v(a) = u(a)$ at some point $a \in D$, the proof shows that $v \equiv u$.

APPLICATION 8.22. (*Comparison with a Poisson integral*). Let v be subharmonic on (a neighborhood of) the closed unit disc $\overline{B}(0,1)$ in $\mathbf{C}$. Then v is majorized on $B = B(0,1)$ by the Poisson integral $u = P[v] \stackrel{\text{def}}{=} P[v|_C]$ of its boundary values on $C(0,1)$:

$$v(re^{i\theta}) \leq u(re^{i\theta}) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} v(e^{it}) dt, \quad 0 \leq r < 1.$$

For the verification one represents v as the limit of a decreasing sequence of finite continuous functions v_k on $\overline{B}$. The associated Poisson integrals $u_k = P[v_k]$ are harmonic functions on B with boundary functions $v_k \mid \partial B$: as $z \in B$ tends to $\zeta \in \partial B$, $u_k(z) \to v_k(\zeta)$. Thus by upper semi-continuity (1d),

$$\limsup_{z \to \zeta} \{v(z) - u_k(z)\} \leq v(\zeta) - v_k(\zeta) \leq 0, \qquad \forall \zeta \in \partial B.$$

Hence by the maximum principle, $v(z) \leq u_k(z)$ throughout B. Now for fixed $z \in B$, the Poisson integrals $u_k(z) = P[v_k](z)$ tend to the Poisson integral $u(z) = P[v](z)$ as $k \to \infty$ [monotone convergence theorem]. Conclusion: $v(z) \leq u(z)$ throughout B.

We will explore various consequences of Application 8.22. First of all, if v is integrable over $C(0,1)$, then $u = P[v]$ is harmonic and by the mean value property of u:

$$\text{(2}b\text{)} \qquad \overline{v}(0; r) \leq \overline{u}(0; r) = u(0) = \overline{v}(0; 1) \qquad 0 \leq r < 1.$$

What if v is not integrable over $C(0,1)$? Our subharmonic v is certainly bounded from above by some real constant M on $C(0,1)$, so that $v(e^{it}) - M \leq 0$ and consequently

$$\frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \{v(e^{it}) - M\} \leq \frac{1 - r}{1 + r} \{v(e^{it}) - M\}.$$

Applying 8.22 to $v - M$ instead of v, we thus obtain

$$v(re^{i\theta}) - M \leq \frac{1 - r}{1 + r} \frac{1}{2\pi} \int_{-\pi}^{\pi} \{v(e^{it}) - M\} dt = \frac{1 - r}{1 + r} \{\overline{v}(0; 1) - M\}.$$

Hence if $\overline{v}(0;1)$ happens to be $-\infty$, then $v \equiv -\infty$ on $B(0,1)$.

Simple transformations give corresponding results for other discs. In particular, if v is subharmonic on $\overline{B}(a,R)$ then

$$\overline{v}(a;r) \leq \overline{v}(a;R), \qquad 0 \leq r < R \tag{2b'}$$

and $\overline{v}(a;R) = -\infty$ implies that $v \equiv -\infty$ on $B(a,R)$. If a subharmonic function v on a connected domain D equals $-\infty$ on a subdomain D_0 then $v \equiv -\infty$. Indeed, let D_1 be the maximal subdomain of D containing D_0 on which $v = -\infty$. If D_1 would have a boundary point a in D, then $\overline{v}(a;r)$ would be $-\infty$ for some small $r > 0$ and hence $v = -\infty$ on a neighborhood of a. This contradiction shows that $D_1 = D$.

COROLLARIES 8.23. Let v be subharmonic on Ω. Then for $a \in \Omega$, the mean value $\overline{v}(a;r)$ is a nondecreasing function of r for $0 \leq r < d(a) = d(a, \partial\Omega)$, see (2b′). The mean value inequality (1c) thus holds for all such r. If D is a connected component of Ω, one either has

$$\overline{v}(a;r) > -\infty \quad \text{for all} \quad a \in D \quad \text{and} \quad 0 < r < d(a),$$

or

$$v \equiv -\infty \quad \text{on} \quad D.$$

Hence if $v \not\equiv -\infty$ on D, one has $v > -\infty$ on a dense subset and then it follows from the sub mean value property that v is locally integrable on D. [Choose any point $a \in D$ such that $v(a) > -\infty$ and then take any compact disc $\overline{B}(a,R) \subset D$. On that disc v is bounded above. On the other hand the integral of v over $\overline{B}(a,R)$ must be $> -\infty$ by (1c) for $0 \leq r \leq R$ and Fubini's theorem.]

We indicate some more applications.

(8.24) *A characterization of harmonic functions.* Any continuous (finite real) function v on Ω with the mean value property (1a) is harmonic [and hence of class C^∞], cf. Application 8.22 and exercise 8.14.

(8.25) *Uniqueness in the Dirichlet problem.* For a bounded domain D and a continuous real function g on ∂D, there is at most one harmonic function u on D with boundary function g: one for which $\lim u(z) = g(\zeta)$ whenever $z (\in D)$ tends to a point $\zeta \in \partial D$. [Apply the maximum principle to $\pm$ the difference of two solutions.]

In the case of unbounded domains one needs a condition at ∞ for uniqueness. For example, the Dirichlet problem for $D = \mathbf{C} - \overline{B}(0,1)$ and boundary function 0 on ∂D has the solutions $c \log|z|$. However, there is only one bounded solution [$u = 0$], and hence also just one solution which has the form $\log|z| + \mathcal{O}(1)$ as $|z| \to \infty$, [$u = \log|z|$]. Indeed, if v is subharmonic on D and

$$\limsup_{z \to \zeta} v(z) \leq 0, \qquad \forall \zeta \in \partial D, \qquad v(z) \leq M \quad \text{on} \quad D,$$

then the modified subharmonic function

$$v_\varepsilon(v) = v(z) - \varepsilon \log|z|, \qquad \varepsilon > 0$$

is majorized by 0 on all of $\partial_e D$. Thus by the maximum principle $v_\varepsilon(z) \le 0$ at every point $z \in D$ and hence, letting $\varepsilon \downarrow 0$, $v(z) \le 0$ throughout D.

There are various problems for which one needs special harmonic functions that behave like $\log|z|$ at ∞:

(8.26) *A bound for polynomials that are bounded by* 1 *on* $[-1,1]$. Let $p(z)$ run over all polynomials such that $|p(x)| \le 1$ on $[-1,1]$. Taking $\deg p = m \ge 1$, does there exist a good upper bound for $|p(z)|^{1/m}$ at the points z in $D = \mathbf{C} - [-1,1]$?

Observe that

$$v(z) \stackrel{\text{def}}{=} \frac{1}{m} \log|p(z)| \tag{2c}$$

is a subharmonic function on $\mathbf{C}$ which is majorized by 0 on $[-1,1]$ and by $\log|z| + \mathcal{O}(1)$ at ∞. Thus if we can find a harmonic function $g(z)$ on D with boundary values 0 on ∂D and such that $g(z) = \log|z| + \mathcal{O}(1)$ at ∞, comparison of $v(z)$ and $(1+\varepsilon)g(z)$ will result in the estimate

$$v(z) \le g(z), \qquad \forall z \in D. \tag{2c$'$}$$

An appropriate "Green function" g can be obtained from $\log|w|$ by $1-1$ holomorphic or conformal mapping of $D' : \{|w| > 1\}$ onto D in such a way that "∞ corresponds to ∞". A suitable map is $z = \frac{1}{2}(w + 1/w)$ [what happens to circles $|w| = r > 1$?]. The inverse map is given by $w = \varphi(z) = z + (z^2-1)^{\frac{1}{2}}$, where we need the holomorphic branch of the square root that behaves like z at ∞; it will give $\varphi(z)$ absolute value > 1 throughout D. [The other branch would give $\varphi(z)$ absolute value < 1 on D.] We may now set

$$g(z) = \log|w| = \log|z + (z^2-1)^{\frac{1}{2}}| \qquad \text{[so that } g(z) > 0 \text{ on } D]. \tag{2c$''$}$$

The example of the Chebyshev polynomials $T_m(z) = \cos mw$ where $\cos w = z$ will show that the upper bound provided by (2c–c$''$) is quite sharp: $|T_m(x)| \le 1$ and

$$\begin{aligned} T_m(z) &= \frac{1}{2}(e^{imw} + e^{-imw}) = \frac{1}{2}(\cos w + i\sin w)^m + \frac{1}{2}(\cos w - i\sin w)^m \\ &= \frac{1}{2}\{z + (z^2-1)^{\frac{1}{2}}\}^m + \frac{1}{2}\{z - (z^2-1)^{\frac{1}{2}}\}^m. \end{aligned}$$

(8.27) *Sets on which a subharmonic function can be* $-\infty$. Let $D \subset \mathbf{C}$ be a connected domain. A subset $E \subset D$ throughout which a locally integrable subharmonic function on D can be equal to $-\infty$ is called a *polar subset.* Polar sets must have planar Lebesgue measure zero, but not every set of measure 0 is polar. For example, line segments I in D are non polar.

Indeed, let v be subharmonic on D and equal to $-\infty$ on I. By coordinate transformation $z' = az + b$ we may assume that $I = [-1,1]$ and by shrinking D if necessary, we may assume that D is bounded and $v \le 0$ throughout D. Let g be the function (2c$''$) and set

$\min_{\partial D} g = c$ so that $c > 0$. For any $\lambda > 0$ the subharmonic function $v - \lambda g$ on $D - [-1, 1]$ has all its boundary values $\le -\lambda c$, hence

$$v(z) \le \lambda\{g(z) - c\}, \qquad \forall z \in D, \quad \forall \lambda > 0.$$

There will be a disc $\overline{B}(0, \delta) \subset D$ throughout which $g(z) < c$. Letting $\lambda \to \infty$ it follows that $v = -\infty$ throughout $B(0, \delta)$, hence v is not integrable over that disc. [In fact, $v \equiv -\infty$, cf. Corollaries 8.23.]

In planar potential theory one introduces the notion of logarithmic *capacity* (cap) to measure appropriate kinds of sets [Section 8.5]. A compact set $K \subset \mathbf{C}$ will be polar relative to $D \supset K$ precisely when $\operatorname{cap} K = 0$, cf. exercise 8.45. For a closed disc and a circle the capacity is equal to the radius.

8.3 Smooth subharmonic functions. Regularization. A real C^2 function g on $\mathbf{R}$ is convex if and only if $g'' \ge 0$. There is a similar characterization for smooth subharmonic functions:

Proposition 8.31. *A (finite) real C^2) function v on Ω in $\mathbf{R}^2$ [or $\mathbf{R}^n$] is subharmonic if and only if its Laplacian Δv is nonnegative throughout Ω.*

PROOF. The simplest way to estimate the deviation of the circular mean $\overline{v}(a; r)$ from $v(a)$ is by integration of the Taylor expansion for v around a. Taking $a = 0$ one has for $(x, y) \to 0$:

$$v(x, y) = v(0) + v_x(0)x + v_y(0)y + \tfrac{1}{2}v_{xx}(0)x^2 + v_{xy}(0)xy + \tfrac{1}{2}v_{yy}(0)y^2 + o(x^2 + y^2).$$

Setting $x = r\cos\theta,\ y = r\sin\theta$ $(r > 0)$ and integrating with respect to θ from $-\pi$ to π, one obtains the formula

$$\overline{v}(0; r) - v(0) = \tfrac{1}{4}\Delta v(0) r^2 + o(r^2) \qquad \text{for} \qquad r \downarrow 0. \tag{3a}$$

Hence if v is subharmonic on a neighborhood of 0, so that $\overline{v}(0; r) \ge v(0)$ for all small r, it follows that $\Delta v(0) \ge 0$. As to the other direction, if $\Delta v(0) > 0$ one finds that $\overline{v}(0; r) > v(0)$ for all sufficiently small r. If one only knows that $\Delta v \ge 0$ on a neighborhood U of 0, one may first consider $v_\varepsilon = v + \varepsilon(x^2 + y^2)$ with $\varepsilon > 0$. Then $\Delta v_\varepsilon \ge 4\varepsilon$, hence the functions v_ε are subharmonic on U. The same will hold for the limit function v of the decreasing family $\{v_\varepsilon\}$ as $\varepsilon \downarrow 0$, cf. Properties 8.12.

REMARKS. An alternative proof may be based on the exact formula

$$v(0) = \overline{v}(0; r) - \frac{1}{2\pi} \int_{B(0,r)} \Delta v(\xi, \eta) \log\frac{r}{\rho}\, d\xi d\eta, \qquad \rho = (\xi^2 + \eta^2)^{\frac{1}{2}}, \tag{3b}$$

cf. exercise 8.20. There are similar proofs for $\mathbf{R}^n$.

In Section 3.1 we have derived a representation formula for smooth functions in terms of boundary values and a (special) first order derivative. Formula (3b) is a particular case of a *general representation formula* for smooth functions in terms of boundary values and the Laplacian, cf. exercise 8.49. Such representations may be obtained with the aid of *Green's formula involving Laplacians*:

$$\int\limits_D (u\Delta v - v\Delta u)dm = \int\limits_{\partial D} (u\,\frac{\partial v}{\partial N} - v\,\frac{\partial u}{\partial N})ds. \tag{3c}$$

Here D is a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^n$ with piecewise smooth boundary, while u and v are functions of class $C^2(\overline{D})$. The symbol dm denotes the "volume" element of D, ds the "area" element of ∂D and $\partial/\partial N$ stands for the derivative in the direction of the outward normal to ∂D [or minus the derivative in the direction of the inward normal]. Formula (3c) may be derived from the Gauss-Green formula for integration by parts, cf. Section 3.1 and exercises 8.46, 8.47 and also Chapter 10.

Arbitrary subharmonic functions v may be characterized by the condition that Δv must be ≥ 0 in the sense of distributions, cf. exercise 8.27.

DEFINITION 8.32. Real C^2 functions v such that $\Delta v > 0$ on Ω (or $v_{z\overline{z}} > 0$ in the case of $\mathbf{C}$) are called *strictly subharmonic.*

EXAMPLES 8.33. Let α be a C^∞ subharmonic function on $\Omega \subset \mathbf{C}$ and let g be a nondecreasing convex C^∞ function on $\mathbf{R}$, or at least on the range of α. Then the composition $\beta = g \circ \alpha$ is also C^∞ subharmonic on Ω. Indeed, $\beta_z = g'(\alpha)\alpha_z$, hence since $\alpha_{\overline{z}} = \overline{\alpha}_z$ [α is real!],

$$\beta_{z\overline{z}} = g''(\alpha)|\alpha_z|^2 + g'(\alpha)\alpha_{z\overline{z}} \geq g'(\alpha)\alpha_{z\overline{z}}. \tag{3d}$$

The function $v(z) = |z|^2$ is strictly subharmonic on $\mathbf{C}$.

Finally, let μ be an absolutely continuous measure on $\mathbf{C}$ with a C^1 density φ on $\mathbf{C}$ of compact support K. Then the potential U^μ in (1f) is of class C^2 and it satisfies Poisson's equation $\Delta U = -2\pi\varphi$, cf. exercise 8.21. If $\varphi \geq 0$, U^μ will be a smooth superharmonic function on $\mathbf{C}$.

Certain properties are easy to obtain for smooth subharmonic functions. For arbitrary subharmonic functions v one may then form so-called regularizations v_ε and try passage to the limit. The regularizations are smooth subharmonic majorants which tend to v as $\varepsilon \downarrow 0$:

Theorem 8.34. *Let v be subharmonic on $\Omega \subset \mathbf{C}$ and not identically $-\infty$ on any component, so that v is locally integrable [Corollaries 8.23.] We let Ω^ε denote the "ε-contraction" of Ω : $\Omega^\varepsilon = \{z \in \Omega : d(z) > \varepsilon\}$, $\varepsilon > 0$. Finally, let $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(|z|/\varepsilon)$ be the standard nonnegative C^∞ approximate identity on $\mathbf{C}$ with circular symmetry; in particular* $\mathrm{supp}\rho_\varepsilon = \overline{B}(0,\varepsilon)$ *and* $\int_{\mathbf{C}} \rho_\varepsilon = 1$ *[Section 3.3]. Then on Ω^ε, the regularization*

$$v_\varepsilon(z) \stackrel{\text{def}}{=} \int\limits_\Omega v(\zeta)\rho_\varepsilon(z-\zeta)dm(\zeta) = \int\limits_{B(0,\varepsilon)} v(z-\zeta)\rho_\varepsilon(\zeta)dm(\zeta)$$

of v is well-defined, of class C^∞, subharmonic and $\geq v$. At each point $z \in \Omega$, the values $v_\varepsilon(z)$ converge monotonically to $v(z)$ as $\varepsilon \downarrow 0$. If v is a finite continuous function, the convergence is uniform on every compact subset of Ω.

PROOF. Since v is locally integrable on Ω, it is clear that the regularization v_ε is well-defined and of class C^∞, cf. Sections 3.3 and 3.1. We may derive mean value inequalities for v_ε from those for v by inverting the order of integration in an appropriate repeated integral. Indeed, for $a \in \Omega^\varepsilon$ and $0 < r < d(a) - \varepsilon$,

$$\begin{aligned}\int_{-\pi}^{\pi} v_\varepsilon(a + re^{it})dt &= \int_{B(0,\varepsilon)} \rho_\varepsilon(\zeta)dm(\zeta) \int_{-\pi}^{\pi} v(a - \zeta + re^{it})dt \\ &\geq \int_{B(0,\varepsilon)} \rho_\varepsilon(\zeta) \cdot 2\pi v(a-\zeta)dm(\zeta) = 2\pi v_\varepsilon(a),\end{aligned}$$

hence v_ε is subharmonic on Ω^ε.

How does $v_\varepsilon(z)$ behave as $\varepsilon \downarrow 0$? This time we will use r and t as polar coordinates, $\zeta = re^{it}$. From the special form of ρ_ε and noting that $\rho(\zeta) = \rho(r)$, we obtain for $\varepsilon < d(z)$:

$$\begin{aligned}v_\varepsilon(z) &= \int_{B(0,\varepsilon)} v(z - \varepsilon\zeta)\rho(\zeta)d\xi d\eta \\ &= \int_0^1 \rho(r)r\,dr \int_{-\pi}^{\pi} v(z - \varepsilon\, re^{it})dt = 2\pi \int_0^1 \rho(r)r\,\overline{v}(z; \varepsilon r)dr.\end{aligned} \tag{3e}$$

Now the mean value $\overline{v}(z; \varepsilon r)$ is monotonically decreasing as $\varepsilon \downarrow 0$ by Corollaries 8.23, hence the same will hold for $v_\varepsilon(z)$. Finally, since $\overline{v}(z; \varepsilon r) \to v(z)$ as $\varepsilon \downarrow 0$ (1e), formula (3e) and the monotone convergence theorem show that

$$v_\varepsilon(z) \downarrow 2\pi \int_0^1 \rho(r)rv(z)\,dr = v(z).$$

For finite continuous v the convergence above will be uniform on compact sets in Ω because in this case, $v(z - \zeta) \to v(z)$ uniformly on compact subsets of Ω as $\zeta \to 0$.

EXAMPLE 8.35. As an application one may show that subharmonic function $v(x, y) = f(x)$ on Ω in $\mathbf{R}^2$ which depends only on x is convex or sublinear on line segments. For smooth v the result is immediate from $\Delta v = f'' \geq 0$. In the general case one finds that the regularization v_ε [on Ω^ε] depends only on x and hence is convex; passage to the limit as $\varepsilon \downarrow 0$ gives the convexity of v. Similarly, if a function $v(z) = \varphi(|z|)$ on an annulus $A(0; \rho, R)$ depends only on $|z| = r$, then $\varphi(r)$ is a convex function of $\log r$. For this and other applications, see exercises 8.23–8.28.

8.4 Plurisubharmonic functions. We have seen already when a continuous function is plurisubharmonic [Section 6.5]. As in the case of subharmonic functions, the requirement of continuity may be relaxed:

DEFINITION 8.41. A *plurisubharmonic* (psh) function on an open set $\Omega \subset \mathbf{C}^n$ is an upper semi-continuous function $v : \Omega \to \mathbf{R} \cup \{-\infty\}$, whose restrictions to the intersections of Ω with complex lines are subharmonic. In other words, for every complex line $z = a + w\zeta$ $(a \in \Omega,\ \zeta \in \mathbf{C}^n - \{0\},\ w \in \mathbf{C}$ variable), the restriction $v(a + w\zeta)$ must have the sub mean value property at the point $w = 0$.

There is a corresponding notion of *pluriharmonic* functions on Ω: they are the real C^2 functions whose restrictions to the intersections with complex lines are harmonic.

EXAMPLES and PROPERTIES 8.42. For holomorphic f on $\Omega \subset \mathbf{C}^n$ both $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ are pluriharmonic, while $|f|$ and $\log |f|$ are plurisubharmonic. Indeed, for $a \in \Omega$, $f(a + w\zeta)$ will be holomorphic in w around $w = 0$. Every convex function v on $\Omega \subset \mathbf{C}^n$ is psh, cf. Examples 8.13.

For psh functions v_1 and v_2 on Ω, the sum $v_1 + v_2$ and the supremum or least common majorant $\sup(v_1, v_2)$ are also psh. The supremum or upper envelope of an infinite family of psh functions is psh provided it is upper semi-continuous. If the latter is not the case, then its *usc regularization* will be psh, cf. the section after the proof of Theorem 8.64, exercise 8.8 and 8.29. The limit function of a decreasing family of psh functions is always psh.

Psh functions v on $\Omega \subset \mathbf{C}^n$ are in particular *subharmonic* in the sense of $\mathbf{R}^{2n}$. Indeed, for $a \in \Omega$ one will have the inequality

$$v(a) \le \int_{-\pi}^{\pi} v(a + e^{it}\zeta)dt/2\pi,\ \forall \zeta \in \mathbf{C}^n \quad \text{with} \quad |\zeta| = r < d(a).$$

Now observe that the transformation $\zeta \to e^{it}\zeta$ (with t fixed) represents a rotation about 0 in $\mathbf{C}^n = \mathbf{R}^{2n}$. Letting ζ run over the sphere $S_r = S(0, r)$ and averaging, Fubini's theorem thus gives the mean value inequality

$$v(a) \le \int_{-\pi}^{\pi} \{ \int_{S_r} v(a + e^{it}\zeta) ds(\zeta)/m(S_r)\} dt/2\pi = \overline{v}(a; r),$$

where $\overline{v}(a; r)$ denotes the average of v over the sphere $S(a, r)$ [we may write $ds(\zeta) = ds(e^{it}\zeta)$]. It follows that psh functions satisfy a maximum principle [Theorem 8.21 for $\mathbf{R}^n$ instead of $\mathbf{C}$], that they are majorized on balls by the Poisson integrals $u = P[v]$ of their boundary values [cf. Applications 8.22 and (1b′)] and that the spherical means $\overline{v}(a; r)$ have the same properties as the circular means in Corollaries 8.23. In particular, if v is psh on $\overline{B}(a, R) \subset \mathbf{C}^n$ then $\overline{v}(a; r)$ is nondecreasing for $0 \le r \le R$ and its limit for $r \downarrow 0$ equals $v(a)$ as in (1e). Furthermore, if v is psh on a connected domain $D \subset \mathbf{C}^n$ and $\not\equiv -\infty$, then $\overline{v}(a; r)$ is finite for all $a \in D$, $0 < r < d(a)$ and v is locally integrable.

Smooth functions. For a C^2 convex function v on $\Omega \subset \mathbf{R}^n$, the restriction to the intersection with any real line $x = a + t\xi$ through $a \in \Omega$ is C^2 convex. Setting $v(a+t\xi) = g(t)$, the characterization $g'' \geq 0$ leads to the necessary and sufficient condition

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial x_j \partial x_k}(a)\xi_j\xi_k \geq 0, \quad \forall a \in \Omega, \quad \forall \xi \in \mathbf{R}^n. \tag{4a}$$

In words: the (real) Hessian matrix or form of v must be positive semidefinite everywhere on Ω.

The characterization 8.31 of smooth subharmonic functions leads to a similar characterization for smooth psh functions v on $\Omega \subset \mathbf{C}^n$. The important quantities now are the *complex Hessians*, that is, the Hermitian matrices

$$[\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}(a)]_{j,k=1,\ldots,n}, \qquad a \in \Omega$$

and the corresponding Hermitian forms, the *complex Hessian* or *Levi forms*

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}(a)\zeta_j\overline{\zeta}_k, \quad \zeta \in \mathbf{C}^n, \quad a \in \Omega. \tag{4a$'$}$$

Proposition 8.43. *A real C^2 function v on $\Omega \subset \mathbf{C}^n$ is plurisubharmonic if and only if its complex Hessian form (4a$'$) is positive semidefinite at every point $a \in \Omega$, or equivalently, if the smallest eigenvalue of the form,*

$$\lambda_v(a) = \min_{|\zeta|=1} \sum_{j,k=1}^{n} D_j\overline{D}_k v(a) \cdot \zeta_j\overline{\zeta}_k \quad \text{is} \quad \geq 0, \quad \forall a \in \Omega. \tag{4b}$$

PROOF. Consider the restriction of v to the intersection of Ω with the complex line $z = a + w\zeta$. This C^2 functions is subharmonic precisely when $\Delta_w v(a + w\zeta) \geq 0$ for all w such that $z = a + w\zeta \in \Omega$. The proof is completed by direct calculation: for $z_j = a_j + w\zeta_j$, $j = 1, \ldots, n$,

$$\frac{\partial v(z)}{\partial w} = \sum_j D_j v(z) \cdot \zeta_j, \quad \tfrac{1}{4}\Delta_w v(z) = \frac{\partial^2 v(z)}{\partial w \partial \overline{w}} = \sum_{j,k} D_j\overline{D}_k v(z) \cdot \zeta_j\overline{\zeta}_k.$$

DEFINITION 8.44. A real function v on Ω is called *strictly plurisubharmonic* if it is of class C^2 and its complex Hessian form is positive definite everywhere on Ω; equivalently, the smallest eigenvalue λ_v must be strictly positive throughout Ω.

EXAMPLES 8.45. Let α be a C^∞ psh function on $\Omega \subset \mathbf{C}^n$ and let g be a nondecreasing convex C^∞ function on $\mathbf{R}$, or at least on the range of α. Then the composition $\beta = g \circ \alpha$ is also C^∞ psh on Ω:

$$D_j\beta = g'(\alpha)D_j\alpha, \quad D_j\overline{D}_k\beta = g''(\alpha) \cdot \overline{D}_k\alpha + g'(\alpha)D_j\overline{D}_k\alpha,$$
$$\sum_{j,k} D_j\overline{D}_k\beta \cdot \zeta_j\overline{\zeta}_k = g''(\alpha)\sum_j \zeta_j D_j\alpha \sum_k \overline{\zeta}_k\overline{D}_k\alpha$$
$$+ g'(\alpha)\sum_{j,k} D_j\overline{D}_k\alpha \cdot \zeta_j\overline{\zeta}_k \geq 0.$$

We record for later use that for the smallest eigenvalues of α and β,

$$\lambda_\beta \geq g'(\alpha)\lambda_\alpha. \tag{4c}$$

The functions

$$|z|^2 - 1, \quad \frac{1}{1-|z|^2}, \quad \log\frac{1}{1-|z|^2}$$

are strictly psh on the ball $B(0,1) \subset \mathbf{C}^n$. Useful psh functions on $\mathbf{C}^n$ are

$$|z|^2, \quad \log|z| = \tfrac{1}{2}\log|z|^2, \quad \log^+|z-a| = \sup(\log|z-a|,\ 0).$$

As subharmonic functions in $\mathbf{R}^{2n}$, psh functions in $\mathbf{C}^n$ may be *regularized* as in Theorem 8.34. The regularizations will also be psh functions:

Theorem 8.46. *Let v be a locally integrable plurisubharmonic function on $\Omega \subset \mathbf{C}^n$ and let $\rho_\varepsilon(z) = \varepsilon^{-2n}\rho(|z|/\varepsilon)$ be the standard nonnegative C^∞ approximate identity on $\mathbf{C}^n$ with spherical symmetry* [Section 3.3]. *Then the regularization $v_\varepsilon = v * \rho_\varepsilon$ is well-defined on $\Omega^\varepsilon = \{z \in \Omega : d(z) > \varepsilon\}$, of class C^∞, psh and $\geq v$. At each point $z \in \Omega$, the values $v_\varepsilon(z)$ converge monotonically to $v(z)$ as $\varepsilon \downarrow 0$; if v is continuous, the convergence is uniform on compact sets in Ω.*

Sketch of PROOF [cf. the proof of Theorem 8.34]. We verify that v_ε is psh: for $a \in \Omega^\varepsilon$ and $\tau \in \mathbf{C}^n,\ 0 < |\tau| < d(a) - \varepsilon$,

$$\int_{-\pi}^{\pi} v_\varepsilon(a + e^{it}\tau)dt = \int_{B(0,\varepsilon)} \rho_\varepsilon(\zeta)dm(\zeta)\int_{-\pi}^{\pi} v(a - \zeta + e^{it}\tau)dt$$
$$\geq \int_{B(0,\varepsilon)} \rho_\varepsilon(\zeta) \cdot 2\pi v(a-\zeta)dm(\zeta) = 2\pi v_\varepsilon(a).$$

Furthermore, for$\varepsilon < d(z)$ and $S_r = \partial B(0,r)$,

$$v_\varepsilon(z) = \int_{B(0,\varepsilon)} v(z - \varepsilon\zeta)\rho(\zeta)dm(\zeta) = \int_0^1 \rho(r)dr\int_{S_r} v(z-\varepsilon\zeta)ds(\zeta)$$
$$= \int_0^1 \rho(r)m(S_r)\overline{v}(z;\varepsilon r)dr \downarrow v(z)\int_B \rho = v(z) \quad \text{as} \quad \varepsilon \downarrow 0.$$

APPLICATION 8.47 (Plurisubharmonic functions and holomorphic maps). Let f be a holomorphic map from a domain $D_1 \subset \mathbf{C}^n$ to a (connected) domain $D_2 \subset \mathbf{C}^p$ and let v be a psh function on D_2. Then the *pull back* V of v to D_1,

$$V = f^* v \stackrel{\text{def}}{=} v \circ f$$

is also psh. Indeed, for a C^2 psh function v the statement may be verified by direct computation of the complex Hessian, cf. Proposition 8.43. An arbitrary psh function $v \not\equiv -\infty$ on D_2 is locally integrable and hence the pointwise limit of a decreasing family of smooth psh functions v_ε as $\varepsilon \downarrow 0$. The pull back f^*v will be the limit of the decreasing family of psh functions f^*v_ε as $\varepsilon \downarrow 0$, hence also psh.

APPLICATION 8.48 *Sets on which a plurisubharmonic function can be* $-\infty$. Let D be a connected domain in $\mathbf{C}^n$, $n \geq 2$. A subset E throughout which a locally integrable psh function on D can be equal to $-\infty$ is called a *pluripolar subset.* In Newtonian potential theory for $\mathbf{R}^n$ or $\mathbf{R}^{2n}$ one works with ordinary subharmonic [or superharmonic] functions and the corresponding small sets are called polar. Whether a set in $\mathbf{C}^n$ is pluripolar or not depends very much on its orientation relative to the complex structure. Any subset of a zero set $Z(f)$, with $f \in \mathcal{O}(D)$ not identically zero, is pluripolar. Thus in $\mathbf{C}^2 \approx \mathbf{R}^4$, the square $-1 \leq x_1, y_1 \leq 1$ in the complex line $z_2 = 0$ is pluripolar, but the square $-1 \leq x_1, x_2 \leq 1$ in the "real" plane $y_1 = y_2 = 0$ is not, cf. exercises 1.16 and 8.39. The two sets are equivalent from the viewpoint of $\mathbf{R}^4$, hence both polar.

For compact sets $K \subset \mathbf{C}^n$ we will introduce a logarithmic capacity. It can be shown that such sets are pluripolar in $\mathbf{C}^n$ precisely when they have capacity zero (cf. [Siciak 1982]).

8.5 Capacities and Green functions: introduction. The mathematical notion of the capacity of a compact set K in $\mathbf{R}^3$ goes back to classical electrostatics and the Newtonian potential, cf. [Wermer 1974]. One would think of K as a conductor [preferably with smooth boundary] which carries a distribution of positive charge, represented by a positive measure μ on K. We suppose that there exists some nonzero distribution μ for which the associated electrostatic potential $\int_K d\mu(\xi)/|x-\xi|$ remains bounded on K [otherwise we say that K has capacity zero]. Question: How much charge can one put on K if the potential is not allowed to exceed a given constant V? The maximal charge $Q = \mu(K)$ is obtained in the case of an equilibrium charge distribution, for which the potential is equal to V (essentially) everywhere on K. The ratio Q/V turns out to be independent of V and gives the capacity. For a closed ball $\overline{B}(a, R)$ is a sphere $S(a, R)$ and in appropriate units, the capacity is equal to the radius.

In the case of arbitrary compact sets K in $\mathbf{C}$ or $\mathbf{R}^2$ one proceeds by analogy. The planar Laplace operator suggests that we now use the logarithmic potential U^μ of a positive measure μ on K (1f). For convenience one normalizes the total charge $\mu(K)$ to 1. One says that K has positive capacity if U^μ is bounded above on K for some μ. Varying μ, the smallest possible upper bound $\gamma = \gamma_K$ is called the *Robin constant* for K. It is attained for the so-called equilibrium distribution μ_0 on K. This measure is concentrated on the outer boundary $\partial_0 K$ and its potential is equal to γ essentially everywhere on K. [The exceptional set will be polar (8.27); it is empty if $\partial_0 K$ is well-behaved; in $\mathbf{C}$, a continuum K is all right.] The constant γ may be negative; for the disc $\overline{B}(a,R)$ or the circle $C(a,R)$ one finds $\gamma = -\log R$, cf. Examples (8.51). It is customary to define the so-called *logarithmic capacity*, $\operatorname{cap} K$, as $e^{-\gamma}$, so the closed discs and circles in the plane have capacity equal to their radius.

There is another way to obtain the Robin constant and thus the capacity for compact sets K in $\mathbf{C}$. Suppose for simplicity that K has well-behaved outer boundary. Then there exists a classical *Green function* on the unbounded component D of $\mathbf{C} - K$ with *"pole" at infinity.* It is the unique harmonic function $g(z)$ on D with boundary values 0 on ∂D and which is of the form $\log|z| + \mathcal{O}(1)$ as $|z| \to \infty$. In terms of the potential of the equilibrium distribution μ_0 on K one will have

$$g(z) = \gamma - U^{\mu_0}(z), \qquad \forall z \in D. \tag{5a}$$

Observe that

$$U^{\mu_0}(z) = -\log|z| - \int_K \log\left|1 - \frac{\zeta}{z}\right| d\mu_0(\zeta) = -\log|z| + o(1)$$

as $|z| \to \infty$, hence

$$\gamma = \min_{|z|\to\infty} \{g(z) - \log|z|\}. \tag{5b}$$

EXAMPLES 8.51. For $K = \overline{B}(a,R)$ and $K = C(a,R)$ in $\mathbf{C}$,

$$g(z) = \log\frac{|z-a|}{R}, \quad |z-a| > R; \quad \gamma = -\log R, \operatorname{cap} K = R.$$

For the line segment $[-1,1]$ in $\mathbf{C}$, cf. (8.26),

$$g(z) = \log|z + (z^2-1)^{\frac{1}{2}}|, \quad \gamma = \log 2, \operatorname{cap}[-1,1] = \frac{1}{2}.$$

Here one has to use the holomorphic branch of $(z^2-1)^{\frac{1}{2}}$ on $\mathbf{C} \setminus [-1,1]$ that behaves like z at ∞.

The Green function $g(z)$ on D may be *extended* to a subharmonic function on $\mathbf{C}$ by setting it equal to 0 on K and throughout bounded components of $\mathbf{C} \setminus K$ [formula (5a) will then hold everywhere]. The extended Green function may also be defined *in terms of polynomials.* The advantage of such an approach is that it provides a Green function for every compact set in $\mathbf{C}$. Using polynomials in $z = (z_1, \ldots, z_n)$, the same definition will work in $\mathbf{C}^n$. Its polynomial origin will make the new Green function directly useful in the study of holomorphic functions in $\mathbf{C}^n$ (cf. [Siciak 1962, 1982]).

8.6 Green functions on $\mathbf{C}^n$ with logarithmic singularity at infinity.

DEFINITION 8.61. For $K \subset \mathbf{C}^n$ compact, the (pre-) *Green function* $g_K(z)$ with "pole" (logarithmic singularity) at infinity is given by

$$g_K(z) = \sup_{m \geq 1} \sup_{\deg\, p_m \leq m} \frac{1}{m} \log \frac{|p_m(z)|}{\|p_m\|_k}, \quad \forall z \in \mathbf{C}^n.$$

Here p_m runs over all polynomials in z of degree $\leq m$ for which $\|p_m\|_K = \sup_K |p_m(\zeta)| > 0$. One defines the *logarithmic capacity* of K in terms of a generalized Robin constant:

$$\gamma = \gamma_K = \limsup_{|z| \to \infty} \{g_K(z) - \log|z|\}, \qquad \operatorname{cap} K = e^{-\gamma}.$$

Simple properties. There is *monotonicity*: if $K \subset K'$ one has $\|p_m\|_K \leq \|p_m\|_{K'}$, hence

$$g_K(z) \geq g_{K'}(z), \qquad \operatorname{cap} K \leq \operatorname{cap} K'.$$

Also cap is invariant under translations and there is homogeneity: $\operatorname{cap}(tK) = t\operatorname{cap}K$, $t \geq 0$. Observe that for any $m \geq 1$ and deg $p_m \leq m$, the function

$$v(z) = \frac{1}{m} \log|p_m(z)|/\|p_m\|_K \qquad (\|p_m\|_K > 0) \tag{6a}$$

is *plurisubharmonic* and satisfies the following conditions:

$$v(z) \leq 0 \quad \text{on} \quad K, \quad v(z) \leq \log|z| + \mathcal{O}(1) \quad \text{as} \quad |z| \to \infty. \tag{6b}$$

[For $|\alpha| \leq m$ and $|z| \geq 1$, $|z^\alpha| \leq |z|^m$.] Clearly $g_K(z) \leq 0$ on K; the special choice $p_m(z) \equiv 1$ (and $m = 1$) shows that

$$g_K(z) \geq 0 \quad \text{on} \quad \mathbf{C}^n, \quad g_K(z) = 0 \quad \text{on} \quad K. \tag{6c}$$

Before we discuss examples it is convenient to prove a simple lemma:

Lemma 8.62. *(i) Let v be any psh function on $\mathbf{C}^n$ which is majorized by* 0 *on the closed ball $\overline{B}(a,R)$ and by $\log|z| + \mathcal{O}(1)$ at ∞. Then*

$$v(z) \leq \log^+ \frac{|z-a|}{R}, \qquad \forall z \in \mathbf{C}^n.$$

(ii) Let $K \subset \mathbf{C}^n$ be such that $g_K(z) \leq M$ on the ball $\overline{B}(a,R)$. Then

$$g_K(z) \leq M + \log^+ \frac{|z-a|}{R}, \qquad \forall z \in \mathbf{C}^n,$$

hence K has finite Robin constant and positive logarithmic capacity.

PROOF. (i) Setting $z = a + wb$ with $w \in \mathbf{C}$, $b \in \mathbf{C}^n$, $|b| = R$ we find

$$v(a+wb) \leq 0 \quad \text{for} \quad |w| \leq 1, \; v(a+wb) \leq \log|w| + \mathcal{O}(1) \quad \text{as} \quad |w| \to \infty.$$

Comparing the subharmonic function $v(a+wb)$ with the harmonic function $(1+\varepsilon)\log|w|$ for $|w|>1$, the maximum principle will show that

$$v(z)=v(a+wb)\le\log|w|=\log|z-a|/R\quad\text{for}\quad|z-a|>R.$$

(ii) By the definition of g_K, the psh functions v of (6a) will be majorized by M on $\overline{B}(a,R)$. Now apply part (i) to $v-M$ instead of v and then use the definition of g_K once again.

EXAMPLES 8.63. (i) Let K be the *closed ball* $\overline{B}(a,R)\subset\mathbf{C}^n$. Setting $z=a+wb$ with $w\in\mathbf{C}, |b|=R$ the Lemma shows that $g_K(a+wb)\le\log^+|w|$, cf. (6c). On the other hand the special choice $p_1(z)=\overline{b}\cdot(z-a)/R^2$ shows that $g_K(a+wb)\ge\log|p_1(a+wb)|=\log|w|$. Conclusion: $g_K(a+wb)\equiv\log^+|w|$ for every $b\in\mathbf{C}^n$ of norm R, hence

$$g_K(z)=\log^+\frac{|z-a|}{R},\quad\gamma=-\log\,R,\quad\operatorname{cap}K=R,$$

in agreement with Example 8.51 when $n=1$. (ii) For the *line segment* $K=[-1,1]$ in $\mathbf{C}$ one will have

$$g_K(z)=g(z)=\log|z+(z^2-1)^{\frac12}|,\qquad\operatorname{cap}K=\frac12,$$

in conformity with 8.51. Indeed, any subharmonic function v on $\mathbf{C}$ which is majorized by 0 on K and by $\log|z|+\mathcal{O}(1)$ at ∞ will be majorized by g on $\mathbf{C}-K$, cf. (8.26), hence $g_K\le g$. On the other hand, if we use the Chebyshev polynomials $T_m(z)$ we find

$$g_K(z)\ge\lim_{m\to\infty}\frac1m\log|T_m(z)|=\log|z+(z^2-1)^{\frac12}|,\qquad z\in\mathbf{C}-[-1,1],$$

see (8.26). Simple transformations will give the Green functions for other compact line segments in $\mathbf{C}$.

(iii) Every non-degenerate *rectangular block* K in $\mathbf{R}^n=\mathbf{R}^n+i0\subset\mathbf{C}^n$:

$$K=\{x\in\mathbf{R}^n:\,a_\nu\le x_\nu\le b_\nu,\quad\nu=1,\dots,n\}\qquad(b_\nu>a_\nu)$$

has *positive capacity* in $\mathbf{C}^n$. This will follow from Lemma 8.62 and the simple inequality

$$g_K(z)\le g_1(z_1)+\ldots+g_n(z_n),$$

where g_ν stands for the one-variable Green function for the real interval $[a_\nu,b_\nu]$ in $\mathbf{C}$ with pole at ∞. We verify the inequality in the case $n=2$. For $v(z_1,z_2)$ as in (6a) we have $v(x_1,x_2)\le0$ whenever $a_\nu\le x_\nu\le b_\nu$. Taking $x_2\in[a_2,b_2]$ fixed, the subharmonic function $v(z_1,x_2)$ will be majorized by 0 on $[a_1,b_1]$ and by $\log|z_1|+\mathcal{O}(1)$ at ∞, hence it is majorized by $g_1(z_1)$ throughout $\mathbf{C}_{z_1}$, cf. (ii). Thus for fixed z_1, the subharmonic function $v(z_1,z_2)-g_1(z_1)$ will be majorized by 0 on $[a_2,b_2]$ and by $\log|z_2|+\mathcal{O}(1)$ at ∞, hence it is majorized by $g_2(z_2)$ throughout $\mathbf{C}_{z_2}$. Conclusion: all admissible functions $v(z_1,z_2)$ are majorized by $g_1(z_1)+g_2(z_2)$ on $\mathbf{C}^2$ and the same will hold for their upper envelope, the Green function $g_K(z_1,z_2)$.

[One actually has $g_K(z)=\sup\{g_1(z_1),\dots,g_n(z_n)\}$, cf. exercise 8.44.]

The following theorem will be used in Section 8.7:

Main Theorem 8.64. (cf. [Siciak 1982]). *For compact K in $\mathbf{C}^n$ the following assertions are equivalent:*

(i) $g_K(z) < +\infty$ *throughout* $\mathbf{C}^n$*;*

(ii) *There are a ball* $\overline{B}(a,r)$ *and a constant* M *such that* $g_K(z) \le M$ *on* $\overline{B}(a,r)$*;*

(iii) *There exist* $a \in \mathbf{C}^n$, $r > 0$ *and* M *such that*

$$g_K(z) \le M + \log^+ \frac{|z-a|}{r}, \qquad \forall z \in \mathbf{C}^n;$$

(iv) $\gamma = \gamma_K < +\infty$ *or* $\operatorname{cap} K = e^{-\gamma} > 0$*;*

(v) *For every bounded set* $H \subset \mathbf{C}^n$ *there is a constant* $C_H = C(H,K)$ *such that for every* $m \le 0$ *and all polynomials* $p(z)$ *of degree* $\le m$,

$$\|p\|_H \le \|p\|_K \, C(H,K)^m.$$

[*In terms of* g_K*, one may take* $C(H,K) = \exp(\sup_H g_K)$.]

PROOF. (i) $\Rightarrow$ (ii). We will use Baire's theorem: If a complete metric space is the union of a countable family of closed sets, at least one of the sets must contain a ball. Assuming (i), define

$$E_s = \{z \in \mathbf{C}^n : g_K(z) \le s\}, \qquad s = 1, 2, \dots .$$

Every E_s is a closed set: it is the intersection of the closed sets $\{z \in \mathbf{C}^n : v(z) \le s\}$ corresponding to the continuous functions v of (6a). Now $\mathbf{C}^n = \bigcup_{s=1}^\infty E_s$, hence by Baire's theorem, some set E_q contains a ball $\overline{B}(a,r)$. We then have (ii) with $M = q$.

(ii) $\Rightarrow$ (iii): apply Lemma 8.62 with $R = r$.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii): use the definition of γ in Definition 8.61.

(iii) $\Rightarrow$ (v) $\Rightarrow$ (ii) or (i): use the definition of g_K in 8.61. *Q.E.D.*

The (pre-) Green function g_K need not be plurisubharmonic even if it is finite, because it need not be upper semi-continuous. For example, if $K = \overline{B}(0,1) \cup \{2\}$ in $\mathbf{C}$ then $g_K(z) = \log^+ |z|$ for $z \ne 2$ but $g_K(2) = 0$. To repair this small defect one may define the "real" Green function g_K^* as the "upper regularization" of g_K:

$$g_K^*(a) = \limsup_{z \to a} g_K(z), \qquad \forall a \in \mathbf{C}^n,$$

cf. exercise 8.8. It follows from Theorem 8.64 that g_K^* is *either identically* $+\infty$ (if $\operatorname{cap} K = 0$) or *finite everywhere* (if $\operatorname{cap} K > 0$). In the latter case one may show that g_K^* is plurisubharmonic. [The regularized upper envelope of a locally bounded family of psh functions is psh, cf. exercise 8.29.]

In the case of $\mathbf{C}$ [but not in $\mathbf{C}^n$!] it may be shown that g_K is harmonic outside K when it is finite. Furthermore

$$g_K^*(z) = \gamma - U^{\mu_0}(z) = \gamma + \int_K \log|z-\zeta| d\mu_0(\zeta),$$

where the positive measure μ_0 of total mass 1 represents the equilibrium distribution on K, cf. (5a). One may deduce from this that the Green function g_K^* on $\mathbf{C}$ satisfies Poisson's equation, $\Delta g_K^* = 2\pi\mu_0$ [at least in the sense of distributions]. There is a corresponding partial differential equation for g_K^* in $\mathbf{C}^n$ $(n \geq 2)$, the so-called complex Monge-Ampère equation, which will be studied in Section 8.8.

The function $\operatorname{cap} K$ has most of the properties usually required of a capacity. It is monotonic and $\operatorname{cap} K_\nu \to \operatorname{cap} K$ if $K_\nu \searrow K$ or $K_\nu \nearrow K$; for bounded sets L that are limits of increasing sequences $\{K_\nu\}$ of compact sets, it makes sense to define $\operatorname{cap} L = \lim \operatorname{cap} K$. See the recent paper [Kołodziej 1989].

8.7 Some applications of $\mathbf{C}^n$ capacities. Our main application will be the partial derivatives lemma ([Korevaar-Wiegerinck 1985], [Korevaar 1986]) which was used already in Sections 3.5, 3.6; for other uses see exercises 8.52, 8.58. Let E be a family of directions ξ in $\mathbf{R}^n$; we think of E as a subset of the unit sphere S^{n-1}. If E is large enough, the partial derivatives of C^∞ functions f in $\mathbf{R}^n$ can be estimated in terms of the directional derivatives of the same order that correspond to the set E. It is remarkable that the best constant $\beta(E)$ in this real variables result is equal to a $\mathbf{C}^n$ capacity for a set closely related to E. The set in question is the closure $\overline{E}_c$ of the circular set $E_c \subset \mathbf{C}^n$ generated by E:

$$E_c \stackrel{\text{def}}{=} \{z = e^{it}\xi \in \mathbf{C}^n : \xi \in E, \quad t \in \mathbf{R}\}. \tag{7a}$$

Theorem 8.71 (Partial derivatives lemma). *(i) For every nonempty open subset E of the real unit sphere S^{n-1} there is a constant $\beta_E > 0$ such that, for any point $a \in \mathbf{R}^n$ and any C^∞ function f in a neighborhood of a,*

$$\max_{|\alpha|=m} \frac{1}{\alpha!}\left|D^\alpha f(a)\right| \leq \sup_{\xi\in E} \frac{1}{m!}\left| \left(\frac{d}{dt}\right)^m f(a+t\xi)\Big|_{t=0}\right| / \beta_E^m, \quad m = 1, 2, \ldots . \tag{7b}$$

(ii) For an arbitrary set $E \subset S^{n-1} \subset \mathbf{R}^n$ there is such a constant $\beta_E > 0$ if and only if the closed circular set $K = \overline{E}_c$ has positive logarithmic capacity in $\mathbf{C}^n$.
(iii) The best (largest possible) constant β_E in (7b) is equal to what may be called a Siciak capacity:

$$\beta_E = \sigma(\overline{E}_c), \quad \sigma(K) \stackrel{\text{def}}{=} \exp(-\sup_{\Delta} g_K), \tag{7c}$$

where g_K is the (pre-) Green function for K in $\mathbf{C}^n$ with logarithmic singularity at ∞ (Definition 8.61) and $\Delta = \Delta_n(0,1)$ is the unit polydisc.

REMARKS. If $\beta_E > 0$ and f is a continuous function on a domain D in $\mathbf{R}^n$ such that the m^{th} order directional derivatives on the right-hand side of (7b) exist at every point $a \in D$ and are uniformly bounded on D for each m, then f is of class C^∞ and (7b) is applicable, cf. exercises 3.22, 8.53.

The proof below will show that the Theorem reduces to a result on polynomials. For the class of all polynomials $f(z)$ in $z = (z_1, \ldots, z_n)$, there exist inequalities (7b) for any bounded set E in $\mathbf{C}^n$ with the property that $\overline{E}_c$ has positive capacity, cf. exercise 8.54.

PROOF of the Theorem. Let $E \subset S^{n-1}$ be given. Taking $a = 0$ as we may, let f be any C^∞ function on a neighborhood of 0 in $\mathbf{R}^n$. We introduce its Taylor expansion

$$(7d) \qquad f(x) \sim \sum_0^\infty q_m(x).$$

where

$$(7d') \qquad q_m(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha, \qquad c_\alpha = D^\alpha f(0)/\alpha!.$$

The homogeneous polynomials $q_m(x)$ may be characterized by the condition that for every integer $N \geq 0$,

$$f(x) - \sum_0^N q_m(x) = o(|x|^N) \quad \text{as} \quad x \to 0.$$

For $x = t\xi$ with $\xi \in E$ fixed, $t \in \mathbf{R}$ variable we also have the expansion

$$f(t\xi) \sim \sum_0^\infty q_m(t\xi) = \sum_0^\infty q_m(\xi) t^m,$$

where for every N, $f(t\xi) - \sum_0^N q_m(\xi)t^m = o(|t|^N)$ as $t \to 0$. It follows that

$$(7d'') \qquad q_m(\xi) = \frac{1}{m!}\left(\frac{d}{dt}\right)^m f(t\xi)\Big|_{t=0}.$$

Thus our problem in (7b) [with $a = 0$] is to estimate the coefficients c_α of homogeneous polynomials q_m in terms of the supremum norm $\|q_m\|_E$.

The coefficients c_α are estimated rather well by the Cauchy inequalities for the closed unit polydisc $\overline{\Delta} = \overline{\Delta}(0,1) \subset \mathbf{C}^n$:

$$|c_\alpha| \leq \|q_m\|_T \leq \|q_m\|_{\overline{\Delta}} = \|q_m\|_\Delta,$$

where T is the torus $T(0,1) = C(0,1) \times \ldots \times C(0,1)$. [Cf. Corollary 1.65. Alternatively, this coefficient inequality may be derived from Parseval's formula for orthogonal representations on T.] On the other hand, since q_m is a homogeneous polynomial,

$$\|q_m\|_E = \|q_m\|_{E_c} = \|q_m\|_K, \qquad K = \overline{E}_c.$$

Question: Can we estimate $\|q_m\|_\Delta$ in terms of $\|q_m\|_K$?

(a) [Proof of half of parts (ii) and (iii).] Suppose first that $\operatorname{cap} K > 0$. Then by Theorem 8.64 part (v),

$$\|q_m\|_\Delta \leq \|q_m\|_K C(\Delta, K)^m, \qquad C(\Delta, K) = \exp(\sup_\Delta g_K).$$

It follows that

$$\max_{|\alpha|=m} |c_\alpha| \leq \|q_m\|_\Delta \leq \|q_m\|_E \, C(\Delta, \overline{E}_c)^m, \qquad m = 1, 2, \ldots .$$

In view of (7d′,″) we have thus proved (7b) with

$$1/\beta_E = C(\Delta, \overline{E}_c) = \exp(\sup_\Delta g_K).$$

The smallest constant $1/\beta_E$ that can be used in (7b) will be $\leq \exp(\sup_\Delta g_K)$.

(b) [Proof of part (i).] Next suppose that E is any nonempty open subset of S^{n-1}. Then the compact truncated cone $E^* = [0,1] \cdot \overline{E}$ in $\mathbf{R}^n$ contains a nondegenerated rectangular block, hence it has positive capacity [see Example 8.63–iii]. Thus by Theorem 8.64 there is a positive constant $C(\Delta, E^*)$ such that for all *homogeneous* polynomials q_m and their coefficients c_α,

$$\max_{|\alpha|=m} |c_\alpha| \leq \|q_m\|_\Delta \leq \|q_m\|_{E^*} \, C(\Delta, E^*)^m = \|q_m\|_E \, C(\Delta, E^*)^m, \quad m = 1, 2, \ldots .$$

(c) [Completion of parts (ii) and (iii).] Finally, suppose that for E there is a positive constant $\beta = \beta_E$ such that (7b) holds [with $a = 0$] for all C^∞ functions $f(x)$. Then for all homogeneous polynomials $q(x) = \Sigma c_\alpha x^\alpha$,

$$|c_\alpha| \leq \beta^{-m} \|q\|_E, \quad \forall \alpha, \quad \text{where} \quad m = \deg q. \tag{7e}$$

We will deduce that $K = \overline{E}_c$ has positive capacity.

From (7e) we obtain the preliminary estimate

$$\begin{aligned} \|q\|_\Delta &= \sup_\Delta \Big| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big| \leq \sum_{|\alpha|=m} |c_\alpha| \\ &\leq \beta^{-m} \|q\|_E \sum_{|\alpha|=m} 1 \leq (m+1)^n \beta^{-m} \|q\|_E. \end{aligned} \tag{7f}$$

[Since $\alpha_1 + \ldots + \alpha_n = m$ there are at most $m+1$ possibilities for each α_j.] By (7f), β must be ≤ 1. Indeed, E belongs to $\overline{\Delta}$ and hence $(m+1)^n \beta^{-m} \geq 1$ or $\beta \leq (m+1)^{n/m}$; now let $m \to \infty$. We will use (7f) to derive a better estimate, valid for *all* polynomials p of degree $\leq m$, namely,

$$\|p\|_\Delta \leq \|p\|_K / \beta^m, \quad \text{where} \quad K = \overline{E}_c. \tag{7f′}$$

For the proof we form powers p^s with $s \in \mathbf{N}$ which we decompose into homogeneous polynomials: $p(z)^s = \sum_0^{ms} q_j(z)$, q_j homogeneous of degree j. Then $p(w\xi)^s = \sum_0^{ms} q_j(\xi) w^j$, hence by the one-variable Cauchy inequalities, taking $\xi \in E$ and letting w run over the circle $C(0,1)$ so that $w\xi \in E_c$,

$$|q_j(\xi)| \leq \big\| p(w\xi)^s \big\|_{C(0,1)} \leq \|p\|_K^s.$$

Thus $\|q_j\|_E \le \|p\|_K^s$ and by (7f)

$$\|p^s\|_\Delta = \Big\|\sum_0^{ms} q_j\Big\|_\Delta \le \sum_0^{ms}(j+1)^n\beta^{-j}\|q_j\|_E \le (ms+1)^{n+1}\beta^{-ms}\|p\|_K^s.$$

Finally, taking the s^{th} root and letting $s \to \infty$ we obtain (7f′).

It follows from (7f′) and Definition 8.61 that the Green function g_K is bounded by $\log 1/\beta$ at each point $z \in \Delta$, hence by Theorem 8.64 part (ii), $K = \overline{E}_c$ has positive capacity. This conclusion completes the proof of part (ii).

By the preceding $\exp g_K \le 1/\beta$ throughout Δ, hence the smallest possible constant $1/\beta(E)$ that works in (7b) must be $\ge \exp(\sup_\Delta g_K)$. In view of part (a), the smallest possible $1/\beta_E$ is thus equal to $\exp(\sup_\Delta g_K)$. In other words, the largest possible constant β_E is equal to the Siciak capacity $\sigma(K)$ in part (iii) of the Theorem.

REMARKS. The definition of $\sigma(K)$ in (7c) may be applied to any compact set K in $\mathbf{C}^n$. Just like $\operatorname{cap} K)$ the function $\sigma(K)$ has most of the properties usually associated with a capacity. In particular, $\sigma(K) \le \sigma(K')$ if $K \subset K'$ and $\sigma(K_\nu) \to \sigma(K)$ if $K_\nu \searrow K$ or $K_\nu \nearrow K$. The function σ is also a $\mathbf{C}^n$ capacity in the sense that $\sigma(K) = 0$ if and only if K is pluripolar or equivalently, $\operatorname{cap} K = 0$. Cf. [Siciak 1982].

In other applications of $\mathbf{C}^n$ potential theory, the precise constant is given by a Siciak capacity involving the unit ball:

$$\rho(K) \overset{\text{def}}{=} \exp(-\sup_B g_K), \qquad B = B(0,1). \tag{7g}$$

For compact circular subsets $K = K_c$ of the closed unit ball, the constant $\rho(K)$ may be characterized geometrically as the radius of the largest ball $\overline{B}(0,r)$ which is contained in the polynomially convex hull $\tilde{K}$ of K, cf. exercises 6.16 and 8.55, 8.56. This property makes $\rho(K)$ the sharp constant in the *Sibony-Wong theorem* on the growth of entire functions in $\mathbf{C}^n$:

Theorem 8.72. *Let $K = K_c$ be a compact circular subset of the unit sphere $\partial B \subset \mathbf{C}^n$ of positive capacity. Then for every polynomial and [hence] for every entire function $F(z)$ in $\mathbf{C}^n$,*

$$\sup_{|z|\le\rho r} |F(z)| \le \sup_{z\in rK} |F(z)|, \quad \text{where} \quad \rho = \rho(K).$$

Cf. exercise 8.57 and [Siciak 1982].

8.8 Maximal functions and the Dirichlet Problem. Let us return to $\mathbf{C}$ for a moment. Suppose we know that we can solve the Dirichlet problem for Δ: given a domain $D \subset \mathbf{C}$ and $f \in C(\partial D)$, there exists a smooth function $u \in C(D)$ such that

$$\Delta u = 0 \text{ on } D$$
$$u|_{\partial D} = f.$$

Now we can describe u. We introduce the *Perron* family:

$$\text{(8a)} \qquad \mathcal{P}_f = \{v \in C(\overline{D}) : v \text{ is subharmonic on } D,\ v|_{\partial D} \le f\}.$$

Clearly $u \in \mathcal{P}_f$ and in view of Theorem 8.21 it is the largest one:

$$\text{(8b)} \qquad u(z) = \sup_{v \in \mathcal{P}_f} v(z).$$

One can write down (8b) even without knowing that Dirichlet's problem is solvable and it is reasonable to expect that this will give some sort of solution. This indeed turns out to be the case as was shown by Perron and as we shall see below.

Working again with several variables we introduce the *Perron-Bremermann* family

$$\text{(8c)} \qquad \mathcal{F}_f = \{u \in C(\overline{D}) : u \text{ is plurisubharmonic on } D,\ u|_{\partial D} \le f\}$$

and form the Perron-Bremermann maximal function:

$$\text{(8d)} \qquad F_f(z) = \sup_{u \in \mathcal{F}_f} u(z).$$

One may expect that this gives rise to the solution of the Dirichlet problem for an analogue of the Laplace operator in some sense. What would this operator look like? The following simple proposition will give an idea.

Proposition 8.81. *Let D be a domain in $\mathbf{C}^n$, $\mathcal{F}_f$ the Perron-Bremermann family for $f \in C(\partial D)$. If $u \in \mathcal{F}_f$ and u is smooth and strictly plurisubharmonic at some point $a \in D$, then $u \ne F_f$.*

PROOF. Let u be smooth and strictly plurisubharmonic on $B(a, r) \subset D$. Choose a smooth real valued cutoff function $\chi \ge 0$ supported in $B(a, r/2)$ with $\chi(a) > 0$. Then for sufficiently small $\epsilon > 0$ the function $u_\epsilon = u + \epsilon\chi$ will be plurisubharmonic, $u_\epsilon \in \mathcal{F}_f$ and $u_\epsilon(a) > u(a)$, which shows that $u \ne F_f$.

Therefore, if F_f would exist and be smooth, it would be a plurisubharmonic function [by 8.42] but nowhere could it be strictly plurisubharmonic. In other words, the least eigenvalue of the complex Hessian of F_f would equal 0. Thus it would be a solution of the *Complex Monge-Ampère equation*

$$\text{(8e)} \qquad M(u) = \det \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0.$$

We are led to the Dirichlet problem for M: Given a domain D in $\mathbf{C}^n$ and a function $f \in C(\partial D)$, find a continuous plurisubharmonic function u on $\overline{D}$ such that

$$M(u) = 0, \quad u|_{\partial D} = f.$$

Note that in the one dimensional case M reduces to a multiple of Δ and we don't need to require that u be subharmonic —it will follow from the equation. In the higher dimensional case there are lots of problems. One can show that the maximal function (8d) need not be C^2, cf. exercise 8.60. Apparently we have the problem of defining $M(u)$ for non smooth u. This can be done, but is much harder than in the one dimensional case where one can use distributions, because Δ is linear. However, M is highly nonlinear in the higher dimensional case.

In what follows we will discuss some aspects of solving the Dirichlet problem for M. The solution will be complete in the one dimensional case only. We refer to the literature for complete proofs and many related interesting results, see [Bedford, Taylor; Bedford; Cegrell; Klimek].

Let D be a bounded domain in $\mathbf{C}^n$ given by a smooth defining function ρ which is plurisubharmonic on a neighborhood of $\overline{D}$. That is,

$$D = \{z \in \mathrm{Dom}(\rho) : \rho(z) < 0\},$$

while $\nabla\rho \neq 0$ on $\{\rho = 0\}$, cf. Chapter 9. In $\mathbf{C}$ one may modify a smooth defining function to be strictly subharmonic, in $\mathbf{C}^n$ this is not true: The condition on D means that the domain is *strictly pseudoconvex*, cf. Chapter 9. In particular D is pseudoconvex. Although strict pseudoconvexity is not a necessary condition to solve the Dirichlet problem for M, pseudoconvexity alone is not enough, cf. exercise 8.62.

Proposition 8.82. *Suppose that $D \subset \mathbf{C}^n$ has a smooth plurisubharmonic defining function ρ and that f is continuous on ∂D and use the notation (8c) and (8d). Then F_f is continuous on $\overline{D}$, plurisubharmonic on D and satisfies $F_f|_{\partial D} = f$.*

PROOF. First we discuss boundary behavior. Let $\epsilon > 0$, and let ϕ be smooth on a neighborhood of $\bar{D}$ such that on ∂D $f - \epsilon < \phi < f$ (One may start with a continuous function with this property defined on a neighborhood of $\bar{D}$ and approximate it uniformly on a compact neighborhood of $\bar{D}$ with smooth functions) For sufficiently large C_1 the function $g_0 = \phi + C_1\rho$ will be strictly plurisubharmonic, thus $g_0 \in \mathcal{F}_f$ and

$$\liminf_{z\to w\in\partial D} F_f(z) \geq \lim_{z\to w\in\partial D} g_0(z) \geq f(w) - \epsilon. \tag{8f}$$

Similarly take $\psi \in C^\infty(\bar{D})$, $f < \psi < f + \epsilon$ on ∂D. Again for sufficiently large $C_2 > 0$ $C_2\rho - \psi$ will be strictly plurisubharmonic. For $g \in \mathcal{F}_f$ we have $C_2\rho - \psi + g < 0$ on ∂D, thus by the maximum principal also on D. Therefore $g < \psi - C_2\rho$ independently of $g \in \mathcal{F}_f$, hence $F_f \leq \psi - C_2\rho$ and

$$\limsup_{z\to w\in\partial D} F_f(z) \leq \lim_{z\to w\in\partial D} (\psi - C_2\rho)(w) \leq f(w) + \epsilon. \tag{8g}$$

Since ϵ was arbitrary, it follows from (8f, g) that F_f is continuous at ∂D and has boundary values f.

Next we investigate continuity in the interior. As a supremum of continuous functions, F_f is lsc. We form the usc regularization F_f^*, which is a plurisubharmonic function, and wish to prove continuity, that is

$$H = F_f^* - F_f \equiv 0.$$

The function H is ≥ 0, usc on $\bar{D}$ and continuous on ∂D with boundary values 0. Let $M = \sup_{z \in D} H(z)$. If $M > 0$ then M is attained at a compact subset K in the interior of D. Let

$$L = L_\delta = \{z \in D, d(z, \partial D) \geq \delta\}$$

Given $\epsilon > 0$ we may take δ small enough such that $K \subset L$ and $H < \epsilon$ as well as $F_f - g_0 < \epsilon$ on ∂L. The function F_f^* can on compact subsets of D be approximated from above by a decreasing sequence of plurisubharmonic functions $\{h_j\}$. We claim that this convergence is almost uniform on ∂L, i.e.

$$\exists m > 0: \; h_m - F_f^* < 2\epsilon \text{ on } \partial L.$$

This is just an elaborate version of Dini's theorem on decreasing sequences of continuous functions, cf. exercise 8.59. Define

$$h(z) = \begin{cases} \max(g_0, h_m - 4\epsilon) & \text{on } L; \\ g_0 & \text{on } D \setminus L. \end{cases}$$

Then $h \in \mathcal{F}_f$, because at ∂L and hence on a tiny neighborhood of ∂L,

$$g_0 > F_f - \epsilon > F_f^* - 2\epsilon > h_m - 4\epsilon.$$

We conclude that $F_f^* - F_f < F_f^* - h < 4\epsilon$ on L. Hence $M < 4\epsilon$, which implies $M = 0$.

Proposition 8.83. *Suppose that F_f is maximal for $\mathcal{F}_f$ on D. Let B be a ball in D and let $g = F_f|_{\partial B}$, then $F_f|_B$ is maximal for $\mathcal{F}_g$.*

PROOF. Proposition 8.82 shows that G is continuous. It is clear that $F_f|_B \in \mathcal{F}_g$, therefore $F_f|_B \leq F_g$. Now form the Poisson modification:

$$\tilde{F} = \begin{cases} F_f & \text{outside } B \\ F_g & \text{on } \bar{B} \end{cases}.$$

This $\tilde{F}$ is indeed an element of $\mathcal{F}_f$; we only have to check plurisubharmonicity on ∂B. Restricting to a complex line l through $a \in \partial B$, we see that the mean value inequality holds: $\tilde{F}(a) = F(a)$, while on a circle about a in l we have $\tilde{F} \geq F_f$. As $\tilde{F} \leq F_f$ by definition of F_f, we obtain $F_f|_B = F_g$.

COROLLARY 8.84. The Dirichlet problem for Δ has a (unique) solution on smooth domains in $\mathbf{C}$.

PROOF. The Poisson integral solves the Dirichlet problem on discs cf. Section 8.1. Proposition 8.83 shows then that the maximal function coincides with a harmonic function on discs. It is therefore harmonic. Uniqueness was shown in (8.2).

In the same fashion one concludes from Proposition 8.83.

COROLLARY 8.85. Suppose that the Dirichlet problem for M is solvable on the unit ball, then it is solvable on every domain D which admits a strictly psh defining function.

Proposition 8.86. *Let $u \in C^2(\bar{D})$, $f = u|_{\partial D}$. If $Mu \equiv 0$, then $u = F_f$.*

Suppose that $\exists v \in \mathcal{F}_f$, $C > 0$, such that $\sup_D v(z) - u(z) = C$ and (hence) $\exists K \subset\subset D$ with $v - u = C$ on K. Adapting C if necessary, we may even assume $v < f - \epsilon$ on ∂D. Hence there exists a compact $K_2 \subset D$ with $v < u - \epsilon$ outside K_2. Then for sufficiently small η, the function

$$v_\eta(z) = v(z) + \eta|z|^2$$

will have the properties $v_\eta - u < 0$ outside K_2 and $v_\eta - u$ will assume its maximum C' close to C on a compact neighborhood K_1 of K. We have

$$K \subset\subset K_1 \subset\subset K_2 \subset\subset D.$$

Approximating v_η uniformly from above on K_2 with psh functions v_j, we find one v', such that $v' - u < 0$ close to the boundary of K_2. Now put

$$h(z) = \begin{cases} u(z) & \text{on } D \setminus K_2, \\ \max\{u(z), v'(z)\} & \text{on } K_2. \end{cases}$$

It is clear that $h \in \mathcal{F}_f$ and that $h - u$ is smooth in a neighborhood of a point z_0 where it assumes its maximum. Let ζ_0 be an eigenvector of $\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)|_{z_0}$ with eigenvalue 0. Then restricted to the complex line $\{z_0 + w\zeta_0\}$, $(h-u)(z_0 + w\zeta_0)$ assumes a maximum at $w = 0$. But $\Delta(h-u)(z_0 + w\zeta_0)|_{w=0} > 0$, a contradiction.

CONCLUSION A function $u \in C^2(\bar{D})$ is maximal in $\mathcal{F}_f$ for $f = u|_\partial D$ if and only if $MA(u) \equiv 0$.

We make some further, extremely sketchy, remarks on the Dirichlet problem for M

Existence. Corollary 8.85 shows that it would be sufficient to show that F_f provides a solution for the Dirichlet problem for $f \in \partial B$, B a (the unit) ball in $\mathbf{C}^n$. Now if one assumes that $f \in C^2(\partial B)$ it can be shown that F_f is almost C^2: second derivatives exist and are locally in L^∞, [this is best possible, cf. exercise 8.60]. The proof exploits the automorphisms of B to perturb f and F_f.

To show that $M(F_f) = 0$, one tries to execute the idea of Proposition 8.81: If $M(F_f) \not\equiv 0$, construct a function v on a small ball $B(z_0, \delta)$ such that v is psh, $v(z_0) > u(z_0)$ and $v < u$ on $\partial B(z_0, \delta)$. Simple as it may sound, it is a difficult and involved step.

One passes to continuous boundary values like this. A key result is the *Chern-Levine-Nirenberg inequality*, a special case of which reads as follows:

For every $K \subset\subset D$ there exists a constant C_K such that for $u \in C^2 \cap PSH(D)$

$$\int_K M(u)\, dV \leq C_K ||u||_\infty^n.$$

It follows that if $u_j \in C^2 \cap PSH(D)$ is a bounded set in L^∞, then $M(u_j)\,dV$ has a subsequence converging to some measure. It can be shown that as long as $u_j \in C^2 \cap PSH(D) \downarrow u \in L^\infty \cap PSH(D)$, this limit measure is independent of the sequence. Thus $M(u)$ or perhaps better $M(u)\,dV$ is defined as a positive measure. Now if $f \in C(\partial D)$ take a sequence of smooth $f_j \downarrow f$. It is clear that $F_{f_j} \downarrow F_f$ and then $M(F_f) = \lim M(F_{f_j}) = 0$.

Finally uniqueness is derived from so called *comparison principles*. An example, of which we don't give a proof, [but see exercise 11.x], is the following

Lemma 8.87. *Let $u, v \in C(\bar{D}) \cap PSH(D)$ and $u \geq v$ on ∂D. Then*

$$\int_{u<v} M(v)\,dV \leq \int_{u<v} M(u)\,dV.$$

Assuming this Lemma, we put $v = F_f$ and let u be an other solution. Then $u \in \mathcal{F}$ and $u(z_0) < v(z_0)$ for some $z_0 \in D$. For suitable $\epsilon, \delta > 0$ the function $\tilde{v}(z) = v(z) - \epsilon + \delta|z|^2$ will satisfy

$$u > \tilde{v} \text{ on } \partial D \quad \text{ while } \quad u(z_0) < \tilde{v}(z_0).$$

Also, $M(\tilde{v}) > \delta^n$ [it suffices to check this for smooth $v \in PSH$]. Thus Lemma 8.87 leads to

$$\delta^n m(\{u < \tilde{v}\}) \leq \int_{u<\tilde{v}} M(\tilde{v})\,dV \leq \int_{u<\tilde{v}} M(u)\,dV = 0.$$

This is a contradiction.

REMARKS 8.88. Lets look back at the definition of the Green function g_K in Definition 8.61. It was defined as the sup of a subset of all psh functions that satisfy (6b). One can show that taking the sup over all functions that satisfy (6b) gives the same Green function. Thus the Green function is a kind of Perron Bremermann function, but now with a growth condition at infinity. Now it should not come as a surprise that $M(g_K^*) = 0$ on the complement of K. This is indeed the case, cf. [Bedford 88, Kołodziej]

We finally remark that one needs to have a good theory of "generalized differential forms", the so called *currents* at one's disposal to complete the proofs, cf. Chapter 10.

Exercises

8.1. Show that the harmonic functions $u(x) = f(r)$ on $\mathbf{R}^n - \{a\}$ which depend only on $|x-a| = r$ have the form

$$u(x) = \begin{cases} c_1 \log \frac{1}{|x-a|} + c_2 & \text{if } n = 2, \\ \\ c_1 \big|x-a\big|^{2-n} + c_2 & \text{if } n \neq 2. \end{cases}$$

$[f'' + \frac{n-1}{r} f' = 0.]$

8.2. (*Poisson integral*). Let u be harmonic on $\overline{B}(0,1) \subset \mathbf{C}$. For $a = re^{i\theta} \in B$, set

$$w = \frac{z-a}{1-\overline{a}z} \quad \text{and} \quad u(z) = U(w).$$

Verify that U is harmonic on $\overline{B}$ and that the mean value property of U furnishes the Poisson integral representation for u:

$$\begin{aligned} u(re^{i\theta}) = u(a) = U(0) &= \frac{1}{2\pi} \int_{C(0,1)} U(w) \frac{dw}{iw} \\ = \frac{1}{2\pi} \int_{C(0,1)} u(z) \frac{1-|a|^2}{|z-a|^2} \frac{dz}{iz} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} u(e^{it}) dt. \end{aligned}$$

8.3. (*Dirichlet problem for disc and ball*). (i) Writing $z = re^{i\theta}$ in $\mathbf{C}$, verify that the Poisson kernel can be written as follows:

$$\frac{1-r^2}{1-2r\cos(\theta-t)+r^2} = \operatorname{Re} \frac{e^{it}+z}{e^{it}-z}.$$

Deduce that the Poisson integral $u = P[g]$ of an integrable function g on $C(0,1)$ is harmonic on the unit disc B in $\mathbf{C}$. Show that for continuous g, $u(z) \to g(\zeta)$ as $z \in B$ tends to $\zeta \in C(0,1)$. [A constant function is equal to its Poisson integral. Now take $\zeta = 1$ and $g(1) = 0$. Split the interval of integration into $[-\delta, \delta]$ and the rest. The kernel is nonnegative.] (ii) Verify that the Poisson kernel for the unit ball in $\mathbf{R}^n$,

$$\frac{|\xi|^2 - |x|^2}{|\xi - x|^n} = \frac{\xi\cdot\xi - x\cdot x}{(\xi\cdot\xi - \xi\cdot x - x\cdot\xi + x\cdot x)^{n/2}},$$

satisfies Laplace's equation relative to x on $\mathbf{R}^n - \{\xi\}$. Then show that for continuous g on $S(0,1) \subset \mathbf{R}^n$, the Poisson integral $u = P[g]$ solves the Dirichlet problem for the unit ball B and boundary function g, cf. (1b′). [How to show that $P[1] \equiv 1$? $P[1](x)$ is harmonic on B and depends only on $|x|$ (why?), hence ....]

8.4. Write down a Poisson integral for harmonic functions on the closed disc [or ball] $\overline{B}(a,R)$. Deduce that harmonic functions are of class C^∞ and show that a uniform limit of harmonic functions on a domain Ω in $\mathbf{C}$ [or $\mathbf{R}^n$] is harmonic.

8.5. Prove that $v(z) = \sum_2^\infty k^{-2} \log|z - 1/k|$ is subharmonic on $B(0, \frac{1}{2}) \subset \mathbf{C}$, but not continuous at 0. [v is, in fact, subharmonic on $\mathbf{C}$.]

8.6. Let v on $E \subset \mathbf{R}^n$ be the limit of a decreasing sequence of upper semi-continuous (usc) functions $\{v_k\}$. Prove that v is usc.

8.7. Let $E \subset \mathbf{R}^n$ be compact and let $v : E \to \mathbf{R} \cup \{-\infty\}$ be such that $\limsup_{x\to a} v(x) \leq v(a)$, $\forall a \in E$. Prove that v assumes a maximum on E and that there is a decreasing sequence of finite continuous functions $\{v_k\}$ which converges to v on E. [First assuming $v > -\infty$, define $v_k(x) = \max_{y \in E}\{v(y) - k|x-y|\}$. Use a value $y_k = y_k(x)$ where the maximum is attained to show that $v_k(x') - v_k(x) \geq -k|x'-x|$, etc. For the proof that $v_k(x) \downarrow v(x)$ it is useful to observe that $y_k(x) \to x$ as $k \to \infty$. Finally, allow also the value $-\infty$ for v.]

8.8. (Usc regularization). Let V be a function $\Omega \to \mathbf{R} \cup \{-\infty\}$ and let V^* be its "regularization":

$$V^*(a) = \limsup_{x \to a} V(x), \qquad \forall a \in \Omega.$$

Supposing $V^* < +\infty$ on Ω, prove that it is upper semi-continuous.

8.9. Prove that for any bounded domain $\Omega \subset \mathbf{C}$, the exhaustion function

$$-\log d(z) = \sup\{-\log|z-b|,\ b \in \partial\Omega\}$$

is subharmonic on Ω. Can you find a subharmonic exhaustion function for arbitrary bounded domains Ω in $\mathbf{R}^n$ $(n \geq 3)$?

8.10. Show that the infimum of the subharmonic functions $v_1(x,y) = x$ and $v_2(x,y) = -x$ on $\mathbf{R}^2$ is not subharmonic.

8.11. Prove the relations (1e) for subharmonic functions.

8.12. Compute the logarithmic potential of (normalized) arc measure on $C(0,1)$: $U(z) = -\int_{-\pi}^{\pi} \log|z - e^{it}|dt/2\pi$. Verify that U is superharmonic on $\mathbf{C}$ and harmonic except on $C(0,1)$.

8.13. (**Maximum principle characterization of subharmonic functions**). Let Ω in $\mathbf{R}^2$ [or $\mathbf{R}^n$] be open and let $v : \Omega \to \mathbf{R} \cup \{-\infty\}$ be upper semi-continuous. Prove that v is subharmonic if and only if it satisfies the following maximum principle:

"For every subdomain $D \subset \Omega$ (or for every disc [or ball]] D with $\overline{D} \subset \Omega$) and every harmonic function u on D which majorizes v on $\partial_e D$, one has $u \geq v$ throughout D".

8.14. Suppose that v on Ω is both subharmonic and superharmonic, or equivalently, that v is finite, real, continuous and has the mean value property on Ω. Prove that v is harmonic.

8.15. (**Hopf's lemma**). Let v be C^1 subharmonic on the closed unit disc $\overline{B}(0,1)$ and < 0 except that $v(1) = 0$. Prove that the outward normal derivative $\partial v/\partial N$ is strictly positive at the point 1. [Let u be the Poisson integral of $v\big|_C(0,1)$. Since $v(r) \leq u(r)$ it is enough to prove that $\overline{\lim}\{v(1) - u(r)\}/(1-r) > 0$.] Extend to other smoothly bounded domains.

8.16. Let v be a subharmonic function on the annulus $A(0;\rho,R) \subset \mathbf{C}$. Prove that $m(r) = \max_\theta v(re^{i\theta})$ is a convex function of $\log r$:

$$m(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} m(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} m(r_2),\ \ \rho < r_1 \leq r \leq r_2 < R.$$

Apply the result to $v(z) = \log|f(z)|$ where f is holomorphic on the annulus $A(0;\, \rho, R)$. The resulting inequality for the "maximum modulus" $M(r) = \max_\theta |f(re^{i\theta})|$ is known as **Hadamard's three circles theorem**.

8.17. Let v be a subharmonic function on $\mathbf{C}$. What can you say if v is bounded above? What if only $\limsup v(z)/\log|z| \le 0$ for $|z| \to \infty$?

8.18. Let v be subharmonic on the infinite strip S: $a < x = \mathrm{Re}z < b,\ -\infty < y = \mathrm{Im}z < \infty$ in $\mathbf{C}$ and bounded above on every interior strip $a + \delta < x < b - \delta,\ \delta > 0$. Prove that $m(x) = \sup_y\ v(x+iy)$ is convex.

8.19. Prove directly and simply that a strictly subharmonic function v on a connected domain $D \subset \mathbf{R}^2$ can not have a maximum at $a \in D$. What about an arbitrary smooth subharmonic function?

8.20. Let v be a C^2 function on the closed disc $\overline{B}(0,r)$. Show that $\forall\theta$,

$$v(r\cos\theta, r\sin\theta) = v(0) + \int_0^r \frac{\partial}{\partial\rho} \ldots\, d\rho$$
$$= v(0) - \int_0^r \frac{\partial v}{\partial\rho}\{\rho\, \frac{\partial v}{\partial\rho}\,(\rho\cos\theta,\ \rho\sin\theta)\}\log\frac{\rho}{r}\, d\rho.$$

8.21. Let φ be a C^1 function on $\mathbf{C}$ of compact support, $U(z) = \int_{\mathbf{C}} \log|z-\zeta| \cdot \varphi(\zeta) d\xi d\eta$. Prove that $\partial U/\partial z$ and $\partial U/\partial\overline{z}$ are of class C^1 and that $\Delta U = 2\pi\varphi$. [Think of Theorem 3.13. Show that

$$\frac{\partial U}{\partial x} = \int_{\mathbf{C}} \log|\zeta| \cdot \frac{\partial}{\partial\xi}\varphi(z+\zeta) d\xi d\eta = -\int_{\mathbf{C}} \mathrm{Re}\,\frac{1}{\zeta} \cdot (z+\zeta) d\xi d\eta,\ \frac{\partial U}{\partial z} = \ldots.]$$

8.22. Let $K \subset \mathbf{C}$ be compact and let μ be a positive measure on K with $\mu(K) = 1$. Prove:
(i) $U_\varepsilon(z) = \frac{1}{2}\int_K \log(|z-\zeta|^2 + \varepsilon^2) d\mu(\zeta),\ \varepsilon > 0$ is C^∞ subharmonic on $\mathbf{C}$:
(ii) $U(z) = \int_K \log|z-\zeta| d\mu(\zeta)$ is subharmonic on $\mathbf{C}$ and harmonic outside K.

8.23. Let $v(z) = \varphi(|z|)$ be a usc function on the annulus $A(0;\, \rho, R) \subset \mathbf{C}$ that depends only on $|z| = r$. Prove that $v(z)$ is subharmonic if and only if $\varphi(r)$ is a convex function of $\log r$. [For smooth φ, this is equivalent to saying that $d\varphi(r)/d\log r$ is nondecreasing.] Can you use the result to show that for arbitrary subharmonic v on $A(0;\, \rho, R)$, both

$$m(r) = m(z) = \sup_\theta\ v(e^{i\theta}z) \quad \text{and} \quad \overline{v}(0;r) = \overline{v}(0;z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(ze^{i\theta}) d\theta$$

are convex functions of $\log r$?

8.24. Let $f: D_1(\subset \mathbf{C}) \to D_2 \subset \mathbf{C}$ be holomorphic and let v be subharmonic on D_2. Prove that $v \circ f$ is subharmonic on D_1.

8.25. Extend Theorem 8.34 to $\mathbf{R}^n$, paying special attention to the case $n = 1$ (regularization of convex functions on $I \subset \mathbf{R}$).

8.26. Let v be subharmonic on $\Omega \subset \mathbf{R}^n$ and let g be a nondecreasing convex function on $\mathbf{R}$. Prove that $g \circ v$ is subharmonic on Ω.

8.27 Let v be locally integrable on Ω. In the theory of distributions the Laplacian Δv is defined by its action on test functions φ on Ω [C^∞ functions of compact support in Ω, cf. Chapter 11]:

$$\langle \Delta v, \varphi \rangle \stackrel{\text{def}}{=} \langle v, \Delta\varphi \rangle \stackrel{\text{def}}{=} \int_\Omega v\, \Delta\varphi.$$

One says that $\Delta v \geq 0$ on Ω in the sense of distributions if $\langle \Delta v, \varphi \rangle \geq 0$ for all test functions $\varphi \geq 0$ on Ω. Prove that a continuous function v on Ω is subharmonic if and only if $\Delta v \geq 0$ in this sense. [$\Delta v_\varepsilon(z) = \int_\Omega v(\zeta)\Delta\rho_\varepsilon(z-\zeta)dm(\zeta)$.]

8.28. Use the regularization of Theorem 8.34 to show that a continuous function with the mean value property is of class C^∞.

8.29. (**Upper envelopes of families of subharmonic functions**). Let $\{v_\lambda\},\ \lambda \in \Lambda$ be a family of subharmonic functions on Ω in $\mathbf{R}^n$ of $\mathbf{C}^n$ whose upper envelope V is locally bounded above. Prove:

(i) V is subharmonic if it is upper semi-continuous;
(ii) The usc regularization V^* of V is subharmonic [cf. exercise 8.8];
(iii) The regularizations V_ε are subharmonic and $\geq V$ [cf. the proof of Theorem 8.34];
(iv) $V_\varepsilon \geq V_\delta$ for $0 < \delta < \varepsilon$ [compare $\rho_\eta * V_\varepsilon$ and $\rho_\eta * V_\delta$];
(v) $\lim\limits_{\varepsilon \downarrow 0} V_\varepsilon = V^*$;
(vi) If the functions v_λ are psh, so is V^*.

8.30. Use Fubini's theorem to prove that a subharmonic function v also has the sub mean value property for balls (or discs if $n = 2$); if v is subharmonic on $\overline{B} = \overline{B}(a,R) \subset \mathbf{R}^n$, then $v(a) \leq \overline{v}_B(a;R)$, the average of v over the ball $B(a,R)$.

8.31. (**Hartog's lemma**). Let $\{v_k\}$ be a sequence of subharmonic functions on $\Omega \subset \mathbf{R}^n$ which is locally bounded above and such that $\limsup v_k(z) \leq A$ at every point $z \in \Omega$. Prove that for every compact subset $E \subset \Omega$ and $\varepsilon > 0$, there is an index k_0 such that $v_k < A + \varepsilon$ throughout E for all $k > k_0$. [Choose a "large" ball $\overline{B} = \overline{B}(a,R)$ in Ω. Use Fatou's lemma to show that $\limsup \int_B v_k \leq \int_B \limsup v_k$. Thus $\int_B v_k < (A + \frac{1}{2}\varepsilon)\operatorname{vol} B,\ \forall k > k_1$. Deduce an inequality for $v_k(z)$ at each point of a small ball $B(a,\delta)$.]

8.32. Let v be C^2 psh on a (connected) domain $D_2 \subset \mathbf{C}^p$ and let f be a holomorphic map from $D_1 \subset \mathbf{C}^n$ to D_2. Prove that $v \circ f$ is psh on D_1.

8.33. Prove that the following functions are strictly psh on $\mathbf{C}^n$;

(i) $|z|^2$;
(ii) $\log(|z|^2 + c^2), \qquad c > 0$;
(iii) $g(|z|^2)$ where g is a real C^2 function on $[0,\infty)$ such that $g' > 0$ and $g' + tg" > 0$.

8.34. Let D be the spherical shell $B(0, R - \overline{B}(0, \rho)$ in $\mathbf{C}^n$, $n \geq 2$. Prove that a usc function $v(z) = \varphi(|z|)$ that depends only on $|z| = r$ is psh on D if and only if $\varphi(r)$ is nondecreasing and convex as a function of $\log r$. [Set $\varphi(r) = g(r^2)$ and start with $g \in C^2$.]

8.35. Prove a Hadamard type "three spheres theorem" for holomorphic functions on a spherical shell in $\mathbf{C}^n$, $n \geq 2$. Do you notice a difference with the case $n = 1$? [Cf. exercise 8.16.]

8.36. Prove that a real C^2 function u on $\Omega \subset \mathbf{C}^n$ is pluriharmonic if and only if

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = 0 \quad \text{on} \quad \Omega, \qquad \forall j, k = 1, \ldots, n.$$

8.37. Prove that a pluriharmonic function u on the unit bidisc $\Delta_2(0, 1) \subset \mathbf{C}^2$ is equal to the real part of a holomorphic function f on Δ_2. [Show first that one has power series representations

$$\frac{\partial u}{\partial z_1} = \sum_{p \geq 1,\, q \geq 0} p a_{pq} z_1^{p-1} z_2^q, \; \frac{\partial u}{\partial z_2} = \sum_{p \geq 0,\, q \geq 1} q b_{pq} z_1^p z_2^{q-1},$$

then compare a_{pq} and b_{pq}. Can you now find f ?]

8.38. Prove that the circle $C(0, r)$ in $\mathbf{C}$ is non polar. Also show that the torus $T(0, r)$ in $\mathbf{C}^n$ is non pluripolar.

8.39. Prove that the square $-1 \leq x_1,\ x_2 \leq 1$ in $\mathbf{R}^2 \subset \mathbf{C}^2$ is non pluripolar. [Cf. 8.27 and exercise 1.16.] Extend to nonempty open subsets of $\mathbf{R}^n \subset \mathbf{C}^n$.

8.40. Let $K \subset \mathbf{C}^n$ be compact. Prove that $g_K(z) \geq \log |z|/R$ for some constant R, so that the Robin constant γ_K is always $> -\infty$ and $\operatorname{cap} K < +\infty$.

8.41. Let $K \subset \mathbf{C}$ be compact and such that there is a positive measure μ on K (with $\mu(K) = 1$) whose logarithmic potential U^μ is bounded above on $\mathbf{C}$, by M say. Prove that $g_K(z) \leq M - U^\mu(z)$ on $\mathbf{C}$ and that $\operatorname{cap} K \geq e^{-M}$.

8.42. Show that a compact line segment in $\mathbf{C}$ of length L has capacity $\frac{1}{4}L$.

8.43. Prove that the Green function $g_K(z)$ with pole at ∞ for the closed unit bidisc $\{|z_1| \leq 1,\ |z_2| \leq 1\}$ in $\mathbf{C}^2$ is equal to $\sup(\log^+ |z_1|, \log^+ |z_2|)$. Also treat the case of the torus $C(0, 1) \times C(0, 1)$ in $\mathbf{C}^2$! [Fix $z \neq 0$ with $|z_1| \geq |z_2|$ and consider the complex line $\zeta = wz$.]

8.44. (i) Prove that a psh function $V(w)$ on $E = \{w \in \mathbf{C}^2 : |w_1| \geq 1,\ |w_2| \geq 1\}$ which is majorized by 0 on the torus $T(0, 1)$ and by $\log |w| + \mathcal{O}(1)$ at ∞ is majorized by $\sup\{\log |w_1|,\ \log |w_2|\}$ throughout E. [First consider $v(w_1, e^{it})$ for $|w_1| > 1$, then $v(w_1, \lambda w_1)$ for $|w_1| > 1/|\lambda|$ where $|\lambda| \leq 1$, etc.
(ii) Prove that the Green function $g_K(z)$ for the closed square $-1 \leq x_1,\ x_2 \leq 1,\ y_1 = y_2 = 0$ in $\mathbf{C}^2$ is equal to $\sup\{g(z_1), g(z_2)\}$, where g is the Green function with pole at ∞

for the interval $[-1,1]$ in $\mathbf{C}$. [Taking both z_1 and z_2 outside $[-1,1]$, one can use (i) and a suitable holomorphic map. The cases where $z_1 = x_1 \in [-1,1]$ or $z_2 = x_2 \in [-1,1]$ may be treated separately.]

8.45. Let K be a compact polar subset of $\mathbf{C}$. Prove that $\operatorname{cap} K = 0$. [Use the fact that g_K is harmonic on $\mathbf{C} - K$ when $\operatorname{cap} K > 0$.] The converse is also true but more difficult. It may be derived with the aid of Hartogs' lemma, exercise 8.31.

8.46. Let D be a bounded domain in $\mathbf{R}^n$ with (piecewise) C^1 boundary and let f be a function of class C^1 on $\overline{D}$. Discuss the classical **Gauss-Green formula** for integration by parts:

$$\int_D \frac{\partial f}{\partial x_j}\, dm = \int_{\partial D} f N_{x_j} ds.$$

Here dm stands for volume element, ds for "area" element and N is the outward unit normal, N_{x_j} its component in the x_j direction.

8.47. Derive **Green's formula involving Laplacians**: for functions u and v of class C^2 on $\overline{D}$,

$$\int_D (u\Delta v - v\Delta u) dm = \int_{\partial D} (u\,\frac{\partial v}{\partial N} - v\,\frac{\partial u}{\partial N}) ds.$$

[Apply exercise 8.46 to $f = u\partial v/\partial x_j$ and $f = v\partial u/\partial x_j$, subtract, etc.]

8.48. (**Representation of smooth functions by potentials**). Let D be a bounded domain in $\mathbf{R}^n$ with (piecewise) C^1 boundary and let u be of class C^2 on $\overline{D}$. Prove that for $n > 2$:

$$\begin{aligned}(n-2)\sigma_n U(a) = &-\int_D |x-a|^{2-n}\Delta u(x) dm(x)\\ &+\int_{\partial D} \{|x-a|^{2-n}\frac{\partial u}{\partial N}(x) - u(x)\frac{\partial}{\partial N}|x-a|^{2-n}\} ds(x),\ \forall a \in D.\end{aligned}$$

Here $\sigma_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$ is the area of the unit sphere $S(0,1)$ in $\mathbf{R}^n$. For $n = 2, |x - a|^{2-n}$ has to be replaced by $\log 1/|x-a|$ and the constant $(n-2)\sigma_n$ by $\sigma_2 = 2\pi$. [Apply Green's formula to $D - \overline{B}(a,\varepsilon)$ and let $\varepsilon \downarrow 0$.]

8.49. (**Representation of smooth functions using the classical Green function with finite pole**). Let D be a smoothly bounded domain in $\mathbf{R}^n$. For $n > 2$, the Green function $g(x,a)$ with pole at $a \in D$ is defined by the following properties:

(i) $g(x,a)$ is continuous on $\overline{D} - \{a\}$ and harmonic on $D - \{a\}$;

(ii) $g(x,a) - |x-a|^{2-n}$ has a harmonic extension to a neighborhood of a;

(iii) $g(x,a) = 0$ for $x \in \partial D$. [For $n = 2$, $|x-a|^{2-n}$ in (ii) must be replaced by $\log 1/|x-a|$.] Assuming that the Green function exists and is of class C^2 on

$\overline{D} - \{a\}$, prove that for every C^2 function u on $\overline{D}$:

$$(n-2)\sigma_n u(a) = -\int\limits_D \Delta u(x) g(x,a) dm(x) - \int\limits_{\partial D} u(x) \frac{\partial g}{\partial N}(x,a) ds(x), \ \forall a \in D.$$

8.50. Prove that the ball $B(0,1) \subset \mathbf{R}^n$ has Green function

$$g(x,a) = \begin{cases} -\log|x-a| + \log(|a|\,|x-a'|) & \text{for } n=2; \\ \\ |x-a|^{2-n} - (|a|\,|x-a'|)^{2-n} & \text{for } n \geq 3. \end{cases}$$

Here a' is the reflection $|a|^{-2}a$ of a in the unit sphere $S(0,1)$ and $|a|\,|x-a'|$ is to be read as 1 for $a = 0$.

8.51. (**Poisson integral for the ball**). Derive the following integral representation for harmonic functions u on the closed unit ball $\overline{B}(0,1)$ in $\mathbf{R}^n$:

$$u(a) = 1\sigma_n \int\limits_{S(0,1)} u(x) \frac{1-|a|^2}{|x-a|^n} ds(x), \forall a \in N(0,1);$$

$$\sigma_n = 2\pi^{\frac{1}{2}n}/\Gamma(\tfrac{1}{2}n).$$

[For the calculation of $\partial g/\partial N$ one may initially set $x = r\tilde{x}$ with $\tilde{x} \in S$, so that $\partial/\partial N = \partial/\partial r$. Note for the differentiation that $|x-a|^2 = (x-a,\, x-a)$.]

8.52. Let D be a convex domain in $\mathbf{R}^n$ and let E be a nonempty open subset of the unit sphere S^{n-1}. For constant $C > 0$, we let $\mathcal{F} = \mathcal{F}(E,C)$ denote the family of all C^∞ functions f on D whose directional derivatives in the directions corresponding to E satisfy the inequalities

$$\sup_{\xi \in E} \frac{1}{m!} \Big| \left(\frac{d}{dt}\right)^m f(a+t\xi)\big|_{t=0} \Big| \leq C^m, \quad m = 0,1,2,\ldots$$

at each point $a \in D$. Prove that there is a neighborhood Ω of D in $\mathbf{C}^n$ to which all functions $f \in \mathcal{F}$ can be extended analytically. [Begin by showing that the power series for f with center a converges throughout the polydisc $\Delta = \Delta_n(a, \beta/C)$, with $\beta = \beta(E)$ as in Theorem 8.71. Does the series converge to f on $\Delta \cap D$?]

8.53. Extend the preceding result to the case where $\mathcal{F}$ consists of the **continuous** functions f on D which have directional derivatives in the directions corresponding to E that satisfy the conditions imposed in exercise 8.52. [Regularize $f \in \mathcal{F}$ and prove a convergence result for analytic extensions of the regularizations f_ε to a neighborhood of a in $\mathbf{C}^n$.]

8.54. Let E be any subset of the closed unit ball $\overline{B} = \overline{B}(0,1)$ in $\mathbf{C}^n$. Prove that there is a constant $\beta(E) > 0$ such that the inequalities (7b) hold for all polynomials $f(z)$ in $z = (z_1, \ldots, z_n)$ if and only if the set $K = \overline{E}_c$ has positive logarithmic capacity. Determine the optimal constant $\beta(E)$.

8.55. Let E be any subset of the closed unit ball $\overline{B} = \overline{B}(0,1)$ in $\mathbf{C}^n$ and let $K = \overline{E}_c$ be the closure of the circular subset E_c generated by E. We define $\alpha(E)$ as the largest nonnegative constant such that

$$\|q_m\|_B \leq \|q_m\|_E/\alpha(E)^m$$

for all $m \geq 1$ and all homogeneous polynomials q_m in $z = (z_1, \ldots, z_n)$n of degree m. Prove that

$$\|p_m\|_B \leq \|p_m\|_K/\alpha(E)^m$$

for all polynomials p_m of degree $\leq m$. Deduce that $\alpha(E)$ is equal to $\rho(\overline{E}_c)$, where ρ is the Siciak capacity defined in (7g). [Cf. the proof of Theorem 8.71.]

8.56. (Continuation). Let K be any compact circular subset of $\overline{B}(0,1) \subset \mathbf{C}^n$. Prove that $\rho(K) = \alpha(K)$ is equal to the radius of the largest ball $\overline{B}(0,r)$ that is contained in the polynomially convex hull $\tilde{K}$ of K.

8.57. Give a proof of the Sibony-Wong theorem, Theorem 8.72. [First consider $G(z) = F(rz)$ where F is a polynomial.]

8.58. (**Helgason's support theorem for Radon transforms**). Let $g(x)$ be a continuous function on $\mathbf{R}^n$ such that $|x^\alpha g(x)|$ is bounded for every multi-index $\alpha \geq 0$ and let $\hat{g}(\xi, \lambda)$ be its Radon transform, obtained by integration over the hyperplanes $x \cdot \xi = \lambda$:

$$\hat{g}(\xi, \lambda) = \int\limits_{x \cdot \xi = \lambda} g(x) ds(x), \quad (\xi, \lambda) \in S^{n-1} \times \mathbf{R}.$$

Prove that g has bounded support whenever $\hat{g}$ does. [Introduce the Fourier transform f of g; clearly $f \in L^2(\mathbf{R}^n)$. Supposing $\hat{g}(\xi, \lambda) = 0$ for $|\lambda| > R$ and all ξ,

$$f(t\xi) = \int\limits_{\mathbf{R}^n} g(x) e^{-it\xi \cdot x} dm(x) = \int\limits_{-R}^{R} \hat{g}(\xi, \lambda) e^{-it\lambda} d\lambda.$$

Now use the partial derivatives lemma to deduce that f can be extended to an entire function of exponential type on $\mathbf{C}^n$. By the so-called Paley-Wiener theorem (or Plancherel-Pólya theorem), such an $f \in L^2$ is the Fourier transform of a function of bounded support, hence $\operatorname{supp} g$ is bounded. For the present proof and an extension of Helgason's theorem, cf. [Wiegerinck 1985] Theorem 1.]

8.59 Let K be compact in $\mathbf{R}^n$, f usc and g lsc on K, $0 < f - g < \epsilon$ on K. Suppose that $\{h_n\}$ is a monotonically decreasing sequence of continuous functions, which converges pointwise to f on K. Prove that $\exists n_0$ with $h_{n_0} - f < 2\epsilon$ on K. Deduce Dini's theorem: if f is continuous on K and $h_n \in C(K) \downarrow f$, then $\{h_n\}$ converges uniformly.

8.60 (Sibony) Let

$$f(z) = (|z_1|^2 - 1/2)^2 = (|z_2|^2 - 1/2)^2 \text{ on } \partial B(0,1).$$

Show that $F_f(z) = \max\{(|z_1|^2 - 1/2)^2, (|z_2|^2 - 1/2)^2\}$ on $B(0,1) - \Delta(0, \frac{1}{2}\sqrt{2})$ and 0 elsewhere. How smooth is F_f?

8.61 Let D be the polydisc $\Delta_2(0,1)$. Show that there is in general no solution for the Dirichlet problem for M on D:

$$M(u) = 0 \text{ on } D, \quad u = f \text{ on } \partial D.$$

[Take $f = 0$ on $|z_2| = 1$, but not identically 0.]

8.62 (Continuation) Find a smoothly bounded pseudoconvex domain with the above property.

CHAPTER 9

Pseudoconvex domains and smooth plurisubharmonic exhaustion functions

Pseudoconvexity was introduced in Chapter 6 where it was shown that domains of holomorphy are pseudoconvex. Here we will further study pseudoconvexity, in particular we will construct smooth strictly plurisubharmonic exhaustion functions of arbitrarily rapid growth. This will be an important ingredient in the solution of the Levi problem in Chapter 11.

Next we will give other characterizations of pseudoconvexity, also in terms of behaviour of the boundary of the domain. The latter is done only after a review of the boundary behaviour of convex domains in terms of the Hessian of the defining function. For smooth pseudoconvex domains the complex Hessian of the defining function has to be positive semidefinite on the complex tangent space at any point of the boundary of the domain. Strict pseudoconvexity is introduced [now the complex Hessian has to be positive definite]. We shall see that this notion is locally biholomorphically equivalent to strict convexity.

9.1 Pseudoconvex domains. According to Definition 6.54, a domain or open set $\Omega \subset \mathbf{C}^n$ is pseudoconvex if the function

$$\log 1/d(z), \quad z \in \Omega, \quad d(z) = d(z, \partial\Omega)$$

is plurisubharmonic. In $\mathbf{C}$, every domain is pseudoconvex, cf. exercise 8.9. In $\mathbf{C}^n$ every convex domain is pseudoconvex. More generally, every domain of holomorphy is pseudoconvex [section 6.5]. A full proof of the converse has to wait until Chapter 11, but for some classes of domains the converse may be proved directly:

EXAMPLE 9.11. (Tube Domains). Let D be a connected tube domain

$$D = H + i\mathbf{R}^n = \{z = x + iy \in \mathbf{C}^n : \ x \in H, y \in \mathbf{R}^n\}$$

Here the *base* H is an arbitrary (connected) domain in $\mathbf{R}^n$. For which domains H will D be a domain of holomorphy?

It may be assumed that the connected domain D is pseudoconvex. Let $[x', x'']$ be any line segment in H; we may suppose without loss of generality that

$$x' = (0, 0, \ldots, 0), \quad x'' = (1, 0, \ldots, 0).$$

Now consider the complex line $z_2 = \cdots = z_n = 0$ through x' and x''. On the closed strip $S : 0 \leq \operatorname{Re} z_1 \leq 1$ in that complex line, the function

$$-\log d(z) = -\log d(z_1, 0, \ldots, 0) = v(z_1) = v(x_1 + iy_1)$$

will be subharmonic. Since the tube D and the strip S are invariant under translation in the y_1 direction, the function $v(z_1)$ must be independent of y_1. Hence v is a sublinear function of x_1 on S, cf. Example 8.35. Varying $[x', x'']$, it follows that $-\log d(x)$ is convex on H. We finally observe that for $x \in H$, $d(x)$ is equal to the boundary distance to ∂H: for $x \in H$, the nearest point of $\partial D = \partial H + i\mathbf{R}^n$ must belong to ∂H by Pythagoras's theorem. The convexity of $-\log d(x)$ now implies that H is convex, cf. exercises 6.7, 8. It follows that D is convex, hence D is a domain of holomorphy [Section 6.1].

A convex tube has a convex base. As final conclusion we have :

Theorem 9.12 (Bochner). *A connected tube domain is a domain of holomorphy if and only if its base is convex.*

Bochner proved more generally that the *hull of holomorphy* of an arbitrary connected tube domain $D = H + i\mathbf{R}^n$ is given by its convex hull, $\mathrm{CH}(D) = \mathrm{CH}(H) + i\mathbf{R}^n$, cf. [BoMa], [Hör]. An elegant proof may be based on the so-called prism lemma, cf. exercises 6.28 and 9.1.

Every pseudoconvex domain $\Omega \subset \mathbf{C}^n$ is *psh exhaustible*: it carries a (continuous) psh exhaustion function α, see Proposition 6.56. As before we will use the notation

$$\Omega_t = \{z \in \Omega : \alpha(z) < t\}, \quad t \in \mathbf{R} \tag{1a}$$

for the associated relatively compact subsets which jointly exhaust Ω. We use the notation

$$\Omega_1 \subset\subset \Omega_2$$

to express that the closure of Ω_1 is a compact subset of the interior of Ω_2. Thus $\Omega_t \subset\subset \Omega_{t+s}$ if $t, s > 0$.

For some purposes, notably for the solution of the $\overline{\partial}$ equation [Chapter 11], we need C^∞ strictly psh exhaustion functions β on Ω which increase rapidly towards the boundary. If one has just one C^∞ strictly psh exhaustion function α for Ω, one can construct others of as rapid growth as desired by forming compositions $\beta = g \circ \alpha$, where g is a suitable increasing convex C^∞ function on $\mathbf{R}$, cf. Example 8.45. Thus the problem is to obtain a first C^∞ psh exhaustion function!

Using regularization by convolution with an approximate identity ρ_ϵ as in Theorem 8.46, one may construct C^∞ psh majorants α_ϵ to a give psh function on $\mathbf{C}^n$. Unfortunately, for given α on a domain $\Omega \neq \mathbf{C}^n$, the function α_ϵ is defined and psh only on the ϵ-contraction Ω_ϵ of Ω. To overcome this difficulty we proceed roughly as follows. For a given psh exhaustion function α on Ω consider the function $v = |z|^2 + \alpha$ on Ω and the exhausting domains $\Omega_t = \{v < t\}$ associated to it. Given any $\tau = (t_1, t_2, t_3, t_4)$, $t_j > 0$, sufficiently large and strictly increasing we can construct a basic building block β_τ which has the following properties: $\beta_\tau \in C^\infty(\Omega)$, $\mathrm{supp}\beta_\tau \subset \Omega_{t_4} \setminus \Omega_{t_1}$, β_τ is psh on Ω_{t_3} and strictly psh on $\Omega_{t_3} \setminus \Omega_{t_2}$. For suitable choice of quadruples τ^k, and $M_k >> 0$ the sum

$$\beta = \sum_k M_k \beta_{\tau^k}$$

will be locally finite (hence smooth) and strictly plurisubharmonic.

The above ideas will be worked out in Section 9.2 to construct the special C^∞ psh exhaustion functions that are required for the solution of the $\bar\partial$ problem and (thus) the Levi problem, cf Chapter 11.

It is always good to keep in mind that in the final analysis, domains of holomorphy are the same as pseudoconvex domains: certain properties are much easier to prove for pseudoconvex domains than for domains of holomorphy. In particular the results of Section 9.3 will carry over to domains of holomorphy. As another useful example of this we have the following

Theorem 9.13.
(i) The interior Ω of the intersection of a family of pseudoconvex domains $\{\Omega_j\}_{j\in J}$ is pseudoconvex.
(ii) The union Ω of an increasing sequence of pseudoconvex domains $\{\Omega_j\}_{j\in\mathbf{N}}$ is pseudoconvex.

[(ii) may be stated for families that are indexed by linearly ordered sets too.]

PROOF. (i): Let d_j denote the boundary distance for Ω_j and d the boundary distance for Ω. Then clearly on Ω we have $d(z) = \inf d_j(z)$. Hence $-\log d(z) = \sup -\log d_j(z)$, and, as $-\log d(z)$ is continuous, it follows from Properties 8.42. that it is plurisubharmonic.
(ii): Observing that $d_j(z) \le d_{j+1}(z)$ and $d_j(z) \uparrow d(z)$. It follows that $-\log d(z)$ is the limit of the decreasing sequence of psh functions $-\log d_j(z)$ and by Theorem 8.42 is a psh function. [The fact that d_j is not defined on all of Ω poses no problem: Every $z \in \Omega$ has a neighborhood $U \subset \Omega_j$ for large enough j and on U we may let the sequence start at j.]

9.2 Special C^∞ functions of rapid growth. We will prove the following important result:

Theorem 9.21. *Let $\Omega \subset \mathbf{C}^n$ be psh exhaustible and let α be a (continuous) psh exhaustion function for Ω. Furthermore, let m and μ be locally bounded real functions on Ω and let $K \subset \Omega$ be compact. Then*
(i) Ω possesses a C^∞ strictly psh exhaustion function $\beta \ge \alpha$.
(ii) More generally there is a C^∞ function $\beta \ge m$ on Ω whose complex Hessian has smallest eigenvalue $\lambda_\beta \ge \mu$ throughout Ω.
(iii) Finally, if α is nonnegative on Ω and zero on a neighborhood of K and if m and μ vanish on a neighborhood N of the zero set $Z(\alpha)$ of α, there is a function β as in (ii) which vanishes on a neighborhood of K.

PROOF. The first statement follows from the second by taking $m = \alpha$ and $\mu > 0$: any continuous function $\beta \ge \alpha$ will be an exhaustion function. The second statement follows from the third by taking K and N empty. We thus turn to the third statement and proceed by constructing the building blocks announced in the previous Section.

Consider the function $v = c|z|^2 + \alpha$ on Ω, $c > 0$ and the exhausting domains $\Omega_t = \{v < t\}$. If $t_1 < t_2$

$$\Omega_{t_1} \subset\subset \Omega_{t_2},$$

if these sets are non-empty. Let $\tau = (t_1, t_2, t_3, t_4)$, $t_j > 0$, sufficiently large and strictly increasing. Choose auxiliary numbers t_5, t_6, with $t_1 < t_5 < t_2$, $t_3 < t_6 < t_4$. We assume that the corresponding Ω_{t_j} are non empty. Consider the function

$$v_\tau = \chi(z) \cdot \max\{v(z) - t_5, 0\},$$

where χ is the characteristic function of Ω_{t_6}. The function v_τ is plurisubharmonic on suppχ and its support is contained in suppχ intersected with the complement of Ω_{t_5}. Let

$$\eta < \frac{1}{2} \min\{d(\partial\Omega_{t_j}, \partial\Omega_{t_k}) : \quad 1 \leq j < k \leq 6\}.$$

We then form the regularization $\beta_\tau = v_\tau * \rho_\eta$, ρ_η belonging to a radial approximate identity and supported on $B(0, \eta)$. This function will be a nonnegative C^∞ function with support in $\Omega_{t_4} \setminus \Omega_{t_1}$. Next β_τ will be psh on Ω_{t_3}, because here it is a smoothened out psh function and it will be strictly psh and strictly positive on $\Omega_{t_3} \setminus \Omega_{t_2}$, because here it can be written as

$$\rho_\eta * |z|^2 + \rho_\eta * (\alpha - t_5)$$

and the first term will have a strictly positive complex Hessian.

Now we choose a sequence $t_j \uparrow \infty$, $j = 1, 2, \ldots$, and form the quadrupels $\tau^j = (t_j, t_{j+1}, t_{j+2}, t_{j+4})$ and corresponding functions β_j. We choose the constant c in the definition of v so small that for a (small) positive t_1 we have $K \subset \Omega_{t_1} \subset N$. We shall choose a sufficiently rapidly increasing sequence of positive numbers $\{M_j\}$ and form

$$\beta(z) = \sum_k M_k \beta_k(z). \tag{2a}$$

For each $z \in \Omega$ there exists a k such that a neigborhood $B(z, r)$ is contained in $\Omega_{t_{k+1}} \setminus \Omega_{t_{k-1}}$. Hence $B(z, r)$ is contained in the support of at most 5 β_k's. It follows that (2a) is a locally finite sum, hence β is well defined and smooth for every choice of M_k.

How to choose M_k? We can choose M_1 such that on Ω_{t_2} the inequalities $M_1\beta_1 > m$ and $\lambda_{M_1\beta_1} > \mu$ hold [and on Ω_{t_1} "everything" vanishes]. Suppose now that we have chosen $M_1, \ldots, M_{k-1}$ such that $\sum_{j=1}^{k-1} M_j \beta_j(z)$ satisfies the requirements of the theorem on Ω_k. As β_k is nonnegative, psh on Ω_{k+1} and positive, strictly psh on $\Omega_{k+1} \setminus \Omega_k$, we can choose $M_k >> 0$ such that $\sum_{j=1}^{k-1} M_j \beta_j(z) + M_k \beta_k(z)$ will have values and Hessian on $\Omega_{k+1} \setminus \Omega_k$ as required. On the rest of Ω_{k+1} the new sum will still meet the requirements. With the M_k as constructed, β will be the function we are looking for: it solves our problem on all Ω_k, hence also on Ω.

9.3 Characterizations of pseudoconvex domains. The following exposition parallels the one for domains of holomorphy in section 6.3-6.5.

DEFINITION 9.31. Let Ω be a domain in $\mathbf{C}^n$, $K \subset \Omega$ nonempty and compact. The *plurisubharmonically* or *psh convex hull* of K relative to Ω is the set

$$\hat{K}^{psh} = \hat{K}^{psh}_\Omega = \{z \in \Omega : \ v(z) \leq \sup_K v(\zeta), \text{ for all psh functions } v \text{ on } \Omega\}.$$

Ω is called *psh convex* if for every compact subset K, the psh convex hull $\hat{K}^{psh}$ has positive boundary distance (or compact closure) in Ω.

$\hat{K}^{psh}$ will be bounded: think of $v(z) = |z|^2$. However, since psh functions need not be continuous, $\hat{K}^{psh}$ might fail to be closed in Ω.

PROPERTIES 9.32.

(i) The psh convex hull $\hat{K}^{psh}$ is contained in the holomorphically convex hull $\hat{K} = \hat{K}_\Omega$: if $v(z) \leq \sup_K v$ for some point $z \in \Omega$ and all psh functions v on Ω, then in particular

$$\log|f(z)| \leq \sup_K \log|f|, \quad \forall f \in \mathcal{O}(\Omega),$$

hence $z \in \hat{K}$. [If Ω is psh convex then $\hat{K}^{psh} = \hat{K}$, cf. [Hör 1], [Ran].]

(ii) Every analytic disc $\bar{\Delta}$ in Ω is contained in the psh convex hull of the edge $\Gamma = \partial\bar{\Delta}$ Example 6.33. Indeed, let $\bar{\Delta} = \varphi(\bar{\Delta}_1)$ with φ continuous on $\bar{\Delta}_1 \subset \Omega$ and holomorphic on Δ_1, and let v be psh on Ω. Then $v \circ \varphi$ is subharmonic on Δ_1 and usc on $\bar{\Delta}_1$, hence by the maximum principle $v \circ \varphi$ is bounded above by its supremum on $C(0,1)$.

We will need the following continuity property for analytic discs relative to psh convex domains:

Proposition 9.33. *Let $\Omega \subset \mathbf{C}^n$ be psh convex and let $\{\bar{\Delta}_\lambda\}$, $0 \leq \lambda \leq 1$ be a family of analytic discs in $\mathbf{C}^n$ which vary continuously with λ, that is, the defining map*

$$\varphi_\lambda(w) = \varphi(w, \lambda) : \bar{\Delta}_1(0,1) \times [0,1] \to \mathbf{C}^n$$

is continuous, while of course $\varphi_\lambda(w)$ is holomorphic on $\{|w| < 1\}$ for each λ. Suppose now that $\bar{\Delta}_0$ belongs to Ω and that $\Gamma_\lambda = \partial\bar{\Delta}_\lambda$ belongs to Ω for each λ. Then $\bar{\Delta}_\lambda$ belongs to Ω for each λ.

REMARK. It follows from exercise 6.26 that there is a corresponding continuity property for analysic discs relative to domains of holomorphy.

PROOF of the proposition. The set $E = \{\lambda \in [0,1] : \ \bar{\Delta}_\lambda \subset \Omega\}$ is nonempty and open. The subset $S = \cup_{0 \leq \lambda \leq 1} \Gamma_\lambda$ of Ω is compact: S is the image of a compact set under a continuous map. Hence by the hypothesis, the psh convex hull $\hat{S}^{psh}$ has compact closure in Ω. By property 9.32-ii,

$$\bar{\Delta}_\lambda \subset \hat{\Gamma}_\lambda^{psh} \subset \text{clos}\hat{S}^{psh}$$

whenever $\bar{\Delta}_\lambda \subset \Omega$, that is whenever $\lambda \in E$. Suppose now that $\lambda_k \to \mu$, where $\{\lambda_k\} \subset E$. Then since $\bar{\Delta}_\lambda$ depends continuously on λ, also $\bar{\Delta}_\mu$ belongs to clos $\hat{S}^{psh} \subset \Omega$, that is $\mu \in E$. Thus E is closed. Conclusion: $E = [0,1]$.

Theorem 9.34. *The following conditions on a domain $\Omega \subset \mathbf{C}^n$ are equivalent:*

(i) Ω *is* PSEUDOCONVEX, *that is the function $-\log d(z) = -\log d(z, \partial\Omega)$ is plurisubharmonic on Ω;*

(ii) Ω *is* LOCALLY PSEUDOCONVEX*: every point $b \in \bar{\Omega}$ has a neighborhood U in $\mathbf{C}^n$ such that the open set $\Omega' = \Omega \cap U$ is pseudoconvex;*

(iii) Ω *is* PSH EXHAUSTIBLE, *that is* Ω *has a psh exhaustion function* α [*Definition 6.23*]
(iv) Ω *carries* C^∞ STRICTLY PSH FUNCTIONS β *of arbitrarily rapid growth towards the boundary* [*cf. Definition 8.44*]
(v) Ω *is* PSH CONVEX [*Definition 9.31*]

PROOF. (i)$\Rightarrow$(ii); a ball is pseudoconvex and the intersection Ω' of two pseudoconvex domains Ω_1 and Ω_2 is pseudoconvex by Theorem 9.13.

(ii)$\Rightarrow$(iii). Take $b \in \partial\Omega$ and U and Ω' as in (ii). Then $d(z) = d'(z)$ for all $z \in \Omega$ close to b. Thus the function$-\log d$ on Ω is psh on some neighborhood of every point of $\partial\Omega$. Hence there is a closed subset $F \subset \Omega$ such that $-\log d$ is psh on $\Omega \setminus F$.

Suppose first that Ω is bounded. Then $-\log d$ is bounded above on F, by M say. One may now define a psh exhaustion function for Ω by setting

$$\alpha = \max(-\log d, M+1).$$

Indeed, $\alpha = M+1$ on a neighborhood of F, hence α is a maximum of continuous psh functions on some neighborhood of each point of Ω. That α is an exhaustion function is clear.

If Ω is unbounded, one may first determine a psh exhaustion function v for $\mathbf{C}^n$ that is larger than $-\log d$ on F. Take $v(z) = g(|z|^2)$ where G is a suitable increasing convex C^2 function on $\mathbf{R}$, or use Theorem 9.21 (iii). A psh exhaustion function for Ω is then obtained by setting

$$\alpha = \sup(-\log d, v).$$

(iii)$\Rightarrow$(iv): see Theorem 9.21.

(iii) or (iv)$\Rightarrow$(v). Let α be a psh exhaustion function for Ω and define subsets Ω_t as in Theorem 9.13. Now take any nonempty compact subset $K \subset \Omega$ and fix $s > M = \sup_K \alpha$. Then $\alpha(z) \le M < s$ for any $z \in \hat{K}^{psh}$, hence $\hat{K}^{psh} \subset \Omega_s$. Thus $\hat{K}^{psh}$ has positive boundary distance in Ω, that is Ω is psh convex.

(v)$\Rightarrow$(i). Let Ω be psh convex. Starting out as in the proof of Theorem 6.55 we choose $a \in \Omega$, $B(a,R) \subset \Omega$ and $\zeta \in \mathbf{C}^n$ with $0 < |\zeta| < R$, so that the flat analytic disc $\bar\Delta = \{z \in \mathbf{C}^n : \; z = a + w\zeta, |w| \le 1\}$ belongs to Ω. Setting $v(z) = -\log d(z)$ we have to prove the mean value inequality $v(a) \le \bar v(a;\zeta)$. To that end we get ready to apply Lemma 6.53 to the continuous real function

$$f(w) \overset{\text{def}}{=} v(a+w\zeta) = -\log d(a+w\zeta), \quad w \in \bar\Delta_1(0,1).$$

Accordingly, let $p(w)$ be any polynomial in w such that

$$\text{(3a)} \qquad \begin{aligned} &\mathrm{Re}\, p(w) \ge f(w) = -\log d(a+w\zeta) \\ &\qquad\qquad \text{or} \quad d(a+w\zeta) \ge |e^{-p(w)}| \text{ on } C(0,1). \end{aligned}$$

We now choose an arbitrary vector $\tau \in \mathbf{C}^n$ with $|\tau| < 1$ and introduce the family of analytic discs

$$\bar\Delta_\lambda = \{z = a + w\zeta + \lambda e^{-p(w)}\tau : \; |w| \le 1\}, \; 0 \le \lambda \le 1.$$

It is clear that $\bar{\Delta}_\lambda$ varies continuously with λ and that $\bar{\Delta}_0 = \bar{\Delta} \subset \Omega$. Furthermore the boundary Γ_λ of $\bar{\Delta}_\lambda$ will belong to Ω for each $\lambda \in [0,1]$. Indeed it follows from (3a) that

$$\begin{aligned} d(a + w\zeta + \lambda e^{-p(w)}\tau) &\geq d(a + w\zeta) - |\lambda e^{-p(w)}\tau| \\ &\geq d(a + w\zeta) - |e^{-p(w)}||\tau| > 0, \quad \forall (w,\lambda) \in C(0,1) \times [0,1]. \end{aligned}$$

The continuity property for analytic discs [Proposition 9.33] now shows that $\bar{\Delta}_\lambda \subset \Omega$, $\forall \lambda \in [0,1]$. Taking $\lambda = 1$ and $w = 0$, we find in particular that

$$a + e^{p(0)}\tau \in \Omega.$$

This result holds for every vector $\tau \in \mathbf{C}^n$ of length $|\tau| < 1$, hence Ω must contain the whole ball $B(a, |e^{p(0)}|)$. In other words, $d(a) \geq |e^{p(0)}|$ or

$$\text{(3}b\text{)} \qquad \mathrm{Re}\, p(0) \geq -\log d(a) = f(0).$$

Summing up: (3a) always implies (3b) so that f has property (Π) of Lemma 6.53. Conclusion:

$$f(0) \leq \bar{f}(0;1) \text{ or } v(a) \leq \bar{v}(a;\zeta).$$

Thus $v = -\log d$ is psh: we have (i).

COROLLARY 9.35. The intersection of a pseudoconvex domain with (affine) complex hyperplanes are also pseudoconvex (as domain in the hyperplanes). [Cf. Proposition 6.56.]

9.4 The boundary of a pseudoconvex domain. By Theorem 9.34, pseudoconvexity of a domain is a local property of the boundary. But how can one tell from the local behavior of $\partial\Omega$ if $\Omega \subset \mathbf{C}^n$ is pseudoconvex? One may first ask more simply how one can tell from local boundary behavior if a domain is convex.

We assume here that Ω is smoothly bounded and discuss boundary smoothness in real coordinates, say for $\Omega \subset \mathbf{R}^n$.

DEFINITION 9.41. We say that the boundary $\partial\Omega$ is of class C^p, $(1 \leq p \leq \infty)$ at $b \in \partial\Omega$ if Ω has a local DEFINING FUNCTION around b of class C^p. This is a real function ρ defined on a neighborhood U of b such that

$$\Omega \cap U = \{x \in U : \rho(x) < 0\} \text{ and } d\rho(x) \neq 0 \text{ or grad } \rho(x) \neq 0), \forall x \in U.$$

One calls $\partial\Omega$ of class C^p if it is of class C^p at each of its (finite) points.

EXAMPLES 9.42. The function $\rho(x) = |x|^2 - 1$ is a global C^∞ defining function for the ball $B(0,1)$. For the unit polydisc $\Delta_n(0,1) \subset \mathbf{C}^n$, the function $\rho(z) = |z_n|^2 - 1$ is a C^∞ defining function for the part $\Delta_{n-1}(0,1) \times C(0,1)$ of the boundary.

On a small neighborhood U of a point b where $\partial\Omega$ is of class C^p ($p \geq 2$), the following signed boundary distance function provides a defining function $\rho \in C^p$:

$$\text{(4}a1\text{)} \qquad \rho(x) = \begin{cases} -d(x, \partial\Omega), & x \in \bar{\Omega} \cap U; \\ d(x, \partial\Omega), & x \in U \setminus \bar{\Omega}. \end{cases}$$

For the verification one may use the local boundary representation $x_n = h(x')$ indicated below, cf. exercise 9.2 and [Kran].

By translation and rotation one may assume in Definition 9.41 that $b = 0$ and that $\operatorname{grad} \rho|_0 = (0, \ldots, 0, \lambda)$ where $\lambda > 0$. Thus with $x = (x', x_n)$,

$$(4a2) \qquad \rho(x', x_n) = \lambda x_n + g(x', x_n), \text{where } g \in C^p \text{ and } g(0) = dg|_0 = 0.$$

By the implicit function theorem there is then a local boundary representation $x_n = h(x')$ with $h \in C^p$ and $h(0) = dh|_0 = 0$. One may finally take $\tilde{x}_n = x_n - h(x')$ as a new nth coordinate so that locally $\partial\Omega = \{\tilde{x}_n = 0\}$ and $\rho(\tilde{x}) = \tilde{x}_n$ is a local defining function.

Lemma 9.43. *Any two local defining functions ρ and σ of class C^p around $b \in \partial\Omega$ are related as follows:*

$$(4b) \qquad \sigma = \omega\rho \quad \text{with } \omega > 0 \text{ of class } C^{p-1}, \ d\sigma = \omega d\rho \text{ on } \partial\Omega.$$

PROOF. Taking $\rho(x) = x_n$ one has by (4a2) applied to σ:

$$\sigma(x) = \lambda x_n + g(x', x_n), \text{with } g(x', 0) = 0,$$

$\lambda > 0$, $g \in C^p$ and $dg|_0 = 0$. One may write g in the form

$$g(x', x_n) = \int_0^{x_n} \frac{\partial g}{\partial x_n}(x', s)\, ds = x_n \int_0^1 \frac{\partial g}{\partial x_n}(x', t x_n)\, dt;$$

the final integral defines a function of class C^{p-1} around 0. On $\partial\Omega$ this function equals $\partial g/\partial x_n(x', 0)$ and there also $dg = \partial g/\partial x_n(x', 0)\, dx_n$.

DEFINITION 9.44. Let ρ be a C^p defining function for $\partial\Omega$ around b. Departing somewhat from the language of elementary geometry, the (real) linear space

$$(4c) \qquad T_b(\partial\Omega) = \{\xi \in \mathbf{R}^n : \sum_1^n \frac{\partial \rho}{\partial x_j}(b)\xi_j = 0\}$$

of real TANGENT VECTORS at b is called the (real) TANGENT SPACE to $\partial\Omega$ at b. [By (4b) it is independent of the choice of defining function.]

Suppose for the moment that $\Omega \subset \mathbf{R}^n$ is *convex* with C^2 boundary. Then the function $v = -\log d$ is convex on Ω and smooth near $\partial\Omega$, say on $\Omega \cap U$. It follows that the Hessian form v is positive semidefinite there cf. (8.4a). A short calculation thus gives the inequality

$$-\frac{1}{d}\sum_{i,j=1}^n \frac{\partial^2 d(x)}{\partial x_i \partial x_j}\xi_i\xi_j + \frac{1}{d^2}\sum_j \frac{\partial d(x)}{\partial x_j}\xi_j \sum_k \frac{\partial d(x)}{\partial x_k}\xi_k \geq 0, \ x \in \Omega \cap U, \ \xi \in \mathbf{R}^n.$$

We now do three things: we introduce the defining function ρ of (4a1) [which equals $-d$ on $\Omega \cap U$ for suitable U], we limit ourselves to what are called tangent vectors ξ at x, that is $\sum \frac{\partial\rho}{\partial x_j}(x)\xi_j = 0$ [which removes the second term above] and we finally pass to the boundary point b of Ω by continuity. The result is

$$(4d) \qquad \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial x_i \partial x_j}(b)\xi_i\xi_j \geq 0, \quad \forall \xi \in T_b(\partial\Omega),\ \forall b \in \partial\Omega.$$

One can show that this condition is independent of the C^2 defining function that is used. [This follows immediately from (4b) in the case of a C^3 boundary and defining functions, but requires some care in general, cf. [Kran], p.102.]

A domain $\Omega \subset \mathbf{R}^n$ with C^2 boundary is called strictly convex at b if the quadratic form in (4d) is strictly positive for $\xi \neq 0$ in $T_b(\partial\Omega)$. There will then be a small ball B around around b such that $\Omega \cap B$ is convex, moreover there exists a large ball B' such that $\Omega \cap B \subset B'$ and $b = \partial(\Omega \cap B) \cap \partial B'$, cf. [Kran].

One can do something similar to the preceding in the case of a *pseudoconvex* domain $\Omega \subset \mathbf{C}^n$ with C^2 boundary. Now the function $v = -\log d$ is psh on Ω and smooth on $\Omega \cap U$. The complex Hessian form of v will be positive semidefinite there [Proposition 8.43]:

$$-\frac{1}{d}\sum_{i,j=1}^{n} \frac{\partial^2 d(z)}{\partial z_i \partial \bar{z}_j}\zeta_i\bar{\zeta}_j + \frac{1}{d^2}\sum_j \frac{\partial d(z)}{\partial z_j}\zeta_j \sum_k \frac{\partial d(z)}{\partial \bar{z}_k}\bar{\zeta}_k \geq 0,\ z \in \Omega \cap U,\ \zeta \in \mathbf{C}^n.$$

Again introducing the defining function ρ of (4a), it is natural to limit oneself to what will be called *complex tangent vectors* ζ to $\partial\Omega$ at z, which are given by

$$(4e) \qquad \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(z)\zeta_j = 0.$$

Passing to the boundary, we find this time that

$$(4f) \qquad \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(b)\zeta_i\bar{\zeta}_j \geq 0, \quad \forall \zeta \text{ as in (4e) with } z = b, \forall b \in \partial\Omega.$$

One can show as before that the condition is independent of the defining function that is used.

One will of course ask what condition (4e) means in terms of the underlying space $\mathbf{R}^{2n}$. Let us write $z_j = x_j + iy_j$, $\zeta_j = \xi_j + i\eta_j$ and carry out the standard identification $z = (x_1, y_1, \ldots)$, $\zeta = (\xi_1, \eta_1, \ldots)$. Then (4e) becomes

$$(4e') \qquad \sum_1^n \left(\frac{\partial\rho}{\partial x_j}\xi_j + \frac{\partial\rho}{\partial y_j}\eta_j\right) = 0, \quad \sum_1^n \left(\frac{\partial\rho}{\partial x_j}\ \eta_j - \frac{\partial\rho}{\partial y_j}\xi_j\right) = 0,$$

where we evaluate the derivatives at $b \in \partial\Omega$. The first condition (4e′) expresses that ζ [or rather, its real representative] is perpendicular to the gradient grad$\rho|_b$ in $\mathbf{R}^{2n}$, cf. (4c), hence ζ belongs to the real tangent space $T_b(\partial\Omega)$. The second condition says that $-i\zeta = (\eta_1, -\xi_1, \ldots)$ also belongs to $T_b(\partial\Omega)$. Interpreting $T_b(\partial\Omega)$ as a subset of $\mathbf{C}^n$, this means that ζ belongs to $iT_b(\partial\Omega)$.

DEFINITION 9.45. The complex linear subspace

$$T_b^{\mathbf{C}}(\partial\Omega) \stackrel{\text{def}}{=} T_b(\partial\Omega) \cap iT_b(\partial\Omega) = \{\zeta \in \mathbf{C}^n : \sum \frac{\partial\rho}{\partial z_j}(b)\zeta_j = 0\}$$

of $\mathbf{C}^n$ is called the *complex tangent space* to $\partial\Omega$ at b. Its elements are *complex tangent vectors.*

As a subset of $\mathbf{R}^{2n}$, $T_b^{\mathbf{C}}(\partial\Omega)$ is a (2n-2)-dimensional linear subspace which is closed under [the operation corresponding to] multiplication by i on $\mathbf{C}^n$. Cf. exercise 2.7.

REMARK 9.46. Instead of the submanifold $\partial\Omega$ we can consider any (real) submanifold S of $\mathbf{C}^n = \mathbf{R}^{2n}$ and form its tangent space $T_b(S)$. Indeed, if S is defined locally by $\rho_1 = \rho_2 = \cdots = \rho_m = 0$, one may define

$$T_b(S) = \{\xi \in \mathbf{R}^{2n} : \sum_{j=1}^{n} \frac{\partial\rho_k}{\partial x_j}(b)\xi_j = 0,\ k = 1, \dots, m\}.$$

Subsequently, we may define the complex tangent space to S at b:

$$T_b^{\mathbf{C}}(S) = T_b(S) \cap iT_b(S).$$

Again this is a complex linear subspace of $\mathbf{C}^n$. One may check that S is a complex submanifold of $\mathbf{C}^n$ if and only if

$$T_b^{\mathbf{C}}(S) = T_b(S) = iT_b(S).$$

On the other hand $T_b^{\mathbf{C}}(S)$ may equal $\{0\}$ for all $b \in S$. Such manifolds are called totally real, the typical example being $\mathbf{R}^n + i\{0\} \subset \mathbf{C}^n$.

DEFINITION 9.47. A domain $\Omega \subset \mathbf{C}^n$ with C^2 boundary is said to be *Levi pseudoconvex* if condition (4f) holds for a certain (or for all) C^2 defining function(s) ρ. Ω is called *strictly (Levi) pseudoconvex* at $b \in \partial\Omega$ if for some local defining function $\rho \in C^2$,

$$\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(b)\zeta_i\bar{\zeta}_j > 0 \quad \text{for all } \zeta \neq 0 \text{ in } T_b^{\mathbf{C}}(\partial\Omega).$$

By our earlier computation, every pseudoconvex domain $\Omega \subset \mathbf{C}^n$ with C^2 boundary is Levi pseudoconvex. For bounded Ω the converse is also true, see below for the strictly pseudoconvex case and compare [Kran], [Ran].

The ball is strictly pseudoconvex, the polydisc $\Delta_n(0,1)$ is not (unless $n = 1$). Indeed, at the points of the distinguished boundary the polydisc is not C^2, at the other boundary points the complex Hessian vanishes. More generally, it is easy to see that if there is an at least strictly one dimensional analytic variety passing through b and contained in $\partial\Omega$ then Ω is not strictly pseudoconvex at b.

How are pseudoconvexity and convexity related? Let Ω be strictly convex at $b \in \partial\Omega$ and let ρ be a defining function at ρ. We may perform an affine change of coordinates and assume that $b = 0$, $\frac{\partial\rho}{\partial z_n} = 1$, $\frac{\partial\rho}{\partial z_j} = 0$, $1 \le j < n$. Thus $T_b(\partial\Omega) = \{\operatorname{Re} z_n = 0\}$ and with $z_j = x_j + iy_j$ we may expand ρ in a Taylor series around 0:

$$\begin{aligned}\rho(x,y) = x_n + \frac{1}{2}\Bigg(&\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial x_i \partial x_j}(0)x_i x_j + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial x_i \partial y_j}(0)x_i y_j \\ &+ \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial y_i \partial x_j}(0)y_i x_j + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial y_i \partial y_j}(0)y_i y_j\Bigg) + o(|(x,y)|^2).\end{aligned} \tag{4g}$$

We rewrite this in terms of z and $\bar z$. Thus after an elementary computation we find:

$$\begin{aligned}\rho(z,\bar z) &= 1/2(z_n + \bar z_n) + \frac{1}{2}\Bigg(\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0)z_i z_j + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial \bar z_i \partial \bar z_j}(0)\bar z_i \bar z_j\Bigg) \\ &\qquad + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar z_j}(0)z_i \bar z_j + o(|z|^2) \\ &= \operatorname{Re} z_n + \operatorname{Re}\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0)z_i z_j + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar z_j}(0)z_i \bar z_j + o(|z|^2).\end{aligned} \tag{4g'}$$

Strict convexity of Ω at 0 is equivalent to positive definiteness of the quadratic part $Q(z)$ of (4g), (4g$'$), that is, $Q(z) \ge c|z|^2$, $c > 0$. Substituting iz for z in the quadratic part of (4g$'$) and adding we find that

$$Q(z) + Q(iz) = 2\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar z_j}(0)z_i \bar z_j \ge 2c|z|^2.$$

Thus strict convexity implies strict pseudoconvexity. In the other direction one can not expect an implication, but the next best thing is true:

Lemma 9.48. (Narasimhan) *Let Ω be strictly pseudoconvex at $b \in \partial\Omega$. Then there is a (local) coordinate transformation at b such that in the new coordinates Ω is strictly convex at b.*

PROOF. We may assume that $b = 0$ and that the defining function ρ of ω has the form (4g$'$). We introduce new coordinates:

$$\begin{aligned} z'_j &= z_j, \quad j = 1, \dots, n-1, \\ z'_n &= z_n + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0)z_i z_j,\end{aligned}$$

and thus $z_n = z'_n + O(|z'|^2)$. In the new coordinates Ω is given by the defining function ρ' which has the following Taylor expansion at 0

$$\rho'(z') = \rho(z(z')) = \operatorname{Re} z'_n + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar z_j}(0)z'_i \bar z'_j + o(|z|^2).$$

We have obtained that in $'$ coordinates Ω is strictly convex at 0.

REMARK. Note that the proof show that at a strictly pseudoconvex boundary point $b \in \partial\Omega$ there exists an at most quadratic polynomial, the so called *Levi polynomial*, $P(z)$ with the property that $P(b) = 0$ and for a small neigborhood U of b, $\mathrm{Re} P(z) < 0$ on $U \cap \bar{\Omega} \setminus \{b\}$. In the notation of the proof the previous lemma $P(z) = \frac{\partial\rho}{\partial z_n}(0) z_n + \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0) z_i z_j$.

Of course one can perform a similar process if $\partial\Omega$ is only pseudoconvex at b. The result will be that one can make the quadratic part of the defining function positve semidefinite. This, of course doesn't guarantee local convexity. Nevertheless it came as a big surprise when Kohn and Nirenberg [KoNi] discovered that there exist smoothly bounded pseudoconvex domains that are *not* locally biholomorphically equivalent to convex domains. See also [Kran] for a more detailed account and [FoSi] for what may be achieved with elaborate changes of coordinates.

Next one may ask how one can relate the defining function to a psh exhaustion function.

Theorem 9.49. *Suppose that Ω is a strictly pseudoconvex domain in $\mathbf{C}^n$ with defining function ρ. Then for sufficiently large M the function*

$$\tilde{\rho}(z) = \frac{e^{M\rho(z)} - 1}{M}$$

is a defining function which is strictly psh in a neighborhood of $\partial\Omega$. Moreover there exists a strictly psh function on Ω which is equal to $\tilde{\rho}$ in a neighborhood of $\partial\Omega$.

PROOF. We know that for $b \in \partial\Omega$

$$\sum_{i,j=1} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(b) z_i \bar{z}_j \geq c(b)|z|^2, \quad z \in T_b^{\mathbf{C}}(\partial\Omega),$$

where $c(b)$ is positive and smoothly depending on b. We have

$$\tilde{\rho}(z) = \frac{e^{M\rho(z)} - 1}{M} = \rho(z)(1 + O(\rho(z))$$

at $\partial\Omega$. Thus it is clear that $\tilde{\rho}$ is a defining function for Ω. The complex Hessian of $\tilde{\rho}$ is given by

$$\frac{\partial^2\tilde{\rho}}{\partial z_i \partial \bar{z}_j} = e^{M\rho}\left(\frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j} + M\frac{\partial\rho}{\partial z_i}\frac{\partial\rho}{\partial \bar{z}_j}\right).$$

We write $z \in \mathbf{C}^n$ as $z = z_t + z_\nu$, $z_t \in T_b^{\mathbf{C}}(\partial\Omega)$, $z_\nu \in T_b^{\mathbf{C}}(\partial\Omega)^\perp$, so that $|z|^2 = |z_t|^2 + |z_\nu|^2$.

Then at b, with $e^{M\rho(b)}=1$ and using matrix notation we obtain:

$$\begin{aligned}
\bar{z}^t\left(\frac{\partial^2\tilde{\rho}}{\partial z_i\partial\bar{z}_j}(b)\right)z &= (\bar{z}_t^t+\bar{z}_\nu^t)\left(\frac{\partial^2\rho}{\partial z_i\partial\bar{z}_j}(b)+M\frac{\partial\rho}{\partial z_i}(b)\frac{\partial\rho}{\partial z_j}(b)\right)(z_t+z_\nu)\\
&= (\bar{z}_t^t+\bar{z}_\nu^t)\left(\frac{\partial^2\rho}{\partial z_i\partial\bar{z}_j}(b)\right)(z_t+z_\nu)+M\bar{z}_\nu^t\frac{\partial\rho}{\partial z_i}(b)\frac{\partial\rho}{\partial z_j}(b)z_\nu\\
&\geq \bar{z}_t^t\left(\frac{\partial^2\rho}{\partial z_i\partial\bar{z}_j}(b)\right)z_t-d(b)|z_t||z_\nu|-e(b)|z_\nu|^2+M|\frac{\partial\rho}{\partial z}|^2|z_\nu|^2\\
&\geq c(b)|z_t|^2-d(b)|z_t||z_\nu|-e(b)|z_\nu|^2+M|\frac{\partial\rho}{\partial z}|^2|z_\nu|^2,
\end{aligned}$$

with d,e,f positive continuous functions of b. Now for any fixed b we can find an $M=M(b)$ such that the last expression $\geq c'(b)(|z_t|^2+|z_\nu|^2)=c'(b)|z|^2$. Thus for every $b\in\partial\Omega$ we have found $M(b)$ such that the corresponding $\tilde{\rho}$ is strictly psh at b, hence also in a neighborhood of b, and by compactness of $\partial\Omega$ there exists an M such that $\tilde{\rho}$ is strictly psh at a neighborhood U of $\partial\Omega$.

To construct a global function σ, first note that for $\delta>0$ sufficiently small, U contains a neighborhood of $\partial\Omega$ of the form $V_\delta=\{z:-\delta<\tilde{\rho}(z)<\delta\}$ with $\partial V_\delta\cap\Omega=\{\tilde{\rho}(z)=-\delta\}$. Now let

$$\sigma_1(z)=\begin{cases}\max\{\tilde{\rho}(z),-\delta/2\} & \text{for } z\in\Omega\cap V_\delta,\\ -\delta/2 & \text{elsewhere on } \Omega.\end{cases}$$

The function σ_1 is clearly psh and continuous. Next we modify it to be C^2. Let $h(t)$ be C^2, convex, non decreasing on $\mathbf{R}$ and equal to $-\delta/4$ for $t<-\delta/3$, equal to T on a neighborhood of 0. Then $\sigma_2=h\circ\sigma_1$ is psh, C^2 and strictly psh close to $\partial\Omega$. Now let $\chi(z)\in C^\infty(\Omega)$ have compact support and be strictly psh on a neighborhood of the set where σ is not strictly psh. We put

$$\sigma(z)=\sigma_2(z)+\epsilon\chi(z).$$

If ϵ is sufficiently small σ will be strictly psh where σ_2 is, and it will always be strictly psh where χ is. We are done.

COROLLARY 9.410. A strictly Levi pseudoconvex domain is pseudoconvex.

PROOF. Let ρ be a strictly psh defining function for the domain. Then $-1/\rho$ is, as a composition of a convex function with a psh one, a plurisubharmonic exhaution function.

REMARK. If Ω is a domain and ρ is a continuous plurisubharmonic function on Ω such that $\rho<0$ on Ω and $\lim_{z\to\partial\Omega}\rho(z)=0$, then ρ is called a *bounded plurisubharmonic exhaustion function* Clearly, by the previous proof, if Ω has a bounded psh exhaustion function, then Ω is pseudoconvex.

Exercises

9.1. Using exercise 6.28, prove Bochners Theorem 9.12.

9.2. Prove that the signed boundary distance (4a1) provides a defining function for C^p domains Ω ($p \geq 2$).
   a. Prove that there exists a neighborhood of $\partial\Omega$ such that for every $x \in U$ there is exactly one $y \in \partial\Omega$ with $d(x, y) = d(x, \partial\Omega)$.
   b. Using a local representation $y_n = h(y')$ for $\partial\Omega$, show that

$$d(x, \partial\Omega) = d(x, y^*) = |x_n - y_n^*|(1 + |\nabla h|^2)^{1/2}$$

   where y^* is the unique element of (a).
   c How smooth is d? (One can show C^p cf. [Krantz])

9.3. Show that for $p < 2$ the domain $\Omega = \{y > |x|^p\} \subset \mathbf{R}^2$ has not the uniqueness property of exercise 9.2a.

9.4. Let M be a submanifold of dimension $k \leq 2n$ of $\mathbf{C}^n$. If $a \in M$, what are the possibilities for dim $T_a^{\mathbf{C}}(M)$?

9.5. Verify the statement made in Remark 9.46: A submanifold is an analytic submanifold if and only if $T^{\mathbf{C}} = T^{\mathbf{R}}$ everywhere on the manifold.

9.6. (Hartogs) Let Ω be a pseudoconvex domain and let h be psh on Ω. Show that

$$D = \{z = (z', z_n) \in \mathbf{C}^n : \ |z_n| < e^{-h(z')}\}$$

is pseudoconvex. [Note that $\log|z_n| + h(z')$ is a bounded psh exhaustion function for D as a subset of $\Omega \times \mathbf{C}$.]

9.7. (Continuation) Suppose that D is a domain of holomorphy. Show that there exist holomorphic functions $a_k(z')$ on Ω such that

$$h(z') = \limsup_{k\to\infty} \frac{\log|a_k(z')|}{k}.$$

[Expand a function $f \in O(D)$ which is nowhere analytically extendable at ∂D in a power series in z_n with coefficients in $\mathcal{O}(\Omega)$.]

9.8. (Continuation) Assuming that the Levi problem can be solved, show that the hull of holomorphy and the psh convex hull of a compact subset of a domain of holomorphy are the same.

9.9. (Range) Show that if Ω is a bounded Levi pseudoconvex domain with C^3 boundary has a strictly psh exhaustion function ρ. [One may take

$$\rho(z) = -(-d)^\eta e^{-N|z|^2},$$

with η sufficiently small and N sufficiently large.] Kerzman and Rosay show that pseudoconvex domains with C^1 boundary admit a bounded psh exhaustion function.

9.10. (Continuation) Show that a Levi pseudoconvex domain with C^3 boundary can be exhausted by strictly pseudoconvex domains, i.e.

$$\Omega = \Omega_j, \quad \Omega_j \subset\subset \Omega_{j+1}.$$

[Readers who are familiar with Sards Lemma may derive this for every pseudoconvex domain using the smooth strictly psh exhaustion function of Theorem 9.21.]

9.11. Let Ω be a pseudoconvex domain in $\mathbf{C}^2$ with smooth boundary. Suppose that $0 \in \partial\Omega$ and $T_0(\partial\Omega) = \{\mathrm{Re}w = 0\}$. Show that close to 0 Ω has a defining function with expansion

$$\rho(z, w) = \mathrm{Re}w + P_k(z, \bar{z}) + O(|w|^2 + |w||z| + |z|^{k+1}),$$

where P_k is a real valued homogeneous polynomial of degree $k \geq 2$ in z and $\bar{z}$. Show that close to 0 the complex tangent vectors have the form

$$\begin{pmatrix} \zeta_1 \\ \\ \zeta_2 \end{pmatrix}, \quad \zeta_2 = \zeta_1 \cdot O(|w| + |z|^{k-1}).$$

Estimate the Hessian and show that P_k is subharmonic. Conclude that k is even.

9.12. (Continuation) Let Ω be a pseudoconvex domain given by

$$\mathrm{Re}w - P_k(z) \leq 0,$$

where P_k is a real valued homogeneous polynomial of degree k on $\mathbf{C}$. Show that if $k = 2$ or 4, then Ω admits a supporting complex hyperplane at the origin, that is, after a holomorphic change of coordinates the complex tangent plane at 0 meets $\bar{\Omega}$ only at 0 [everything taking place in a sufficiently small $\mathbf{C}^2$-neighborhood of 0].

9.13. (Continuation) Give an example of a polynomial P_6 such that Ω is pseudoconvex, but every disc of the form $w = f(z)$, $f(0) = 0$ meets Ω. Conclusion?

CHAPTER 10

Differential forms and integral representations

Differential forms play an important role in calculus involving surfaces and manifolds. If one only needs the theorems of Gauss, Green and Stokes in $\mathbf{R}^2$ or $\mathbf{R}^3$, an elementary treatment may suffice. However, in higher dimensions it is hazardous to rely on geometric insight and intuition alone. Here the nice formalism of differential forms comes to the rescue. We will derive the so-called general Stokes theorem which makes many calculations almost automatic.

The purpose of this chapter is to obtain general integral representations for holomorphic and more general smooth functions. We will start in $\mathbf{R}^n$, where things are a little easier than in $\mathbf{C}^n$. A representation for test functions will lead to a good kernel α, resulting in formulas with and without differential forms. Proceeding to $\mathbf{C}^n$, we are led to the related Martinelli-Bochner kernel β and the corresponding integral representation. Final applications include the Szegö integral for the ball and explicit continuation of analytic functions across compact singularity sets.

In $\mathbf{C}^n$ with $n \geq 2$ there are now many kernels for the representation of holomorphic functions. The relatively simple Martinelli-Bochner kernel $\beta(\zeta - z)$ has the advantage of being independent of the domain, but it is not holomorphic in z and in general does not solve the $\overline{\partial}$ problem. Fundamental work of Henkin and Ramirez (1969-70) for strictly pseudoconvex domains has led to many new integral representations which do not have the above drawbacks and give sharp results for the $\overline{\partial}$ problem. However, the subject has become extremely technical and we refer to the literature for details, cf. the books [Aiz-Yuz], [Hen-Leit], [Kerz], [Ran] and [Rud 4], where further references may be found. Rudin's book provides a very readable introduction.

10.1 Differential forms in $\mathbf{R}^n$. A domain $\Omega \subset \mathbf{R}^n$ will carry differential forms of any order $p \geq 0$. The class of p-forms will be denoted by Λ^p; throughout this chapter it is assumed that the coefficients of the forms are at least continuous on Ω. A p-form may be considered as a complex-valued function whose domain consists of all (oriented) smooth surfaces of dimension p in Ω. The functions is given by an integral and the notation (1a) for p-forms serves to show how the function is evaluated for different p-surface and in different coordinate systems. We start with $p = 1$.

Λ^1 consists of the 1-forms, symbol

$$f = \sum_{j=1}^{n} f_j(x) dx_j.$$

A 1-form f assigns a number to every C^1 arc $\gamma: \quad x = \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, $0 \leq t \leq 1$ in Ω, called the *integral of f over γ*:

$$\int_\gamma f \stackrel{\text{def}}{=} \int_{[0,1]} \sum_{1}^{n} f_j \circ \gamma(t) \, \frac{d\gamma_j}{dt} \, dt.$$

Such an integral is independent of the parametrization of γ.

Λ^2 consists of the 2-forms, symbol

$$f = \sum_{j,k=1}^{n} f_{jk}(x) dx_j \wedge dx_k.$$

The symbols $dx_j \wedge dx_k$ are so-called wedge products. A smooth surface X in Ω is given by a parametric representation of class C^1,

$$x = X(t) = (X_1(t), \ldots, X_n(t)), \quad t = (t_1, t_2) \in D,$$

where D is a compact parameter region in $\mathbf{R}^2$, such as the closed unit square $[0,1] \times [0,1]$. A form f in Λ^2 assigns a number to every C^1 surface X in Ω, the *integral of f over X*:

$$\int_X f \stackrel{\text{def}}{=} \int_D \sum_{j_1,j_2=1}^{n} f_{j_1 j_2} \circ X(t) \frac{\partial(X_{j_1}, X_{j_2})}{\partial(t_1, t_2)} dm(t).$$

Here $dm(t)$ denotes Lebesgue measure on D and $\frac{\partial(X_{j_1}, X_{j_2})}{\partial(t_1,t_2)}$ the determinant of the Jacobi matrix with entries $\frac{\partial X_{j_k}}{\partial t_l}$, $k, l = 1, 2$. In general we have for any $p \geq 0$:

Λ^p, the *p-forms*, symbol

$$f = \sum_{j_1,\ldots,j_p=1}^{n} f_{j_1 \ldots j_p}(x) dx_{j_1} \wedge \ldots \wedge dx_{j_p}. \tag{1a}$$

For $p \geq 1$ a smooth p-surface X in Ω is given by a C^1 *map*

$$x = X(t) = (X_1(t), \ldots, X_n(t)), \quad t = (t_1, \ldots, t_p) \in D,$$

where D is a compact parameter region in $\mathbf{R}^n$ such as the closed unit cube, $D = [0,1]^p$. A form f in Λ^p assigns a number to every smooth p-surface X in Ω, the *integral of f over X*:

$$\int_X f \stackrel{\text{def}}{=} \int_D \sum f_{j_1 \ldots j_p} \circ X(t) \frac{\partial(X_{j_1}, \ldots, X_{j_p})}{\partial(t_1, \ldots, t_p)} dm(t). \tag{1b}$$

Again $dm(t)$ denotes Lebesgue measure on D and $\frac{\partial(X_{j_1},\ldots,X_{j_p}}{\partial(t_1,\ldots,t_p)}$ the determinant of the Jacobi matrix with entries $\frac{\partial X_{j_k}}{\partial t_l}$, $k, l = 1, \ldots, p$. Note that the integral of a differential form is always taken over a *map*. The integral is invariant under orientation preserving coordinate transformations in $\mathbf{R}^n$.

For $p = n$ we have the important special case where X is the identity map, id, on the closure of a (bounded) domain Ω, while $f = \varphi(x) dx_1 \wedge \ldots \wedge dx_n$, with φ continuous on (a neighborhood of) $\overline{\Omega}$. Taking $D = \overline{\Omega}$ and $X = \text{id}$, so that $X_j(t) = t_j$, one obtains

$$\int_{\text{id}|\overline{\Omega}} f = \int_{\text{id}|\overline{\Omega}} \varphi(x) dx_1 \wedge \ldots \wedge dx_n = \int_{\overline{\Omega}} \varphi(t) dm(t) = \int_{\Omega} \varphi dm. \tag{1b'}$$

Taking $\varphi = 1$, it will be clear why the form

$$(1b'') \qquad \omega(x) \stackrel{\text{def}}{=} dx_1 \wedge \ldots \wedge dx_n$$

is called the *volume form* in $\mathbf{R}^n$. [Incidentally, in the case of the identity map id $| \, \overline{\Omega}$, one often simply writes Ω under the integral sign.]

Observe that the (continuous) 0-forms are just the continuous functions on Ω.

Anticommutative relation and standard representation. Two p-forms f and g on Ω are called *equal*: $f = g$ if

$$\int_X f = \int_X g$$

for all smooth p-surfaces X in Ω. Formula (1b) involves the determinant of a Jacobi matrix, *not* the absolute value of the determinant! If one interchanges two rows, the sign is reversed. Considering special p-forms $dx_{j_1} \wedge \ldots \wedge dx_{j_p}$ and a permutation $(k_1, \ldots, k_p)$ of $(j_1, \ldots, j_p)$, one will have

$$\int_X dx_{k_1} \wedge \ldots \wedge dx_{k_p} = \int_X \varepsilon dx_{j_1} \wedge \ldots \wedge dx_{j_p}$$

for all X, where ε equals 1 for an even, -1 for an odd permutation. Thus by the definition of equality,

$$(1c) \qquad dx_{k_1} \wedge \ldots \wedge dx_{k_p} = \varepsilon dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

As a special case one obtains the anticommutative relation for wedge products:

$$(1c') \qquad dx_k \wedge dx_j = -dx_j \wedge dx_k.$$

This holds also for $k = j$, hence $dx_j \wedge dx_j = 0$. Whenever some index in a wedge product occurs more than once, that product is equal to 0. In particular all p-forms in $\mathbf{R}^n$ with $p > n$ are zero.

With the aid of (1c$'$) we can arrange the indices in every nonzero product $dx_{k_1} \wedge \ldots \wedge dx_{k_p}$ in *increasing order*. Combining terms with the same subscripts, we thus obtain the *standard representation* for p-forms,

$$(1d) \qquad f = \Sigma_J \, f_J(x) dx_J.$$

One sometimes writes Σ'_J to emphasize that the summation is over (all) increasing p-indices $J = (j_1, \ldots, j_p)$, $1 \le j_1 < \ldots < j_p \le n$. dx_J is a so-called *basic p-form*,

$$(1d') \qquad dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

For p-forms in standard representation one has $f = g$ if and only if $f_J = g_J$ for each J, cf. exercise 10.2.

Exterior or wedge product. The sum of two p-forms is defined in the obvious way. The wedge product of two basic forms dx_J and dx_K of orders p and q is defined by

$$dx_J \wedge dx_K = dx_{j_1} \wedge \ldots \wedge dx_{j_p} \wedge dx_{k_1} \wedge \ldots \wedge dx_{k_q}.$$

This product is of course equal to 0 if J and K have a common index. For general p- and q-forms f and g one sets, using their standard representations,

$$f \wedge g = \Sigma f_J dx_J \wedge \Sigma g_K dx_K \overset{\text{def}}{=} \Sigma f_J g_K dx_J \wedge dx_K. \tag{1e}$$

The product of a function φ and a form f is written without wedge:

$$\varphi f = f\varphi = \Sigma \varphi f_J dx_J.$$

The multiplication of differential forms is associative and distributive, but not commutative.

Upper bound for integrals. Using the standard representation of f one defines, cf. (1b),

$$\int_X |f| = \int_D \left| \Sigma'_J f_J \circ X(t) \, \frac{\partial(X_{j_1}, \ldots, X_{j_p})}{\partial(t_1, \ldots, t_p)} \right| dm(t). \tag{1f'}$$

One the has the useful inequality

$$\left|\int_X \varphi f\right| \le \int_X |\varphi f| \le \sup_X |\varphi| \cdot \int_X |f|. \tag{1f''}$$

Differentiation. There is a differential operator d from p-forms of class C^1 [that is, with C^1 coefficients] to $(p+1)$-forms. By definition it is *linear* and

$$\begin{cases} \text{for 0-forms } \varphi \text{ one has } d\varphi \overset{\text{def}}{=} \Sigma_1^n \frac{\partial \varphi}{\partial x_j} \, dx_j, \\ \text{for special } p\text{-forms } f = \varphi dx_J \text{ one has } df \overset{\text{def}}{=} d\varphi \wedge dx_J. \end{cases} \tag{1g}$$

Thus if $f = \Sigma f_L dx_L$, then

$$df = \Sigma \, df_L \wedge dx_L = \sum_{j,L} \frac{\partial f_L}{\partial x_j} \, dx_j \wedge dx_L. \tag{1g'}$$

Applying the definition to 1-forms $f = \Sigma f_k dx_k$, it is easy to obtain the standard representation for df:

$$df = \Sigma_k df_k \wedge dx_k = \sum_{j,k} \frac{\partial f_k}{\partial x_j} \, dx_j \wedge dx_k = \underset{(j<k)}{\Sigma'} \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right) dx_j \wedge dx_k.$$

For C^2 functions one thus finds

$$d^2\varphi = d(d\varphi) = \Sigma' \left(\frac{\partial}{\partial x_j} \frac{\partial \varphi}{\partial x_k} - \frac{\partial}{\partial x_k} \frac{\partial \varphi}{\partial x_j} \right) dx_j \wedge dx_k = 0.$$

Going to p-forms $f = \Sigma f_L dx_L$ and using associativity, there results

$$d^2 f = d(df) = d(\Sigma df_L \wedge dx_L) = \Sigma d^2 f_L \wedge dx_L = 0.$$

PROPOSITION (10.11). For all p-forms f of class C^2 one has $d^2 f = 0$, hence

$$d^2 = 0.$$

For the derivative of a product $f \wedge g$ (1e) of C^1 forms there is the "Leibniz formula"

$$\begin{aligned} d(f \wedge g) &= \Sigma d(f_J g_K) \wedge dx_J \wedge dx_K \\ &= \Sigma\{(df_J)g_K + f_J dg_K\} \wedge dx_J \wedge dx_K \\ &= df \wedge g + (-1)^p f \wedge dg \quad \text{if} \quad f \in \Lambda^p. \end{aligned} \tag{1h}$$

10.2 Stokes' theorem. *Green's theorem* for integration by parts in the plane may be interpreted as a result on differential forms. We recall that for appropriate *closed* regions $D \subset \mathbf{R}^2$ and functions f_1, f_2 in $C^1(D)$,

$$\int_{\partial D} f_1 dx_1 + f_2 dx_2 = \int_D \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dm(x) \tag{2a}$$

[Section 3.1]. Setting $f_1 dx_1 + f_2 dx_2 = f$, the formula may be rewritten as

$$\int_{\partial(\mathrm{id})} f = \int_{\mathrm{id}} \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2 = \int_{\mathrm{id}} df \qquad [\mathrm{id} = \mathrm{id} \mid D]. \tag{2a'}$$

This is a special case of the general Stokes theorem below.

Something similar may be done with *Gauss's theorem* for integration by parts in $\mathbf{R}^3$ [easily extended to $\mathbf{R}^p$] or the related *divergence theorem*:

$$\int_D \frac{\partial \varphi}{\partial x_j} dm = \int_{\partial D} \varphi N_{x_j} d\sigma, \qquad \int_D \operatorname{div} \vec{v}\, dm = \int_{\partial D} \vec{v} \cdot \vec{N}\, d\sigma. \tag{2b}$$

Here φ is a function in $C^1(D)$ and $\vec{N}$ is the exterior unit normal to ∂D; N_{x_j} or N_j is the component of $\vec{N}$ in the x_j-direction, while $d\sigma$ is the "area element" of ∂D. Finally, $\vec{v}$ is a vector field,

$$\operatorname{div} \vec{v} = \sum_1^3 \frac{\partial v_j}{\partial x_j} \qquad \left[\text{or} \quad \sum_1^p \frac{\partial v_j}{\partial x_j}\right].$$

The special case of the closed unit cube $D = [0,1]^p$ in $\mathbf{R}^p$ is basic for the proof of the general Stokes theorem. Just as in formula (2a′) we consider the identity map id on D, thus obtaining a special p-surface. To obtain a formula like (2a′), we have to give a suitable definition for the *oriented boundary* $\partial(\text{id})$. It will be defined by a formal sum or chain of "oriented faces". The faces are the following maps on the closed unit cube $[0,1]^{p-1}$ to $\mathbf{R}^p$:

$$(2b') \qquad \begin{cases} V^{1,0}(t) = (0, t_1, \ldots, t_{p-1}), & V^{1,1}(t) = (1, t_1, \ldots, t_{p-1}), \\ V^{2,0}(t) = (t_1, 0, t_2, \ldots, t_{p-1}), & V^{2,1}(t) = (t_1, 1, t_2, \ldots, t_{p-1}), \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ V^{p,0}(t) = (t_1, \ldots, t_{p-1}, 0), & V^{p,1}(t) = (t_1, \ldots, t_{p-1}, 1), \\ & \qquad 0 \le t_1, \ldots, t_{p-1} \le 1. \end{cases}$$

q-forms assign numbers to q-surfaces; conversely, q-surfaces assign numbers to q-forms. Since one can add $\mathbf{C}$ valued functions, one can formally add q-surfaces V_j as functions defined on q-forms. Saying that X is a chain $V_1 + \ldots + V_m$ means that for all q-forms f, $\int_X f$ is defined by

$$(2c) \qquad \int_X f = \int_{V_1} f + \ldots + \int_{V_m} f; \quad \text{similarly} \int_{-V} f = -\int_V f.$$

Lemma 10.21. *Let f be any $(p-1)$-form of class C^1 on the closed unit cube D in* $\mathbf{R}^p$, $\text{id} = \text{id} \mid D$. *Then the chain*

$$(2c) \qquad \partial(\text{id}) \stackrel{\text{def}}{=} \sum_{j=1}^{p} (-1)^j (V^{j,0} - V^{j,1}) \qquad [\text{cf.}(2b')]$$

provides the correct (oriented) boundary to yield the desired formula

$$(2c') \qquad \int_{\text{id}} df = \int_{\partial(\text{id})} f \qquad \left[= \sum_{j=1}^{p} (-1)^j \left(\int_{V^{j,0}} f - \int_{V^{j,1}} f \right) \right].$$

.

PROOF. To verify (2c′) it will be enough to consider the representative special case $f = \varphi(x) dx_2 \wedge \ldots \wedge dx_p$:

$$\int_{\partial(\text{id})} f = \int_{\partial(\text{id})} \varphi(x) dx_2 \wedge \ldots \wedge dx_p = \left(\int_{V^{1,1}} - \int_{V^{1,0}} \right) - \left(\int_{V^{2,1}} - \int_{V^{2,0}} \right) + \ldots.$$

Observe that

$$\text{for } x = V(t) = V^{1,1}(t) \text{ or } V^{1,0}(t),\ \frac{\partial(V_2,\ldots,V_p)}{\partial(t_1,\ldots,t_{p-1})} = \frac{\partial(t_1,\ldots,t_{p-1})}{\partial(t_1,\ldots,t_{p-1})} = 1,$$
$$\text{for } x = V(t) = V^{2,1}(t),\ \frac{\partial(V_2,\ldots,V_p)}{\partial(t_1,\ldots,t_{p-1})} = \frac{\partial(1,t_2,\ldots,t_{p-1})}{\partial(t_1,t_2,\ldots,t_{p-1})} = 0, \text{ etc.}$$

Thus

$$\int\limits_{\partial(\mathrm{id})} f = \int\limits_{[0,1]^{p-1}} \{\varphi(1,t_1,\ldots,t_{p-1}) - \varphi(0,t_1,\ldots,t_{p-1})\}dt_1,\ldots dt_{p-1} + 0$$
$$= \int\limits_{[0,1]^p} \frac{\partial\varphi}{\partial x_1}(t_0,t_1,\ldots,t_{p-1})dt_0dt_1\ldots dt_{p-1} = \int\limits_{\mathrm{id}} \frac{\partial\varphi}{\partial x_1}\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$$
$$= \int\limits_{\mathrm{id}} \left(\frac{\partial\varphi}{\partial x_1}\, dx_1 + \ldots + \frac{\partial\varphi}{\partial x_p}\, dx_p\right) \wedge dx_2 \wedge \ldots \wedge dx_p = \int\limits_{\mathrm{id}} df.$$

The general Stokes theorem may be obtained from the special case in the lemma by the machinery of *pull backs*.

DEFINITION 10.22. Let $y = T(x) : y_j = T_j(x),\ j = 1,\ldots,n$ be a C^1 map from Ω_1 in $\mathbf{R}^m$ to Ω_2 in $\mathbf{R}^n$. Given a p-form [in standard representation]

$$f = \Sigma f_J(y)dy_J \quad \text{on} \quad \Omega_2,$$

its pull back T^*f is the p-form on Ω_1 given by

$$\begin{aligned} T^*f &= \Sigma\{f_J \circ T(x)\}dT_J, \qquad J = (j_1,\ldots,j_p), \\ dT_J &= dT_{j_1} \wedge \ldots \wedge dT_{j_p}, \qquad dT_j = \Sigma \frac{\partial T_j}{\partial x_k}\, dx_k. \end{aligned} \tag{2d}$$

Proposition 10.23. *Let T be as above and let $S,\ x_k = S_k(u)$ be a C^1 map from Ω_0 in $\mathbf{R}^\ell$ to Ω_1. Let f be a p-form, g a q-form on Ω_2, X a p-surface in Ω_1. Then one has the following properties of pull backs:*
(i) $T^*(f+g) = T^*f + T^*g$ *if* $q = p$;
(ii) $T^*(f \wedge g) = T^*f \wedge T^*g$;
(iii) $S^*(T^*f) = (TS)^*f$;
(iv) $\int_{T\circ X} f = \int_X T^*f$;
(v) $dT^*f = T^*(df)$ *when* $f \in C^1,\ T \in \mathbf{C}^2$. *In particular:*
(v′) *if* $df = 0$, *then* $d(T^*f) = 0$.

PROOF. (i), (ii): these follow directly from the definition. (iii) This is obvious for 0-forms, hence by (i) and (ii) it is enough to consider the case of 1-forms $f = dy_j$. Here, using the chain rule in the second line,

$$T^*f = dT_j = \sum_k \frac{\partial T_j}{\partial x_k}\, dx_k,$$

$$S^*(T^*f) = \sum_k \frac{\partial T_j}{\partial x_k} \circ S(u) \sum_p \frac{\partial S_k}{\partial u_p}\, du_p = \sum_p \frac{\partial}{\partial u_p}(T_j \circ S) du_p = (TS)^* f.$$

(iv) Let be the parameter domain for X, hence also for $Y = T \circ X$ and let id be the identity map on D. It will be enough to prove

$$(iv') \qquad \int_Y f = \int_{Y \circ \mathrm{id}} f = \int_{\mathrm{id}} Y^* f.$$

Indeed, from this step and (iii) it will follow that

$$\int_{T \circ X} f = \int_{\mathrm{id}} Y^* f = \int_{\mathrm{id}} X^*(T^* f) = \int_{X \circ \mathrm{id}} T^* f = \int_X T^* f.$$

For the proof of (iv′) we may take $f = \varphi dY_J$. Now in self-explanatory notation, using the expansion formula for a determinant on the way,

$$\begin{aligned} dY_{j_1} \wedge \ldots \wedge dY_{j_p} &= \sum_{k_1,\ldots,k_p} \frac{\partial Y_{j_1}}{\partial t_{k_1}} \cdots \frac{\partial Y_{j_p}}{\partial t_{k_p}}\, dt_{k_1} \wedge \ldots \wedge dt_{k_p} \\ &= \sum_{k_1,\ldots,k_p} \frac{\partial Y_{j_1}}{\partial t_{k_1}} \cdots \frac{\partial Y_{j_p}}{\partial t_{k_p}}\, \varepsilon(k_1,\ldots,k_p) dt_1 \wedge \ldots \wedge dt_p \\ &= \frac{\partial(Y_{j_1},\ldots,Y_{j_p})}{\partial(t_1,\ldots,t_p)}\, {}''dm(t)'', \end{aligned}$$

cf. (1c), (1b″). Hence

$$\begin{aligned} \int_{\mathrm{id}} Y^* f &= \int_{\mathrm{id}} (\varphi \circ Y) dY_{j_1} \wedge \ldots \wedge dY_{j_p} = \int_D \varphi \circ Y(t) \frac{\partial(Y_{j_1},\ldots,Y_{j_p})}{\partial(t_1,\ldots,t_p)}\, dm(t) \\ &= \int_Y \varphi dY_J = \int_Y f. \end{aligned}$$

(v) For C^1 functions φ on ω_2, using the chain rule

$$\begin{aligned} d(T^*\varphi) &= d\{\varphi(T \circ x)\} = \Sigma_k \frac{\partial}{\partial x_k}\, \varphi(T \circ x) dx_k \\ &= \Sigma_k \Sigma_j \frac{\partial \varphi}{\partial Y_j}(T \circ x) \frac{\partial T_j}{\partial x_k}\, dx_k = \Sigma_j \frac{\partial \varphi}{\partial Y_j}\, (T \circ x) dT_j = T^*(d\varphi). \end{aligned}$$

For $dY_J = dY_{j_1} \wedge \ldots \wedge dY_{j_p}$ one has $T^*(dY_J) = dT_{j_1} \wedge \ldots \wedge dT_{j_p}$ and hence by (1h) and Proposition (10.11)

$$d\{T^*(dY_J)\} = d^2T_{j_1} \wedge dT_{j_2} \wedge \ldots \wedge dT_{j_p} - dT_{j_1} \wedge d^2T_{j_2} \wedge \ldots \wedge dT_{j_p} + \ldots = 0.$$

Finally, for $f = \varphi dY_J$ one has $T^*f = \varphi(T \circ x)T^*(dY_J)$ and thus by the preceding,

$$\begin{aligned} d(T^*f) &= d\{\varphi(T \circ x)\} \wedge T^*(dY_J) + \varphi(T \circ x)d\{T^*(dY_J)\} \\ &= T^*(d\varphi) \wedge T^*(dY_J) = T^*(d\varphi \wedge dY_J) = T^*(df). \end{aligned}$$

Q.E.D.

We can now prove the general Stokes theorem. Let $X : D \to \Omega \subset \mathbf{R}^n$ be a p-surface of class C^2, where D is the closed unit cube in $\mathbf{R}^p$. The (oriented) *boundary* of X is defined by the chain

$$\partial X = X \circ \partial(\mathrm{id}) = \sum_{j=1}^{p} (-1)^j (X \circ V^{j,0} - X \circ V^{j,1}), \tag{2e}$$

where id is the identity map on D and $\partial(\mathrm{id})$ is as in Lemma (10.24).

Theorem 10.24 (Stokes). *Let X be as above and let f be a $(p-1)$-form of class $C^1(\Omega)$. Then with ∂X defined by (2e) and (2b′),*

$$\int_{\partial X} f = \int_X df. \tag{2f}$$

PROOF. One has $X = X \circ \mathrm{id}$, hence by the properties of pull-backs (10.23) and by Lemma 10.21,

$$\begin{aligned} \int_{\partial X} f &= \int_{X \circ \partial(\mathrm{id})} f = \int_{\partial(\mathrm{id})} X^*f = \int_{\mathrm{id})} d(X^*f) \\ &= \int_{\mathrm{id}} X^*(df) = \int_{X \circ \mathrm{id}} df = \int_X df. \end{aligned}$$

Formula (2f) readily extends to chains $X = V_1 + \ldots + V_m$ of smooth p-surfaces, for which one defines $\partial X = \partial V_1 + \ldots + \partial V_m$ [cf. (2c)].

REMARKS 10.25. The important special case $p = n$ may be called the general *Gauss-Green* or *divergence theorem*. Taking $D = \overline{\Omega}$, where Ω is bounded domain in $\mathbf{R}^n$ with oriented C^2 boundary $\partial\Omega$ and assuming that f is an $(n-1)$-form of class C^1 on (a neighborhood of) $\overline{\Omega}$, the formula becomes

$$\int_{\partial\Omega} f = \int_\Omega df \tag{2g}$$

where we have carelessly written Ω instead of id $| \overline{\Omega}$.

The name "Stokes' theorem" for the general case stems from the fact that Kelvin and Stokes considered the important case $p = 2,\ n = 3$.

10.3 **Integral representations in $\mathbf{R}^n$.** We first derive an integral formula for test functions φ on $\mathbf{R}^n$, that is, C^∞ functions of compact support. By calculus,

$$\varphi(0) = -\{\varphi(\infty) - \varphi(0)\} = -\int_0^\infty \frac{\partial \varphi}{\partial x_1}(r, 0, \ldots, 0)dr.$$

Instead of $\frac{\partial \varphi}{\partial x_1}(r, 0, \ldots, 0)$ may write $(\partial/\partial r)\varphi(re_1)$, where e_1 is the unit vector in the x_1-direction. Of course, we can go to infinity in any direction ξ, where ξ denotes a unit vector. Thus

$$\varphi(0) = -\int_0^\infty \frac{\partial}{\partial r}\varphi(r\xi)dr, \quad \forall \xi \in S_1 = S(0,1) \subset \mathbf{R}^n.$$

We now take the *average* over S_1:

$$\varphi(0) = -\frac{1}{\sigma(S_1)}\int_{S_1} d\sigma(\xi)\int_0^\infty \frac{\partial}{\partial r}\varphi(r\xi)dr, \tag{3a}$$

where

$$\sigma(S_1) = \sigma_n(S_1) = \text{“area of unit sphere”} = 2\pi^{n/2}/\Gamma(n/2). \tag{3a$'$}$$

The integral (3a) may be transformed into an integral over $\mathbf{R}^n$ by appropriate use of Fubini's theorem. One first inverts the order of integration, then substitutes $r\xi = x$. Next observe that area elements of spheres $S_r = S(0,r)$ transform according to the rule of similarity: $d\sigma(x) = r^{n-1}d\sigma(x/r)$. However, in polar coordinates the product of dr and $d\sigma(x)$ gives the n-dimensional volume element:

$$dr\, d\sigma(x) = dm(x), \quad |x| = r.$$

Writing out some of the steps, there results

$$\begin{aligned} -\sigma(S_1)\varphi(0) &= \int_0^\infty dr \int_{x/r\in S_1} \frac{\partial}{\partial r}\varphi(x)d\sigma\left(\frac{x}{r}\right) = \int_0^\infty dr \int_{x\in S_r} \frac{\partial}{\partial r}\varphi(x)\frac{d\sigma(x)}{r^{n-1}} \\ &= \int_{\mathbf{R}^n} \frac{\partial}{\partial r}\varphi(x)\cdot\frac{1}{r^{n-1}}\,dm(x). \end{aligned}$$

One finally rewrites the radial derivative $\partial\varphi/\partial r$ as

$$\frac{\partial}{\partial r}\varphi(x) = \frac{\partial \varphi}{\partial x_1}\frac{x_1}{r} + \ldots + \frac{\partial \varphi}{\partial x_n}\frac{x_n}{r}.$$

Thus we obtain the following

Proposition 10.31. *For test functions φ on $\mathbf{R}^n$ [and in fact, for all functions φ in C_0^1 $(\mathbf{R}^n)$],*

$$\varphi(0) = -\frac{1}{\sigma(S_1)} \int_{\mathbf{R}^n} \left(\frac{\partial\varphi}{\partial x_1} x_1 + \ldots + \frac{\partial\varphi}{\partial x_n} x_n\right) \frac{1}{|x|^n}\, dm(x).$$

[Note that the final integral is (absolutely) convergent: r^{1-n} is integrable over a neighborhood of 0 in $\mathbf{R}^n$.]

We wish to obtain representations for smooth functions on bounded domains and for that we will use Stokes' theorem. As a first step we rewrite (10.31) in terms of differential forms. Besides $d\varphi$ and the volume form,

$$d\varphi = \Sigma_1^n \frac{\partial\varphi}{\partial x_j}\, dx_j \quad \text{and} \quad \omega(x) = dx_1 \wedge \ldots \wedge dx_n,$$

we need the auxialiary forms

(3*b*) $$\omega_k(x) \stackrel{\text{def}}{=} (-1)^{k-1} dx_1 \wedge \ldots [k] \ldots \wedge dx_n, \quad k = 1, \ldots, n$$

where $[k]$ means that the differential dx_k is absent. Observe that

(3*b*′) $$dx_j \wedge \omega_k(x) = \begin{cases} 0 & \text{if } k \neq j \;\; [dx_j \text{ will occur twice}], \\ \\ \omega(x) & \text{if } k = j \;\; [\text{thanks to } (-1)^{k-1}]. \end{cases}$$

Thus the following product gives the differential form corresponding to the integrand in (10.31):

(3*b*″) $$\sum_j \frac{\partial\varphi}{\partial x_j}\, dx_j \wedge \sum_k (-1)^{k-1} x_k \frac{1}{|x|^n}\, dx_1 \wedge \ldots [k] \ldots \wedge dx_n = \sum_j \frac{\partial\varphi}{\partial x_j} x_j \frac{1}{|x|^n}\, \omega(x).$$

Apparently the "good kernel" to use in conjunction with $d\varphi$ is

(3*c*) $$\alpha(x) = \alpha_n(x) \stackrel{\text{def}}{=} \frac{1}{\sigma(S_1)} \sum_{k=1}^n x_k \frac{1}{|x|^n}\, \omega_k(x) \qquad [\sigma(S_1) = \sigma_n(S_1) = 2\pi^{n/2}/\Gamma(n/2)].$$

Proposition 10.32. *The values of a function φ in $C_0^1(\mathbf{R}^n)$ may be obtained from $d\varphi$ with the aid of the $(n-1)$-form α of (3c), (3b):*

$$\varphi(0) = -\int_{\mathrm{id}|D} d\varphi \wedge \alpha, \quad \varphi(a) = -\int_{\mathrm{id}|D} d\varphi \wedge \alpha(x-a).$$

Here D may be any closed cube about 0 *or* a *that contains* $\operatorname{supp}\varphi$.

The first formula follows from (10.31) in view of (3c), (3b$''$) and (1b$'$). For the last formula, one need only apply the first to $\varphi(x+a)$.

Having identified a candidate kernel α we will derive a more general representation theorem. For this we need some

Properties 10.33 *of* α*:*

(i) $d\alpha(x) = 0$ *for* $x \neq 0$*;*

(ii) $\int_{S_r} \alpha = \int_{S_1} \alpha = 1,\ \forall r > 0,$ *where* S_r *or* $S(0,r)$ *stands for* $\mathrm{id} \mid S(0,r)$*;*

(iii) $|\int_{S_r} u\alpha| \le \sup\limits_{S_r} |u| \int_{S_r} |\alpha| = c \sup\limits_{S_r} |u|,$

where u *is any continuous function and* $c = \int_{S_1} |\alpha|$.

PROOF. (i) This may be verified by computation: by (3c) and (3b$'$),

$$\begin{aligned}\sigma(S_1) d\alpha &= \Sigma_k d(x_k|x|^{-n}) \wedge \omega_k = \Sigma_k \Sigma_j \frac{\partial}{\partial x_j}(x_k|x|^{-1}) dx_j \wedge \omega_k \\ &= \Sigma_k \frac{\partial}{\partial x_k}(x_k|x|^{-1})\omega = \Sigma_k(|x|^{-1} - n|x|^{-n-2}x_k^2)\omega = 0, \quad x \neq 0.\end{aligned}$$

[The result will be less surprising if one observes that, for $n \ge 3$, $x_k|x|^{-1} = \text{const} \cdot (\partial/\partial x_k)|x|^{2-n}$; the relation $d\alpha = 0$ is equivalent to $\Delta|x|^{2-n} = 0$.]

(ii) We can now apply Stokes' theorem or the divergence theorem to the spherical shell $\overline{\Omega}$ bounded by S_r and S_1. Taking $0 < r < 1$,

$$\int_{S_1} \alpha - \int_{S_r} \alpha = \int_{\overline{\Omega}} d\alpha = 0.$$

An alternative is to remark that α is invariant under change of scale: $\alpha(\lambda x) = \alpha(x)$, cf. (iii) below.

The constant value of the integral may also be derived from Stokes' theorem. On S_1, $\alpha = |x|^n\alpha$ and by (3c), (3b),

$$\sigma(S_1) d(|x|^n \alpha) = \Sigma_k \Sigma_j \frac{\partial}{\partial x_j}(x_k) dx_j \wedge \omega_k = \Sigma_k dx_k \wedge \omega_k = n\omega.$$

Thus

$$\int_{S_1} \alpha = \int_{S_1} |x|^n \alpha = \int_{B_1} d(|x|^n\alpha) = \frac{n}{\sigma(S_1)} \int_{B_1} \omega = \frac{n}{\sigma(S_1)} m(B_1) = 1$$

[cf. exercise 10.10].

(iii) The first part follows from (1f$''$). The constancy of $\int_{S_r} |\alpha|$ is due to the fact that $\alpha(\lambda x) = \alpha(x),\ \forall \lambda > 0$. Indeed, using the parameter region S_1 for S_λ, the sum in definition (1f$'$) will be independent of λ.

We can now prove the following general representation of smooth functions:

Theorem 10.34. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with C^2 boundary, u a function of class C^1 on $\overline{\Omega}$. Then*

$$u(a) = \int\limits_{\partial\Omega} u(x)\alpha(x-a) - \int\limits_{\Omega} du(x) \wedge \alpha(x-a), \qquad \forall a \in \Omega$$

where we have written Ω under the integrals instead of id $| \overline{\Omega}$.

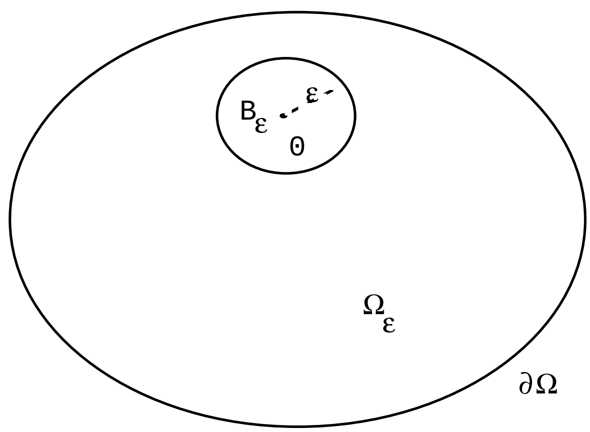
For a formulation free of differential forms, cf. exercises 10.7, 10.8. The result may be extended to domains Ω with piecewise C^1 boundary by approximation from inside.

Proof of the theorem. It may be assumed by translation that $a = 0$; the general formula readily follows. Taking $\varepsilon > 0$ so small that $\overline{B}_\varepsilon = \overline{B}(0,\varepsilon)$ belongs to Ω, we will apply Stokes' theorem to $du \wedge \alpha$ on $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$.
On Ω_ε, cf. (10.33),

$$du \wedge \alpha = d(u\alpha) - ud\alpha = d(u\alpha).$$

Thus

$$\int\limits_{\Omega_\varepsilon} du \wedge \alpha = \int\limits_{\Omega_\varepsilon} d(u\alpha) = \int\limits_{\partial\Omega_\varepsilon} u\alpha = \int\limits_{\partial\Omega} u\alpha - \int\limits_{\partial B_\varepsilon} u\alpha. \tag{3d}$$



Again using (10.33),

$$\begin{aligned} &\int\limits_{\partial B_\varepsilon} u\alpha = u(0) + \int\limits_{\partial B\varepsilon} \{u(x) - u(0)\}\alpha, \\ &|\int\limits_{\partial B_\varepsilon} \{u(x) - u(0)\}\alpha| \le \sup_{\partial B_\varepsilon} |u(x) - u(0)| \int\limits_{S_1} |\alpha| \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \end{aligned} \tag{3e}$$

On the other hand

$$\int\limits_{\Omega_\varepsilon} du \wedge \alpha \to \int\limits_{\Omega} du \wedge \alpha \quad \text{as} \quad \varepsilon \downarrow 0, \tag{3e$'$}$$

since every coefficient in the form $du \wedge \alpha$ is bounded by const.$|x|^{1-n}$ on B_ε, cf. (3b$''$). Thus letting $\varepsilon \downarrow 0$ in (3d), there results

$$\int_\Omega du \wedge \alpha = \int_{\partial\Omega} u\alpha \;\; - u(0).$$

10.4 Differential forms in $\mathbf{C}^n$. One may consider $\mathbf{C}^n$ as $\mathbf{R}^{2n}$ with coordinates $x_j = \mathrm{Re} z_j$ and $y_j = \mathrm{Im} z_j$, but for the application of differential forms in $\mathbf{C}^n$, it is advantageous to use not dx_j and dy_j, but their complex counterparts

$$dz_j = dx_j + i dy_j, \qquad d\overline{z}_j = dx_j - i dy_j. \tag{4a}$$

with the aid of the inverse formulas

$$dx_j = \tfrac{1}{2}(dz_j + d\overline{z}_j), \quad dy_j = \tfrac{1}{2i}(dz_j - d\overline{z}_j), \tag{4a$'$}$$

every s-form f in $\mathbf{C}^n = \mathbf{R}^{2n}$ can be written *in exactly one way* as

$$f = \sum_{J,K} f_{JK}(z) dz_J \wedge d\overline{z}_K, \tag{4b}$$

where $J = (j_1, \ldots, j_p)$ and $K = (k_1, \ldots, k_q)$ run over all increasing $p-$ and q-indices, $1 \le j_1 < \ldots < j_p \le n,\ 1 \le k_1 < \ldots < k_q \le n$, with variable p and q such that $p + q = s$. Naturally,

$$dz_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}, \quad d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_q}.$$

The class $\Lambda^{p,q}$. A sum (4b) in which every J is a p-index [with p fixed] and every K a q-index [with q fixed] defines a (p,q)-*form*, or a form of *type* or *bidegree* (p,q). The class of (p,q)-forms [with continuous coefficients] is denoted by $\Lambda^{p,q}$.

For a C^1 function φ on $\Omega \subset \mathbf{C}^n$ we saw in Section 1.3 that

$$d\varphi = \partial\varphi + \overline{\partial}\varphi, \quad \text{where} \quad \partial\varphi = \Sigma_1^n \frac{\partial\varphi}{\partial z_j} dz_j,\ \overline{\partial}\varphi = \Sigma_1^n \frac{\partial\varphi}{\partial \overline{z}_j} d\overline{z}_j. \tag{4c}$$

Similarly, for a C^1 form f in $\Lambda^{p,q}$,

$$df = \sum_{J,K} df_{JK} \wedge dz_J \wedge d\overline{z}_K = \partial f + \overline{\partial} f, \tag{4c$'$}$$

where

$$\partial f = \sum_{J,K} \partial f_{J,K} \wedge dz_J \wedge d\overline{z}_K, \quad \overline{\partial} f = \sum_{J,K} \overline{\partial} f_{J,K} \wedge dz_J \wedge d\overline{z}_K. \tag{4c$''$}$$

Notice that ∂f is a $(p+1,q)$-form and $\overline{\partial} f$ is a $(p, q+1)$ form.

Since $d = \partial + \overline{\partial}$ on $\wedge^{p,q}$, the equation $d^2 = 0$ becomes

$$\partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2 = 0. \tag{4d}$$

If f is a C^2 form in $\wedge^{p,q}$, the forms $\partial^2 f$, $(\partial\overline{\partial} + \overline{\partial}\partial)f$ and $\overline{\partial}^2 f$ are of the respective types $(p+2, q)$, $(p+1, q+1)$ and $(p, q+2)$. Thus the vanishing of their sum implies that each of these forms must be 0:

$$\partial^2 = 0, \quad \partial\overline{\partial} = -\overline{\partial}\partial, \quad \overline{\partial}^2 = 0 \tag{4e}$$

on $\wedge^{p,q}$ [and hence generally on $\wedge^s$]. The last relation confirms the local integrability condition $\overline{\partial}v = 0$ for the equation $\overline{\partial}u = v$.

We finally remark that for (n, q)-forms f in $\mathbf{C}^n$ of class C^1, always

$$\partial f = 0, \quad df = \overline{\partial} f.$$

Such forms f are said to be *saturated* with differentials dz_j. A similar remark applies to (p, n)-forms in $\mathbf{C}^n$.

The volume form in $\mathbf{C}^n$. For $n = 1$, writing $z = x + iy$,

$$d\overline{z} \wedge dz = d(x - iy) \wedge d(x + iy) = 2i dx \wedge dy,$$

hence in $\mathbf{C}^n = \mathbf{R}^{2n}$

$$\wedge_{j=1}^n (d\overline{z}_j \wedge dz_j) = (2i)^n \wedge_{j=1}^n (dx_j \wedge dy_j).$$

Thus, using an equal number of transpositions on each side,

$$d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \wedge dz_1 \wedge \ldots dz_n = (2i)^n dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n.$$

Using the customary notation $\omega(z) = dz_1 \wedge \ldots \wedge dz_n$, cf. (1b″), the *volume form* for $\mathbf{C}^n$ becomes

$$(2i)^{-n}\omega(\overline{z}) \wedge \omega(z) = dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n = \text{“}dm_{2n}\text{”}, \tag{4f}$$

provided we orient our $\mathbf{R}^{2n}$ as $\mathbf{R}^n \times \mathbf{R}^n$. [The more natural choice “dm_{2n}” $= dx_1 \wedge dy_1 \wedge dx_2 \wedge \ldots$, made by many authors, has the drawback that it introduces an unpleasant factor $(-1)^{n(n-1)/2}$ into formula (4f).]

10.5 Integrals in $\mathbf{C}^n$ involving the Martinelli-Bochner kernel. We begin once more with test functions φ, but now in $\mathbf{C}^n = \mathbf{R}^{2n}$. By Proposition (10.31),

$$\varphi(0) = -\frac{1}{\sigma(S_1)} \int_{\mathbf{C}^n} \left(\frac{\partial\varphi}{\partial x_1} x_1 + \frac{\partial\varphi}{\partial y_1} y_1 + \ldots + \frac{\partial\varphi}{\partial x_n} x_n + \frac{\partial\varphi}{\partial y_n} y_n \right) |z|^{-2n} dm(z). \tag{5a}$$

Using the definition of the derivatives $\partial\varphi/\partial z_j$ and $\partial\varphi/\partial\overline{z}_j$,

$$\text{(5}b\text{)} \qquad \frac{\partial\varphi}{\partial x_j}x_j + \frac{\partial\varphi}{\partial y_j}y_j = \frac{\partial\varphi}{\partial z_j}z_j + \frac{\partial\varphi}{\partial\overline{z}_j}\overline{z}_j = D_j\varphi\cdot z_j + \overline{D}_j\varphi\cdot\overline{z}_j.$$

Now there is a little miracle:

$$\text{(5}b'\text{)} \qquad \int_{\mathbf{C}^n} D_j\varphi\cdot z_j|z|^{-2n}dm = \int_{\mathbf{C}^n}\overline{D}_j\varphi\cdot\overline{z}_j|z|^{-2n}dm!$$

Indeed, for $n\geq 2$,

$$z_j|z|^{-2n} = c_n\overline{D}_j(|z|^2)^{1-n}, \quad \overline{z}_j|z|^{-2n} = c_nD_j(|z|^2)^{1-n}.$$

[For $n=1$ one has to use $\log|z|^2$.] Thus, using distributional bracket notation for convenience, $\int f\psi = \langle f,\psi\rangle$, $\langle\overline{D}_jf,\psi\rangle = -\langle f,\overline{D}_j\psi\rangle$ etc.,

$$\begin{aligned}\int z_j|z|^{-2n}D_j\varphi dm &= c_n\langle\overline{D}_j|z|^{2-2n},\ D_j\varphi\rangle\\ \text{(5}b''\text{)} \qquad &= -c_n\langle|z|^{2-2n},\ \overline{D}_jD_j\varphi\rangle - c_n\langle|z|^{2-2n},\ D_j\overline{D}_j\varphi\rangle\\ &= c_n\langle D_j|z|^{2-2n},\ \overline{D}_j\varphi\rangle = \int\overline{z}_j|z|^{-2n}\overline{D}_j\varphi dm.\end{aligned}$$

[This is basically ordinary integration by parts, cf. exercise 10.17.]

By the preceding, the integral in (5a) splits as a sum of two equal integrals, one involving $\Sigma D_j\varphi\cdot z_j$ and the other involving $\Sigma\overline{D}_j\varphi\cdot\overline{z}_j$. Choosing the latter, one obtains

Proposition 10.51. *For test functions φ on $\mathbf{C}^n$ [and in fact, by approximation, for all functions φ in $C_0^1(\mathbf{C}^n)$],*

$$\varphi(0) = -\frac{2}{\sigma(S_1)}\int_{\mathbf{C}^n}\left(\frac{\partial\varphi}{\partial\overline{z}_1}\overline{z}_1 + \ldots + \frac{\partial\varphi}{\partial\overline{z}_n}\overline{z}_n\right)|z|^{-2n}dm(z)\ [\sigma_{2n}(S_1) = 2\pi^n/\Gamma(n)].$$

As before, we wish to formulate the result with the aid of differential forms.

$$\overline{\partial}\varphi = \Sigma_j\frac{\partial\varphi}{\partial\overline{z}_j}d\overline{z}_j \quad\text{and}\quad \omega(z) = dz_1\wedge\ldots\wedge dz_n$$

we need

$$\omega_k(\overline{z}) \stackrel{\text{def}}{=} (-1)^{k-1}d\overline{z}_1\wedge\ldots[k]\ldots\wedge d\overline{z}_n.$$

Observe that

$$d\overline{z}_j\wedge\omega_k(\overline{z})\wedge\omega(z) = \begin{cases}0 & \text{if } k\neq j \quad [d\overline{z}_j \text{ will occur twice}],\\ \\ \omega(\overline{z})\wedge\omega(z) & \text{if } k=j \quad [\text{thanks to } (-1)^{k-1}],\end{cases}$$

so that by (4f),

$$\Sigma_j \frac{\partial\varphi}{\partial\overline{z}_j} d\overline{z}_j \wedge \Sigma_k \overline{z}_k |z|^{-2n}\omega_k(\overline{z}) \wedge \omega(z) = \Sigma_k \frac{\partial\varphi}{\partial\overline{z}_k}\overline{z}_k|z|^{-2n}\omega(\overline{z}) \wedge \omega(z)$$
$$= (2i)^n \Sigma_k \frac{\partial\varphi}{\partial\overline{z}_k}\overline{z}_k|z|^{-2n} dm_{2n}.$$

Thus by (10.51) a "good kernel" for use in conjunction with $\overline{\partial}\varphi$ is

$$\beta(z) = \beta_n(z) \stackrel{\text{def}}{=} b_n \sum_{k=1}^{n} \overline{z}_k |z|^{-2n}\omega_k(\overline{z}) \wedge \omega(z), \tag{5c}$$

where

$$b_n = 2(2i)^{-n}/\sigma_{2n}(S_1) = (n-1)!/(2\pi i)^n. \tag{5c'}$$

The $(n, n-1)$-forms β is called the *Martinelli-Bochner kernel* [Mart 1938], [Boch 1943].

Proposition 10.52. *The values of a function φ in $C_0^1(\mathbf{C}^n)b$ may be obtained from $\overline{\partial}\varphi$ with the aid of the kernel β:*

$$\varphi(0) = -\int_{\mathrm{id}|D} \overline{\partial}\varphi \wedge \beta, \quad \varphi(a) = -\int_{\mathrm{id}|D} \overline{\partial}\varphi(z) \wedge \beta(z-a).$$

Here D may be any closed cube about 0 or a that contains $\operatorname{supp}\varphi$

The properties of β are very similar to those of α, as are their proofs.

Properties 10.53 *of β:*
(i) $d\beta(z) = 0$ *for* $z \neq 0$ *(also* $\overline{\partial}\beta = 0$*);*
(ii) $\int_{S_r} \beta = \int_{S_1} \beta = 1, \quad \forall r > 0$*;*
(iii) $|\int_{S_r} u\beta| \leq \sup_{S_r} |u| \int_{S_1} |\beta|$ *for all continuous functions u.*

Theorem 10.54. *Let $\Omega \subset \mathbf{C}^n$ be a bounded domain with C^2 boundary, u a function of class C^1 on $\overline{\Omega}$. Then*

$$u(a) = \int_{\partial\Omega} u(z)\beta(z-a) - \int_{\Omega} \overline{\partial}u(z) \wedge \beta(z-a), \quad \forall a \in \Omega. \tag{5d}$$

In particular, for *holomorphic functions* f on $\overline{\Omega}$ ([Mart 1938], [Boch 1943]),

$$f(z) = \int_{\partial\Omega} f(\zeta)\beta(\zeta - z), \quad \forall z \in \Omega. \tag{5e}$$

Strictly speaking, we should have written id $|\,\overline{\Omega}$ under the integrals instead of just Ω. For the proof, observe that

$$\overline{\partial} u \wedge \beta = du \wedge \beta = d(u\beta)$$

since β is saturated with dz_j's; now proceed as in the case of Theorem (10.34). As in that result a piecewise C^1 boundary $\partial\Omega$ will suffice.

REMARKS 10.55. The integral (5e) expresses a holomorphic function f on Ω in terms of its boundary values. What sort of formula do we obtain for $n = 1$? In that case the product ω_k is empty, hence $\equiv 1$ and $\omega(z) = dz$. Thus

$$\beta(z) = \frac{1}{2\pi i}\frac{\overline{z}}{|z|^2}dz = \frac{1}{2\pi i}\frac{dz}{z}, \quad \beta(\zeta - z) = \frac{1}{2\pi i}\frac{d\zeta}{\zeta - z},$$

so that (5e) reduces to the familiar Cauchy integral formula when $n = 1$.

The kernel $\beta(\zeta - z)$ is the same for every domain $\Omega \subset \mathbf{C}^n$, but for $n \geq 2$ it is *not holomorphic* in z. Hence, except in the case $n = 1$, the integral (5e) does not generate holomorphic functions on Ω. Neither does formula (5d) solve the $\overline{\partial}$ equation when $n \geq 2$. On strictly pseudoconvex domains the kernel may be modified to remedy the situation, but the resulting kernels depend on the domain, cf. Section 10.8. In Section 10.6 we will obtain a holomorphic kernel for the case of the unit ball.

10.6 Szegö's integral for the ball. We begin with a lemma that can be used to write the Martinelli-Bochner theorem (10.54) in classical notation. Let $N_{x_k}(z)$ and $N_{y_k}(z)$ denote the x_k- and y_k -component of the outward unit normal $\vec{N}$ to $\partial\Omega$ at z and set

$$\nu_k(z) = N_{z_k} = N_{x_k} + iN_{y_k}, \quad k = 1, \ldots, n.$$

Ignoring constant factors, $\omega(\overline{z}) \wedge \omega(z)$ represents the volume element dm of Ω; similarly, the $(n, n-1)$-form

$$\sum_{k=1}^{n} \overline{\nu}_k(z)\omega_k(\overline{z}) \wedge \omega(z) \tag{6a}$$

will represent the "*area element*" $d\sigma$ of $\partial\Omega$. The precise result is as follows:

Lemma 10.61. *Let $\Omega \subset \mathbf{C}^n$ be a bounded domain with C^2 boundary and let φ be any C^1 function on $\partial\Omega$. Then*

$$\int_{\partial(\mathrm{id})|\overline{\Omega}} \varphi(z)\omega_k(\overline{z}) \wedge \omega(z) = \frac{1}{2}(2i)^n \int_{\partial\Omega} \varphi\nu_k d\sigma$$

$$\int_{\partial(\mathrm{id})|\overline{\Omega}} \varphi\Sigma_1^n\overline{\nu}_k\overline{\omega}_k \wedge \omega = \frac{1}{2}(2i)^n \int_{\partial\Omega} \varphi d\sigma.$$

PROOF. Since $\partial\Omega$ is smooth one can extend φ to a C^1 function on $\overline{\Omega}$. [One may use local parametrization to "straighten" $\partial\Omega$, cf. Section 9.4 and to make φ equal to zero except in a neighborhood of $\partial\Omega$.] Applying Stokes' theorem one obtains, writing id for id $|\,\overline{\Omega}$,

$$\int_{\partial(\mathrm{id})} \varphi\overline{\omega}_k \wedge \omega = \int_{\mathrm{id}} d(\varphi\overline{\omega}_k \wedge \omega) = \int_{\mathrm{id}} (\overline{D}_k\varphi)\overline{\omega} \wedge \omega = (2i)^n \int_\Omega \overline{D}_k\varphi dm,$$

cf. (4f). Now by Gauss's formula (2b),

$$\begin{aligned}\int_\Omega \overline{D}_k\varphi dm &= \int_\Omega \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \frac{1}{i}\frac{\partial}{\partial y_k}\right)\varphi dm \\ &= \frac{1}{2}\int_{\partial\Omega} \varphi(N_{x_k} + iN_{y_k})d\sigma = \frac{1}{2}\int_{\partial\Omega} \varphi\nu_k d\sigma.\end{aligned}$$

The second formula in the lemma follows by applying the first to $\varphi\overline{\nu}_k$,
$k = 1, \ldots, n$ and by adding:

$$\begin{aligned}\int_{\partial(\mathrm{id})} \sum \varphi\overline{\nu}_k\overline{\omega}_k \wedge \omega &= \frac{1}{2}(2i)^n \int_{\partial\Omega} \sum \varphi\overline{\nu}_k\nu_k d\sigma \\ &= \frac{1}{2}(2i)^n \int_{\partial\Omega} \varphi d\sigma. \quad [\Sigma|\nu_k|^2 = |\vec{N}|^2]\end{aligned}$$

Observe that for the case of the *unit ball* $B = B(0,1)$ and $S = \partial B$, one has

$$\nu_k(z) = x_k + iy_k = z_k.$$

Let f be in $\mathcal{O}(\overline{B})$; we will obtain an integral formula for f with holomorphic kernel. To this end we set

$$\beta(z,w) \stackrel{\text{def}}{=} b_n \frac{1}{(z\cdot w)^n} \Sigma_1^n w_k\omega_k(w) \wedge \omega(z), \quad [b_n \text{ as in } (5c')] \tag{6b}$$

so that $\beta(z,\overline{z})$ is equal to the Martinelli-Bochner kernel $\beta(z)$. Recall that $z\cdot w$ was defined as $z_1w_1 + \cdots + z_nw_n$. Define

$$g(z,w) = \int_S f(\zeta)\beta(\zeta - z, \overline{\zeta} - w). \tag{6c}$$

For $\zeta \in S$ and small $|z|, |w|$, the denominator of $\beta(\zeta - z, \overline{\zeta} - w)$ can not vanish, hence $g(z,w)$ is *holomorphic* on $B_r \times B_r$ if r is small [for example, $r = 1/3$].

By the Martinelli-Bochner theorem 10.54,

$$g(z,\overline{z}) = f(z), \quad z \in B. \tag{6d}$$

We will deduce that the power series

$$\Sigma a_{\lambda\mu} z^\lambda w^\mu \quad \text{for } g(z,w)$$

on $B_r \times B_r$ can not contain terms involving w. Indeed, replacing z by tz ($t \in \mathbf{R}$) and differentiating with respect to t, one finds that the following equality for power series:

$$g(z, \overline{z}) = \Sigma a_{\lambda\mu} z^\lambda \overline{z}^\mu = f(z) = \Sigma c_\nu z^\nu$$

implies equality of the homogeneous polynomials of the same degree:

$$\sum_{|\lambda|+|\mu|=k} a_{\lambda\mu} z^\lambda \overline{z}^\mu = \sum_{|\nu|=k} b_\nu z^\nu.$$

From this it readily follows by special choices of the variables that $a_{\lambda\mu} = 0$ for all $\mu \neq 0$.

In conclusion, the function $g(z, w)$ is independent of w on $B_r \times B_r$, so that for $|z| < r$, cf. (6b) and (6d),

$$\begin{aligned} f(z) &= g(z, \overline{z}) = g(z, 0) = \int_S f(\zeta)\beta(\zeta - z, \overline{\zeta}) \\ &= b_n \int_S \frac{f(\zeta)}{((\zeta - z) \cdot \overline{\zeta})^n} \Sigma_1^n \overline{\zeta}_k \omega_k(\overline{\zeta}) \wedge \omega(\zeta). \end{aligned} \tag{6e}$$

By the uniqueness theorem for holomorphic functions, the representation will hold for all $z \in B$. Applying the second formula of Lemma 10.61 to $\varphi(\zeta) = f(\zeta)/(1 - z \cdot \overline{\zeta})^n$ on $S = \partial B$ where $\nu_k(\zeta) = \zeta_k$, (6e) gives

Theorem 10.62 (*Szegö*). *Let f be holomorphic on $\overline{B}(0,1) \subset \mathbf{C}^n$. Then*

$$f(z) = \frac{(n-1)!}{2\pi^n} \int_{\partial B} \frac{f(\zeta)}{(1 - z \cdot \overline{\zeta})^n} d\sigma(\zeta), \quad \forall z \in B.$$

The constant in front of the integral comes from $b_n \cdot \frac{1}{2}(2i)^n$. As a check one may take $f \equiv 1$ and $z = 0$ which shows that the constant must equal $1/\sigma_{2n}(S_1)$. The representation (10.62) will actually hold for all continuous functions f on $\overline{B}$ that are holomorphic on B. [Applying the formula to $f(\lambda z)$ and let $\lambda \uparrow 1$.]

There is a related result for any convex domain with smooth boundary, cf. [Rud 4].

10.7 The Martinelli-Bochner transform and analytic continuation. For $n = 1$ and a smooth arc γ in $\mathbf{C}$, we know that the Cauchy transform

$$\hat{f}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

is holomorphic on the complement of γ for every continuous function f on γ. For $n \geq 2$ and a smooth $(2n-1)$-surface X in $\mathbf{C}^n$, one may ask for which functions f on X the *Martinelli-Bochner transform*

$$\hat{f}(z) \sup_{S_r} \int_X f(\zeta)\beta(\zeta - z) \tag{7a}$$

is holomorphic on the complement of X. This question will not be very interesting when $\mathbf{C}^n - X$ is connected. [Why not? Cf. exercise 10.30.] Thus let X be a "closed" surface.

Proposition 10.71. *Let X be a compact $(2n-1)$-surface of class C^2 in $\mathbf{C}^n$ without boundary. Let f be a C^1 function on X that satisfies the tangential Cauchy-Riemann equations, that is,*

$$d_X f(z) \wedge \omega(z) = 0, \tag{7b}$$

where $d_X f$ is df computed tangentially. Then the Martinelli-Bochner transform $\hat{f}$ is holomorphic on the complement of X. Furthermore, if $n \geq 2$ then $\hat{f} = 0$ on the unbounded component of $\mathbf{C}^n - X$.

Note that the tangential Cauchy-Riemann equations are certainly satisfied if f is holomorphic on a neighborhood of X. If the coordinate system is chosen such that the real tangent space at $a \in X$ is given by $\mathrm{Im} z_n = 0$, the tangential $C-R$ equations will reduce to

$$\frac{\partial f}{\partial \overline{z}_1} = \ldots = \frac{\partial f}{\partial \overline{z}_{n-1}} = 0. \tag{7c}$$

For the proof of the proposition we need some additional facts about β.

Lemma 10.72. *(i) For $\zeta_1 \neq z_1$ one has*

$$\beta(\zeta - z) = d_\zeta \beta^{(1)}(\zeta - z) = \overline{\partial}_\zeta \beta^{(1)}(\zeta - z),$$

where

$$(n-1)\beta^{(1)}(\zeta - z) = b_n \sum_{k=2}^{n} \frac{\overline{\zeta}_k - \overline{z}_k}{\zeta_1 - z_1} |\zeta - z|^{2-2n} \omega_{1k}(\overline{\zeta}) \wedge \omega(\zeta),$$
$$\omega_{1k}(\overline{\zeta}) = (-1)^k [1] d\overline{\zeta}_2 \wedge \ldots [k] \ldots \wedge d\overline{\zeta}_n$$

(the differentials $d\overline{\zeta}_1$ and $d\overline{\zeta}_k$ are absent).
(ii) For fixed z, the derivative $(\partial/\partial\overline{z}_1)\beta^{(1)}(\zeta - z)$ has a smooth extension $\{\ldots\}$ to $\mathbf{C}^n - \{z\}$:

$$\left\{ \frac{\partial}{\partial \overline{z}_1} \beta^{(1)}(\zeta - z) \right\} = b_n \sum_{k=2}^{n} (\overline{\zeta}_k - \overline{z}_k) |\zeta - z|^{-2n} \omega_{1k}(\overline{\zeta}) \wedge \omega(\zeta).$$

In terms of that extension,

$$\frac{\partial}{\partial \overline{z}_1} \beta(\zeta - z) = d_\zeta \left\{ \frac{\partial}{\partial \overline{z}_1} \beta^{(1)}(\zeta - z) \right\}.$$

For part (i) it is all right to take $z = 0$. The verification is similar to the computation that shows $d\beta = 0$ or $d\alpha = 0$. Part (ii) is simple. There are of course corresponding results with ζ_1 replaced by one of the other variables ζ_j.

PROOF of Proposition 10.71. Take $z \in \mathbf{C}^n - X$. By the definition of $\hat{f}$, the lemma, Stokes' theorem and the tangential $C-R$ equations (7b),

$$\begin{aligned}\frac{\partial}{\partial \overline{z}_1}\hat{f}(z) &= \int_X f(\zeta)\frac{\partial}{\partial \overline{z}_1}\beta(\zeta - z) = \int_X f(\zeta) d_\zeta \left\{\frac{\partial}{\partial \overline{z}_1}\beta^{(1)}(\zeta - z)\right\} \\ &= \int_X d_\zeta \left[f(\zeta)\left\{\frac{\partial}{\partial \overline{z}_1}\beta^{(1)}(\zeta - z)\right\}\right] - \int_X d_\zeta f(\zeta) \wedge \left\{\frac{\partial}{\partial \overline{z}_1}\beta^{(1)}(\zeta - z)\right\} \\ &= \int_{\partial X} f(\zeta)\left\{\frac{\partial}{\partial \overline{z}_1}\beta^{(1)}(\zeta - z)\right\} - 0 = 0. \qquad [\partial X = \emptyset]\end{aligned}$$

Similarly for the other derivatives $\partial/\partial\overline{z}_j$.

The final statement of the proposition follows from the fact that $\hat{f}$ is holomorphic on a connected domain $\mathbf{C}^n - E$, E compact and that $\hat{f}(z) \to 0$ as $|z| \to \infty$, cf. exercise 1.22.

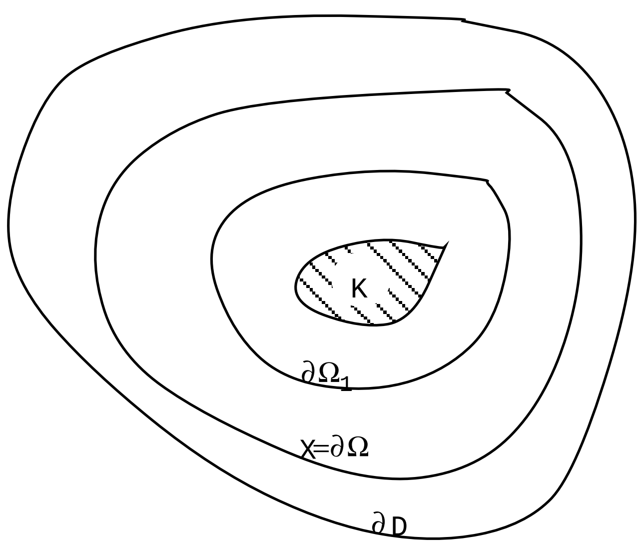
We can now give an integral formula for analytic continuation across a compact singularity set, resulting in another proof of the Hartogs-Osgood-Brown theorem (3.41).

Theorem 10.73. *Let D be a connected domain in $\mathbf{C}^n$, $n \geq 2$ and let K be a compact subset of D such that $D-K$ is connected. Let f be holomorphic on $D-K$. Taking Ω to be any bounded subdomain of D containing K and with connected C^2 boundary $X = \partial\Omega$ in D, the Martinelli-Bochner transform $\hat{f}$ of f corresponding to $X = \partial\Omega$ provides an analytic continuation of f across K.*

Proof. Besides Ω we consider a similar subdomain Ω_1 containing K, with $\overline{\Omega}_1 \subset \Omega$. By Proposition 10.71, the Martinelli-Bochner transforms $\hat{f}$ and $\hat{f}_1$ of f corresponding to $\partial\Omega$ and $\partial\Omega_1$ will be holomorphic on the complements of $\partial\Omega$ and $\partial\Omega_1$, respectively.

On the other hand, by the Martinelli-Bochner theorem 10.54,

$$f = \hat{f} - \hat{f}_1 \quad \text{on} \quad \Omega - \overline{\Omega}_1.$$



However, $\hat{f}_1 = 0$ outside $\overline{\Omega}_1$ (10.71), hence $\hat{f} = f$ on $\Omega - \overline{\Omega}_1$. This $\hat{f}$ provides an analytic continuation of f to Ω.

With a little more work one can obtain a stronger result: For Ω as above and $f \in C^1(\partial\Omega)$ satisfying the tangential Cauchy-Riemann equations, the transform $\hat{f}$ provides a holomorphic extension of f to Ω which is C^1 on $\overline{\Omega}$. For this and other results on Martinelli-Bochner transforms, see [Ran].

10.8. Good integral representations. In this section we will see how integral representations which are good in the sense that the boundary integral has a holomorphic kernel are obtained. Among other things, analogues of the Szegö formula will be found for general convex domains.

To obtain such integral representations we analyze the Martinelli-Bochner form (6b)

$$\beta(z,w) = b_n \frac{1}{(z \cdot w)^n} \sum_{k=1}^{n} w_k \omega_k(w) \wedge \omega(z).$$

The properties 10.53 are all that is needed to get a Martinelli-Bochner type integral formula. The most important (and hardest to achieve) is clearly 10.53 (i):

$$d\beta(z,\bar z) = \partial\beta(z,\bar z) + \bar\partial\beta(z,\bar z) = 0. \tag{8a}$$

One sees that $\partial\beta = 0$ because ω is saturated with dz_j; computation shows that $\bar\partial$ falls essentially on the second component of the argument of β i.e. the "$\overline{z}$ part" and does not "see" the first one. Also, from Section 10.6 we get the impression that the integral formula remains valid under some changes of the second component. Inroducing $F = \{(\zeta,\eta) \in \mathbf{C}^n \times \mathbf{C}^n$ such that $(\zeta \cdot \eta) = 0\}$, this suggests

Lemma 10.81. *The Martinelli-Bochner form β has the property that*

$$d\beta(\zeta,\eta) = 0 \quad (\zeta,\eta) \notin F.$$

[Here $d = d_\zeta + d_\eta = \partial_\zeta + \bar\partial_\zeta + \partial_\eta + \bar\partial_\eta$.]
PROOF. Observe that $\beta(\zeta,\eta)$ is saturated with $d\zeta$, hence $\partial_\zeta\beta = 0$; the coefficients of β are holomorphic in ζ and η, hence $\bar\partial_\zeta\beta = \bar\partial_\eta\beta = 0$. Finally we compute, using $d\eta_k \wedge \omega_j(\eta) = 0$ if $j \neq k$,

$$\partial_\eta\beta(\zeta,\eta) = b_n \sum_j \left(\frac{-n\zeta_j}{(\zeta\cdot\eta)^{n+1}} \eta_j d\eta_j + \frac{1}{(\zeta\cdot\eta)^n} d\eta_j \right) \wedge \omega_j(\eta) \wedge \omega(\zeta) = 0.$$

A *Leray map* will be a (smooth) map $\eta = \eta(z,\zeta)$ to $\mathbf{C}^n$ defined on a subset of $\mathbf{C}^n \times \mathbf{C}^n$ such that

$$(\zeta - z)\cdot\eta \neq 0 \qquad (z \neq \zeta). \tag{8b}$$

To a Leray map η we associate the kernel [$(n,n-1)$ form]

$$K^\eta(z,\zeta) = \beta(\zeta - z, \eta(z,\zeta)), \tag{8c}$$

here z is a (fixed) parameter.

COROLLARY 10.82. Let z be fixed. Suppose that $\eta(\zeta, z)$ satisfies (8b) as a function of ζ on a domain D. Then
$$d_\zeta K^\eta(z,\zeta) = 0 \quad \text{on } D.$$

PROOF. For fixed z, the form $K^\eta(z,\zeta)$ is the pull back of β under the map $\zeta \mapsto (\zeta - z, \eta(z,\zeta))$. Since the image of this map doesn't meet F, combination of Proposition 10.24 (v′) and previous Lemma gives $d_\zeta K^\eta(z,\zeta) = 0$ as the pull back of a closed form.

Proposition 10.83. *Let Ω be a domain in $\mathbf{C}^n$, $z \in \Omega$ fixed and $d(z) = d(z, \partial\Omega) > \epsilon$. Suppose that there exists a Leray map $\eta(z,\zeta)$ on $\{z\} \times \bar{\Omega}$, such that $\eta(z,\zeta) = \overline{\zeta - z}$ for $|\zeta - z| < \epsilon$. Then for all $f \in C^1(\bar{\Omega})$*

$$f(z) = \int_\Omega K^\eta(z,\zeta) \wedge \bar{\partial} f + \int_{\partial\Omega} K^\eta(z,\zeta) f$$

PROOF. For $|\zeta - z| < \epsilon$ we have $K^\eta(z,\zeta) = \beta(\zeta - z, \overline{\zeta - z})$ the Martinelli-Bochner form. Hence

$$f(z) = \int_{B(z,\epsilon)} K^\eta(z,\zeta) \wedge \bar{\partial} f + \int_{\{|\zeta|=\epsilon\}} K^\eta(z,\zeta) f. \tag{8d}$$

We apply Stokes' theorem to the last integral on the domain $\Omega \setminus B(z,\epsilon)$ and obtain

$$\begin{aligned}\int_{\{|\zeta|=\epsilon\}} K^\eta(z,\zeta) f &= \int_{\partial\Omega} K^\eta(z,\zeta) f - \int_{\Omega\setminus B(z,\epsilon)} d(K^\eta(z,\zeta) f) \\ &= \int_{\partial\Omega} K^\eta(z,\zeta) f + \int_{\Omega\setminus B(z,\epsilon)} (K^\eta(z,\zeta) \wedge \bar{\partial} f),\end{aligned} \tag{8e}$$

because $dK = 0$ on $\Omega \setminus B(z,\epsilon)$, K is saturated with $d\zeta$ and anticommutativity. Substitution of (8e) in (8d) completes the proof.

We also need a homogeneity property of the form

$$\omega'(w) = \sum_k w_k \omega_k(w). \tag{8d}$$

Lemma 10.84. *For every smooth function $f(w)$*

$$\omega'(f(w)w) = f^n(w)\omega'(w).$$

PROOF.

$$\begin{aligned}\omega'(f(w)w) &= \sum_k (-1)^{k-1} f(w) w_k \, d(f(w)w_1) \wedge \ldots [k] \ldots \wedge d(f(w)w_n) \\ &= \sum_k (-1)^{k-1} f(w) w_k \, (w_1 df(w) + f(w) dw_1) \wedge \ldots \\ &\qquad \ldots [k] \ldots \wedge (w_n df(w) + f(w) dw_n) \, .\end{aligned}$$

As $df(w) \wedge df(w) = 0$, this amounts to

$$\sum_k (-1)^{k-1} f(w)^n w_k dw_1 \wedge \ldots [k] \ldots \wedge dw_n$$

$$+ \sum_k (-1)^{k-1} f(w) w_k \left(\sum_{j<k} w_j (-1)^{j-1} df(w) \wedge dw_1 \wedge \ldots [j] \ldots [k] \ldots \wedge dw_n \right.$$

$$\left. + \sum_{j>k} w_j (-1)^{j-2} df(w) \wedge dw_1 \wedge \ldots [k] \ldots [j] \ldots \wedge dw_n \right).$$

The second part equals 0! The forms in this sum with coefficient $(-1)^{k-1+j-1} f(w) w_k w_j$ and $(-1)^{k-1+j-2} f(w) w_k w_j$ cancel. This proves the Lemma.

For a "clever" proof based on properties of determinants in non commutative rings, see [Hen-Lei].

Again, let Ω be a domain in $\mathbf{C}^n$ and U a neighborhood of $\partial\Omega$; let $\eta(z,\zeta) : \Omega \times U \to \mathbf{C}^n$ be a Leray map and let $\chi(z,\zeta) \in C^\infty(\Omega \times \Omega)$ be a nonnegative function such that $\chi(z,\zeta) = 1$ on a neighborhood of the diagonal $\{\zeta = z\} \subset \Omega \times \Omega$, while for fixed z, $\chi(z,\zeta) \in C_0^\infty(\Omega)$. Then we may form a Leray map $\tilde{\eta}$ on $\Omega \times \bar{\Omega}$:

$$\tilde{\eta}(z,\zeta) = \|\zeta - z\|^2 \left(\frac{1 - \chi(z,\zeta)}{(\zeta - z) \cdot \eta(z,\zeta)} \eta(z,\zeta) + \frac{\chi(z,\zeta)}{\|\zeta - z\|^2} \overline{(\zeta - z)} \right).$$

Clearly $(\zeta - z) \cdot \tilde{\eta} = \|\zeta - z\|^2$ and $\tilde{\eta} = \overline{\zeta - z}$ as a function of ζ close to $\zeta = z$. Thus we may apply Proposition 10.83 and we obtain that for any C^1 function f on $\bar{\Omega}$

$$f(z) = \int_{\partial\Omega} f(\zeta) K^{\tilde{\eta}}(z,\zeta) + \int_\Omega K^{\tilde{\eta}}(z,\zeta) \wedge \bar{\partial} f.$$

We write $\phi(z,\zeta) = \|\zeta - z\|^2 / (\zeta - z) \cdot \eta(z,\zeta)$ and observe that on $\partial\Omega$

$$K^{\tilde{\eta}} = b_n \frac{\omega'(\phi\eta)}{((\zeta - z) \cdot \phi\eta)^n} \wedge \omega(\zeta) = b_n \frac{\omega'(\eta)}{((\zeta - z) \cdot \eta)^n} \wedge \omega(\zeta) = K^\eta,$$

by Lemma 10.84. We have proved

Proposition 10.85. *Using notation as above, we have for every C^1 function f on Ω*

$$f(z) = \int_{\partial\Omega} f(\zeta) K^\eta(z,\zeta) + \int_\Omega K^{\tilde{\eta}}(z,\zeta) \wedge \bar{\partial} f.$$

If we can choose the Leray map to depend holomorphically on z, we have achieved with Proposition 10.85 a good analogue of the Cauchy Pompeiu formula. Indeed as with the usual Cauchy kernel we have a Cauchy type transform which yields holomorphic functions:

$$g(z) = \int_{\partial\Omega} f(\zeta) K^\eta(z,\zeta) \tag{8f}$$

is holomorphic for every continuous function f. Also we have

COROLLARY 10.86. Suppose that Ω admits a holomorphic Leray map. If the equation $\bar\partial u = v,\ (\bar\partial v = 0)$ admits a solution $u \in C^1(\bar\Omega)$ then the function

$$\tilde u(z) = \int_\Omega K^{\tilde\eta}(z,\zeta)\wedge v$$

also satisfies $\bar\partial \tilde u = v$.

PROOF. Express u by means of Proposition 10.85 and observe that the boundary integral represents a holomorphic function, hence $u - \tilde u$ is holomorphic.

Holomorphic Leray maps exist if Ω is C^1 and convex: Let $\Omega = \{\rho < 0\}$, take $\eta_j(\zeta) = \frac{\partial\rho}{\partial\zeta_j}$ independent of z. Then $(z-\zeta)\cdot\eta = \sum_j \frac{\partial\rho}{\partial z_j}(z_j - \zeta_j) \neq 0$ on Ω. In case of the unit ball this reduces the Cauchy type transform (8f) to the Szegö Kernel. It will be clear from Narasimhan's Lemma 9.48 that one can find such Leray maps at least locally on strictly pseudoconvex domains. The integral representation of Proposition 10.85 may be used to solve the Cauchy Riemann equations on strictly pseudoconvex domains, (without assuming solvability as we do in Corollary 10.86). This gives a solution of the Levi problem, cf. [Øvrelid]. However, to obtain estimates for solutions of the $\bar\partial$ equation, say in Hölder norms Proposition 10.85 admits much freedom and there is an other approach, which we will now discuss. Notice that the kernel $K^{\tilde\eta}$ is obtained by continuously deforming the Martinelli-Bochner kernel into our neat holomorphic kernel at the boundary. We can do a similar trick but now the deformation will take place on the boundary itself. Thus we consider the domain $Z = \partial\Omega \times [0,1]$, with coordinates (ζ,λ). Its boundary equals $\partial Z = \partial\Omega\times\{1\} - \partial\Omega\times\{0\}$.

Again assume that we have a Leray map η for Ω. We introduce the following Leray map on $\Omega\times\partial\Omega\times[0,1]$:

$$\tilde\eta(z,\zeta,\lambda) = \lambda\frac{\eta(z,\zeta)}{(\zeta - z)\cdot\eta(z,\zeta)} + (1-\lambda)\frac{\overline{\zeta - z}}{\|\zeta - z\|^2}, \quad (z \text{ fixed in } \Omega)$$

and the kernel

$$K^{\tilde\eta}(z,\zeta,\lambda) = \omega'(\tilde\eta)\wedge\omega(\zeta - z).$$

As before, $(\zeta - z)\cdot\tilde\eta(z,\zeta,\lambda) \neq 0$ and $K^{\tilde\eta}$ is a pull back of β, so that

$$dK = (\bar\partial_\zeta + \partial_\zeta + d_\lambda)K = 0.$$

Theorem 10.87. *Suppose that the domain Ω admits a Leray map $\eta(z,\zeta) : \Omega\times\partial\Omega \to \mathbf{C}^n$. With the kernel $K^{\tilde\eta}$ as defined above, the following integral representation is valid for $f \in C^1$.*

$$f(z) = \int_\Omega \beta(\zeta - z, \overline{\zeta - z})\wedge\bar\partial f + \int_{\partial\Omega\times[0,1]} K^{\tilde\eta}(z,\zeta,\lambda)\wedge\bar\partial_\zeta f + \int_{\partial\Omega} fK^\eta(z,\zeta).$$

PROOF. Starting with the Martinelli-Bochner representation for f, we only have to show that

$$\int_{\partial\Omega} f\beta(\zeta - z, \overline{\zeta - z}) = \int_{\partial\Omega\times[0,1]} K^{\tilde\eta}(z,\zeta,\lambda)\wedge\bar\partial_\zeta f + \int_{\partial\Omega} fK^{\eta}(z,\zeta). \tag{8g}$$

We apply Stokes' theorem to obtain

$$\int_{\partial\Omega\times[0,1]} d_{\zeta,\lambda}(f\wedge K^{\tilde\eta}(z,\zeta,\lambda)) = \int_{\partial\Omega\times\{1\}} fK^{\tilde\eta}(z,\zeta,1) - \int_{\partial\Omega\times\{0\}} fK^{\tilde\eta}(z,\zeta,0). \tag{8h}$$

The lefthand side is equal to $\int_{\partial\Omega\times[0,1]}\bar\partial_\zeta f\wedge K^{\tilde\eta}(z,\zeta,\lambda)$, because $d_{\zeta,\lambda}K = 0$, f does not depend on λ and K is saturated with $d\zeta$. For the righthand side, observe that $K^{\tilde\eta}(z,\zeta,0) = b_n\omega'(\frac{\zeta-z}{\|\zeta-z\|^2})\wedge\omega(\zeta-z)$ and $K^{\tilde\eta}(z,\zeta,1) = b_n\omega'(\frac{\eta(z,\zeta)}{(\zeta-z)\cdot\eta(z,\zeta)})\wedge\omega(z-\zeta)$. Lemma 10.85 gives that $K^{\tilde\eta}(z,\zeta,0) = \beta(\zeta-z,\overline{\zeta-z})$ and $K^{\tilde\eta}(z,\zeta,1) = K^{\eta}(z,\zeta)$. Substitution of all this in (8h) gives (8g). We are done.

How is Theorem 10.87 used to solve the Cauchy Riemann equations?. Assume that we have a C^1 $\bar\partial$ closed form v and that we know that a solution u with $\bar\partial u = v$ exists. Theorem 10.87 represents u and as before we see that

$$\tilde u = \int_\Omega v\wedge\beta(\zeta-z,\overline{\zeta-z}) + \int_{\partial\Omega\times[0,1]} v\wedge K^{\tilde\eta}(z,\zeta,\lambda)$$

is another solution. One easily checks that the integrand in $\int_{\partial\Omega\times[0,1]}$ is a polynomial in λ, hence we can integrate with respect to λ and this integral is reduced to an integral of v over $\partial\Omega$. It turns out that a situation has been reached in which one can make good estimates of $\tilde u$ in terms of v. The details are rather technical and we refer the reader to the literature cited in the beginning of this chapter.

Exercises

10.1. Equivalent parametrizations of a smooth arc are obtained by smooth maps of one parameter interval onto another with strictly positive derivative. How would one define equivalent parametrizations of a smooth 2-surface? A p-surface? The integral of a p-form over a smooth p-surface must have the same value for equivalent parametrizations.

10.2. Let f be a continuous p-form in $\Omega\subset\mathbb{R}^n$ in standard representation $\Sigma'_J f_J dx_J$. Suppose that $\int_X f = 0$ for all smooth p-surfaces X in Ω. Prove that $f_J = 0$ for every multi-index $J = (j_1,\ldots,j_p)$. [Choose $a\in\Omega$ and $\varepsilon > 0$ so small that $a+\varepsilon D\subset\Omega$, where D is the closed unit cube in $\mathbb{R}^n$. Now define X as follows:

$$X_{j_1}(t) = a_{j_1}+\varepsilon t_1,\ldots,X_{j_p}(t) = a_{j_p}+\varepsilon t_p,$$
$$X_k(t)\equiv a_k \quad\text{for}\quad k\neq j_1,\ldots,j_p;\ 0\le t_j\le 1.]$$

10.3. Let $f = \Sigma' f_J(x)dx_j$ be a continuous $(k-1)$-form on $\mathbb{R}^p - \{0\}$ such that $f(\lambda x) = f(x)$ for all $\lambda > 0$. Let S_r be the sphere $S(0,r)$ in $\mathbb{R}^p$, $r > 0$. Prove that the integrals $\int_{S_r} f$ and $\int_{S_r}|f|$ are independent of r.

10.4. Let f be a p-form in $\Omega \subset \mathbb{R}^n$, g a q-form. Prove that
(i) $g \wedge f = (-1)^{pq} g \wedge g$; (ii) $d(f \wedge g) = df \wedge g + (-1)^p f \wedge dg$.

10.5. Suppose one wants to apply Stokes' theorem to a disc and its boundary. Can one represent the disc as a smooth 2-surface with $[0,1] \times [0,1]$ as parameter domain? How would you deal with the annulus $A(0; \rho, R)$? If f is a smooth 1-form on $\overline{A}$, how would you justify wrting

$$\int_A df = \int_{C(0,R)} f - \int_{C(0,\rho)} f\,?$$

10.6. Go over the proofs of properties (iii)-(v) of pull backs. Could you explain the proofs to somebody else?

10.7. Use Theorem (10.34) to obtain the following $\mathbb{R}^2$ formula which is free of differential forms: for $a \in \Omega$,

$$u(a) = \frac{1}{2\pi} \int_{\partial\Omega} u(x) \frac{\partial}{\partial N} \log|x-a| ds - \frac{1}{2\pi} \int_{\Omega} \operatorname{grad} u(x) \cdot \operatorname{grad} \log|x-a| dm.$$

10.8. Apply the Gauss-Green theorem (2b) to $\vec{v} = u \operatorname{grad} E$ to obtain the $\mathbb{R}^n$ formula

$$\int_{\Omega} (\operatorname{grad} u \cdot \operatorname{grad} E + u \operatorname{div} \operatorname{grad} E) dm = \int_{\partial\Omega} u \frac{\partial E}{\partial N} d\sigma.$$

Then use a fundamental solution E of Laplace's equation in $\mathbb{R}^n$ to obtain in a form of Theorem (10.34) that is free of differential forms. [Start with $a = 0$.]

10.9. Let Ω be a bounded domain $\mathbb{R}^n$ with C^2 boundary, $N_k(x)$ the component of the outward unit normal $\vec{N}$ to $\partial\Omega$ at x. Show that the $(n-1)$-form $\Sigma_1^n N_k(x)\omega_k(x)$ represents the area element of $\partial\Omega$ in the sense that for all smooth functions φ on $\partial\Omega$,

$$\int_{\partial(\mathrm{id})|\overline{\Omega}} \varphi \Sigma_1^n N_k \omega_k = \int_{\partial\Omega} \varphi d\sigma.$$

[Extend $\varphi \Sigma N_k \omega_k$ smoothly to $\overline{\Omega}$.]

10.10. Use differentiation to relate $m(B_r)$ in $\mathbb{R}^n$ to $\sigma(S_r)$ and deduce that
$\sigma(S_1) = n \cdot m(B_1)$

10.11. Prove that the different forms $dz_J \wedge d\overline{z}_K$ in $\wedge^{p,q}$, where J and K run over the increasing p-indices and q-indices, are linearly independent over $\mathbb{C}$. [Start with the real situation.]

10.12. Prove that $\wedge^s$ decomposes uniquely in $\mathbb{C}^n$ as

$$\wedge^s = \wedge^{s,0} + \wedge^{s-1,0} + \ldots + \wedge^{0,s}.$$

10.13. Calculate ∂f, $\overline{\partial} f$ and df when $f = \overline{z}_1 dz_2 + z_2 d\overline{z}_1$.

10.14. Prove that for a C^1 form f in $\wedge^{p,q}$, one has $df = 0$ only if $\partial f = \overline{\partial} f = 0$. Does this also hold in $\wedge^s$?

10.15. Prove that for $f \in \wedge^{p,q}$, one has $\bar\partial \bar f = \overline{\partial f}$.

10.16. Determine g such that $E = g(z \cdot \overline{z})$ satisfies Laplace's equation on $\mathbb{C}^n - \{0\}$:

$$\Delta E = 4 \sum_1^n \frac{\partial^2 E}{\partial z_j \partial \overline{z}_j} = 0.$$

10.17. Prove that for test functions φ on $\mathbb{C}^n = \mathbb{R}^{2n}$,

$$\int_{\mathbb{C}^n} \left(\frac{\partial \varphi}{\partial x_1} x_1 + \frac{\partial \varphi}{\partial y_1} y_1 \right) |z|^{-2n} dm = 2 \int_{\mathbb{C}^n} \frac{\partial \varphi}{\partial \overline{z}_1} \overline{z}_1 |z|^{-2n} dm$$

by showing first that for all $z' = (z_2, \ldots, z_n) \neq 0$,

$$\int_{\mathbb{R}^2} \left(\frac{\partial \varphi}{\partial x_1} y_1 - \frac{\partial \varphi}{\partial y_1} x_1 \right) |z|^{-2n} dx_1\, dy_1 = 0.$$

10.18. Let φ be a test function on $\mathbb{C}^n$. Show that

$$\begin{aligned} \varphi(0) &= -\frac{1}{n} \int_{\mathbb{C}} \frac{\partial \varphi}{\partial z_1}(z_1, 0, \ldots, 0) \frac{1}{z_1} dm(z_1) \\ &= \Big(-\frac{1}{n} \Big)^n \int_{\mathbb{C}^n} \frac{\partial^n \varphi}{\partial \Sigma_1 \ldots \partial \overline{z}_n} \, \frac{1}{z_1 \ldots z_n} \, dm(z). \end{aligned}$$

10.19. Show that the above formula can be extended to include the case $\varphi(z) = (1 - |z|^2)^n$ for $|z| \leq 1,\ \varphi(z) = 0$ for $|z| > 1$. Deduce that in $\mathbb{C}^n = \mathbb{R}^{2n},\ m(B_1) = \pi^n / n!$.

10.20. (*A Sobolev-type lemma*). Let u be an L^2 function on $\mathbb{C}^n$ of bounded support and such that the distributional derivative $\partial^n u / \partial \overline{z}_1 \ldots \partial \overline{z}_n$ is equal to an L^p function where $p > 2$. Prove that u is a.e. equal to a continuous function. [From exercise 10.18 and Hölder's inequality it may be derived that $|\varphi(0)| \leq C \|\partial^n \varphi / \partial \overline{z}_1 \ldots \partial \overline{z}_n\|_p$ and similarly $\sup |\varphi| \leq \ldots$. Deduce that $u \star \rho_\varepsilon$ tends to a limit function uniformly as $\varepsilon \downarrow 0$.]

10.21. Verify the properties of the Martinelli-Bochner kernel β in (10.56).

10.22. Supply the details in the proof of the Martinelli-Bochner theorem (10.57), starting with the case $a = 0$ and then passing on to the case of general $a \in \Omega$.

10.23. Let f be holomorphic on $\overline{\Omega}$ where Ω is as in Theorem (10.54). Prove directly [without using Section 10.7] that

$$\int_{\partial \Omega} f(\zeta) \beta(\zeta - z) = 0 \quad \text{for} \quad z \in \mathbb{C}^n - \overline{\Omega}.$$

10.24. What representation for C^1 functions does Theorem (10.54) give in the case $n = 1$?

10.25. Show that for holomorphic functions f on $\overline{B} = \overline{B}(0,1) \subset \mathbb{C}^n$,

$$f(z) = \frac{(n-1)!}{2\pi^n} \int_{\partial B} f(\zeta) \frac{1 - \overline{z} \cdot \zeta}{|\zeta - z|^{2n}} d\sigma(\zeta).$$

10.26. Derive a form of Theorem (10.57) that is free of differential forms.

10.27. Show that the following forms can serve as area element $d\sigma$ on the unit sphere $S = \{z \in \mathbb{C}^2 : z_1\overline{z}_1 + z_2\overline{z}_2 = 1\}$:

$$-\frac{1}{2}(\overline{z}_1 d\overline{z}_2 - \overline{z}_2 d\overline{z}_1) \wedge dz_1 \wedge dz_2, \ \frac{1}{2z_2} d\overline{z}_1 \wedge dz_1 \wedge dz_2, \ \frac{1}{2\overline{z}_2} d\overline{z}_1 \wedge d\overline{z}_2 \wedge dz_1.$$

10.28. In $\mathbb{C}^2$ the following automorphism of $B = B(0,1)$ takes the point $c = (c_1, 0) \in B$ to the origin:

$$\zeta_1^1 = \frac{\zeta_1 - c_1}{1 - \overline{c}_1\zeta_1}, \ \zeta_2^1 = \frac{(1-|c_1|^2|)^{\frac{1}{2}}}{1 - \overline{c}_1\zeta_1}\zeta_2.$$

Use the mean value theorem for holomorphic functions f on $\overline{B}$ to derive that

$$f(c_1, 0) = \frac{1}{2\pi^2}\int_S \frac{(1-|c_1|^2)^2}{|1 - c_1\overline{\zeta}_1|^4} f(\zeta)d\sigma(\zeta).$$

Deduce the so-called invariant Poisson integral for $f(z)$:

$$f(z) = \frac{1}{2\pi^2}\int_S \frac{(1-|z|^2)^2}{|1 - z\cdot\overline{\zeta}|^4} f(\zeta)d\sigma(\zeta), \quad z \in B.$$

10.29. Express the "invariant Poisson Kernel" $P(z, \overline{\zeta})$ of exercise 10.28 in terms of the Szegö kernel $S(z, \overline{\zeta}) = (1 - z\cdot\overline{\zeta})^{-2}$. Now use the Szegö integral to derive the preceding formula. Extend the latter to $\mathbb{C}^n$.

10.30. Let X be a smooth $(2n-1)$-surface in $\mathbb{C}^n$, $n \geq 2$ such that $X^c = \mathbb{C}^n - X$ is connected. Let $f \in C(X)$ be such that the Martinelli-Bochner transform $\hat{f}$ is holomorphic on X^c. Prove that $\hat{f} \equiv 0$.

10.31. Prove that statements about the tangential Cauchy-Riemann equations made after Proposition (10.71).

10.32. Let X be a smooth $(2n-1)$-surface in $\mathbb{C}^n$ with real defining function ρ. [$\rho \in C^p$ on a neighborhood of X for some $p \geq 1$, $\rho = 0$ on X, $d\rho \neq 0$ on X.] Prove that $f \in C^1(X)$ satisfies the tangential $C-R$ equations if and only if

$$\overline{\partial} f \wedge \overline{\partial}\rho = 0 \quad \text{or} \quad \frac{\partial f}{\partial \overline{z}_j}\frac{\partial \rho}{\partial \overline{z}_k} - \frac{\partial f}{\partial \overline{z}_k}\frac{\partial \rho}{\partial \overline{z}_j} = 0, \quad \forall j, k.$$

CHAPTER 11

Solution of the $\bar\partial$ equation on pseudoconvex domains

In this chapter we discuss Hörmander's ingenious L^2 method with weights for the global solution of the inhomogeneous Cauchy-Riemann equations

$$\bar\partial u = v \quad \text{or} \quad \frac{\partial u}{\partial \bar z_j} = v_j, \quad j = 1, \ldots, n$$

on domains in $\mathbf{C}^n$.

The method applies to all domains Ω which possess a plurisubharmonic exhaustion function. It proves the existence of weak or distributional solutions u on Ω that are locally equal to L^2 functions when the v_j's are. If the functions v_j are of class C^p $(1 \le p \le \infty)$, it readily follows that the solutions are also of class C^p. Taking $p = \infty$, one concludes that every psh exhaustible domain is a $\bar\partial$ domain [as defined in Chapter 7] and hence is a Cousin -I domain [the holomorphic Cousin-I problem is generally solvable on Ω]. Applying the results also to the intersections of Ω with affine subspaces of $\mathbf{C}^n$, one obtains the solution of the Levi problem: Every pseudoconvex domain is a domain of holomorphy [cf. Section 7.7]. Some remarks are in order here. One began to search for analytic approaches to the $\bar\partial$ problem around 1950. There where important contributions contributions by many authors, notably Morrey and Kohn, before Hörmander obtained his weighted L^2 results in 1964. Estimating solutions of $\bar\partial$ equations has remained in active area of research up till the present time. It has turned out that while Hörmander's introduction of weights in the problem gives a fast and clean solution by sweeping all unpleasant boundary behavior under the rug, the approach of Kohn and his students, notably Catlin, although very involved, is more fundamental and gives much more precise results, with wider applications.

In a sense, postulating pseudoconvexity of Ω for the solution of the "first order" $\bar\partial$ problem is too much: for $n \ge 3$, $\bar\partial$ domains or Cousin-I domains need not be pseudoconvex [cf. Section 7.2]. However, Hörmander's method also gives solutions to the "higher order" $\bar\partial$ equations on pseudoconvex domains, cf Section 11.8 and [Hör 1]. The general solvability of the $\bar\partial$ equations of every order on Ω is equivalent to the property of pseudoconvexity; we will return to this matter in Chapter 12. A more important benefit of Hörmander's method is that it provides useful growth estimates for the solution of the $\bar\partial$-equation [see Section 11.7]. Such estimates are finding applications even in the case $n = 1$; further applications in $\mathbf{C}^n$ may be expected.

For $n = 1$ the principal existence theorem may be derived in a more or less straightforward manner [Section11.3], but for $n \ge 2$ the proof remains rather involved. The ideas in our exposition are of course Hörmander's, with some small modifications. We do not explicitly use any results on unbounded operators. The principal tool is F. Riesz's theorem to the effect that every continuous linear functional on a Hilbert space is represented by an inner product function.

11.1 Distributions and weak solutions. Let Ω be a domain or open set in $\mathbf{R}^n$ or $\mathbf{C}^n$. Test functions ϕ on Ω are complex-valued functions of class $C_0^\infty(\Omega)$, that is, C^∞ functions whose supports are compact subsets of Ω. Examples of test functions have been encountered in section 3.3, namely:

(i) the standard C^∞ approximation of the identity ρ_ϵ on $\mathbf{R}^n$ whose support is the ball $\bar{B}(0,\epsilon)$: $\rho_\epsilon(x) = \epsilon^{-n}\rho_1(|x|/\epsilon)$, $\int \rho_\epsilon = 1$, $\rho_\epsilon \geq 0$;

(ii) for compact F, a C^∞ cutoff function ω on $\mathbf{R}^n$ which equals 1 on F and 0 at distance $\geq \epsilon$ from F, cf. Proposition (3.).

DEFINITION 11.11. A *distribution*

$$T : \phi \mapsto T(\phi) = \langle T, \phi \rangle$$

on Ω is a *linear functional* on the space of test functions $C_0^\infty(\Omega)$:

$$\langle T, \lambda\phi + \mu\psi \rangle = \lambda\langle T, \phi\rangle + \mu\langle T, \psi\rangle.$$

We say that distributions T_ν are (weakly or distributionally) *convergent* to the distribution T as $\nu \to \nu_0$ or $\nu \to \infty$ if

$$\langle T_\nu, \phi\rangle \;\to\; \langle T, \phi\rangle, \quad \forall \phi \in C_0^\infty(\Omega).$$

The symbol $\langle T, \phi\rangle$ denotes a (bi)linear functional; the symbol (,) is reserved for an inner product which is conjugate linear in the second factor. It is customary to impose a weak continuity condition on the functionals $\langle T, \phi\rangle$ as functions of ϕ: starting with a strong notion of convergence $\phi_\nu \to \phi$ in the vector space of test functions, one requires that $\langle T, \phi_\nu\rangle \to \langle T, \phi\rangle$ whenever $\phi_\nu \to \phi$. Since such continuity of the functional T is not important in the present context, we do not go into detail. Cf., [Schwa], [Hör 2].

EXAMPLES 11.12. Every continuous or locally integrable function f on Ω defines a distribution by the formula

$$\langle f, \phi\rangle = \int_\Omega f\phi \, dm = \int_{\mathrm{supp}\phi} f\phi \, dm.$$

In the sequel we often omit the Lebesgue measure or "volume element" dm on Ω. Locally uniform (or locally L^1) convergence of functions f_ν to f on Ω implies distributional convergence:

$$|\langle f, \phi\rangle - \langle f_\nu, \phi\rangle| \leq \int_{\mathrm{supp}\phi} |f - f_\nu| \cdot \sup|\phi| \to 0 \quad \text{as} \quad \nu \to \nu_0.$$

Derivatives of test functions are again test functions. Repeated integration by parts will show that

$$f_\nu(x) = \nu^{100} e^{i\nu x} \to 0 \qquad \text{weakly on } \mathbf{R} \text{ as } \nu \to \infty.$$

The famous *delta distribution* on $\mathbf{R}^n$ is given by the formula

$$\langle \delta, \phi\rangle = \phi(0).$$

A distribution T is said to *vanish on an open subset* $\Omega_0 \subset \Omega$ if $\langle T, \phi \rangle = 0$ for all test functions with support in Ω_0. Taking Ω_0 equal to the maximal open subset of Ω on which T vanishes, the complement $\Omega \setminus \Omega_0$ is called the *support* of T. Two distributions are said to be equal on an open subset if their difference vanishes there. For continuous functions these definitions agree with the usual ones. This will follow from

Proposition 11.13. *Let f be a continuous or locally integrable function on $\Omega \subset \mathbf{R}^n$ such that $\langle f, \phi \rangle = 0$ for all test functions ϕ on Ω. Then $f(x) = 0$ almost everywhere on Ω, and in particular at all points x where f happens to be continuous.*

PROOF. Let $\{\rho_\epsilon\}$ be the standard nonnegative approximate identity on $\mathbf{R}^n$ with $\mathrm{supp}\rho_\epsilon = \bar{B}(0,\epsilon)$ [Section 3.3]. Then for any compact subset $K \subset \Omega$ and $0 < r < d(K, \partial\Omega)$, the function $\rho_\epsilon(x-y)$ with $x \in K$ fixed and y variable will be a test function on Ω whenever $\epsilon \le r$. Hence since $\langle f, \rho_\epsilon(x - \cdot) \rangle = 0$

$$\begin{aligned} 0 = \langle f, \rho_\epsilon(x-\cdot)\rangle &= \int_\Omega f(y)\rho_\epsilon(x-y)\,dy = f * \rho_\epsilon(x) = \int_{B(0,\epsilon)} f(x-z)\rho_\epsilon(z)\,dz \\ &= \int_{B(0,1)} f(x-\epsilon y)\rho_1(y)\,dy, \quad \forall x \in K. \end{aligned}$$

Now for continuous f, using uniform continuity on the [closure of the] r-neighborhood K_r of K,

$$\int_K |f(x) - f(x-\epsilon y)|\,dx \to 0 \quad \text{as } \epsilon \downarrow 0,$$

uniformly for $y \in B = B(0,1)$. This holds more generally for all locally integrable f: such functions may be approximated in L^1 norm on $\bar{K}_r$ by continuous functions.

By the preceding we have

$$\begin{aligned} \int_K |f(x)|\,dx &= \int_K dx \left| \int_B \{f(x) - f(x-\epsilon y)\}\rho_1(y)\,dy \right| \le \int_K dx \int_B |\ldots|\,dy \\ &= \int_B \{\int_K |f(x) - f(x-\epsilon y)|\,dx\}\rho_1(y)\,dy, \end{aligned}$$

where the final member tends to 0 as $\epsilon \downarrow 0$. [Since we deal with positive functions the inversion of the order of integration is justified by Fubini's theorem.] Hence one has $\int_K |f(x)|\,dx = 0$, and since K may be any compact subset of Ω, the proposition follows.

Proposition 11.14. *The test functions ϕ on Ω are dense in $L^1(\Omega)$ and $L^2(\Omega)$.*

PROOF. Let $\Omega \subset \mathbf{R}^n$. The continuous functions with compact support are dense in $L^1(\Omega)$ and also in $L^2(\Omega)$. Any continuous function f with compact support is a uniform limit of functions ϕ_ν in C_0^∞ with support in a fixed compact set K, e.g., through convolution with an approximate identity. Finally,

$$\|f - \phi_\nu\|_p^p \le \mathrm{vol}(K) \cdot (\sup_K |f - \phi_\nu|)^p, \quad p = 1, 2,$$

Hence uniform convergence leads to L^p convergence.

The most important notion in distribution theory is that of *distributional derivatives*:

DEFINITION 11.15. The partial derivatives of the distribution T on $\Omega \subset \mathbf{R}^n$ are defined by formal integration by parts:

$$\langle \frac{\partial T}{\partial x_j}, \phi \rangle = -\langle T, \frac{\partial \phi}{\partial x_j} \rangle.$$

In $\mathbf{C}^n$ this leads to formulas for

$$D_j T = \frac{\partial T}{\partial z_j} = \frac{1}{2}\left(\frac{\partial T}{\partial x_j} + \frac{1}{i}\frac{\partial T}{\partial y_j} \right), \qquad \bar{D}_j T = \frac{\partial T}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial T}{\partial x_j} - \frac{1}{i}\frac{\partial T}{\partial y_j} \right):$$

$$\langle D_j T, \phi \rangle = -\langle T, D_j \phi \rangle, \qquad \langle \bar{D}_j T, \phi \rangle = -\langle T, \bar{D}_j \phi \rangle.$$

Observe that the distributional derivatives are again distributions. For functions f of class C^1, integration by parts gives precisely the result of the definition:

$$\langle \frac{\partial f}{\partial x_j}, \phi \rangle = \int_\Omega \frac{\partial f}{\partial x_j} \phi \, dx_1 \dots dx_n = -\int_\Omega f \frac{\partial \phi}{\partial x_j} \, dx_1 \dots dx_n = -\langle f, \frac{\partial \phi}{\partial x_j} \rangle.$$

The boundary integrals vanish because ϕ has compact support. It follows that for such functions the (first order) distributional derivatives agree with the ordinary derivatives in their action on test functions. Defining the product of a C^∞ function ω and a distribution T by $\langle \omega T, \phi \rangle = \langle T\omega, \phi \rangle = \langle T, \omega\phi \rangle$, one has the usual rule for differentiation of ωT. In higher order distributional derivatives, the *order of differentiation is immaterial* since this is so for test functions. Distributional differentiation is a *continuous* operation: if $T_\nu \to T$ in the distributional sense then $\frac{\partial T_\nu}{\partial x_j} \to \frac{\partial T}{\partial x_j}$:

$$\langle \frac{\partial T_\nu}{\partial x_j}, \phi \rangle = -\langle T_n u, \frac{\partial \phi}{\partial x_j} \rangle \to -\langle T, \frac{\partial \phi}{\partial x_j} \rangle = \langle \frac{\partial T}{\partial x_j}, \phi \rangle.$$

We can now define a weak (locally integrable) solution of the $\bar{\partial}$ problem

$$\bar{\partial} u = v = \sum_1^n v_j \, d\bar{z}_j \quad \text{or} \quad \bar{D}_j = v_j, \; j = 1, \dots, n$$

on $\Omega \subset \mathbf{C}^n$. Here it is assumed that the coefficients v_j of the (0,1) form v are locally integrable functions.

DEFINITION 11.16. A locally integrable function u on Ω is called a weak solution of the equation $\bar{\partial} u = v$ if the distributional derivatives $\bar{D}_j u$ are equal to the functions v_j, considered as distributions on Ω. That is, for each $j = 1, \dots n$ and for all test functions $\phi \in C_0^\infty(\Omega)$,

$$\langle \bar{D}_j u, \phi \rangle = -\int_\Omega u \bar{D}_j \phi \, dm = \int_\Omega v_j \phi \, dm = \langle v_j, \phi \rangle.$$

Observe that the equation $\bar{\partial} u = v$ can have a weak solution u only if

$$\bar{D}_j v_k = \bar{D}_j \bar{D}_k u = \bar{D}_k \bar{D}_j = u = \bar{D}_k v_j, \quad \forall j, k$$

in the sense of distributions. In terms of the (0,2) form or "tensor"

$$\bar\partial_1 v \stackrel{\text{def}}{=} \sum_{1\le j<k\le n} (\bar D_j v_k - \bar D_k v_j)\, d\bar z_j \wedge d\bar z_k, \tag{1b}$$

the resulting (local) *integrability conditions* may be summarized by

$$\bar\partial_1 v = 0 \qquad \text{(often written } \bar\partial v = 0\text{)}. \tag{1c}$$

11.2 When weak solutions are ordinary solutions. The L^2 method will provide weak (global) solutions of the equation $\bar\partial u = v$. Here we will show that for smooth forms v such weak solutions are (almost everywhere) equal to ordinary smooth solutions. We begin with an auxiliary result for the homogeneous equation $\bar\partial u = 0$.

Proposition 11.21. *Let u be an integrable function on the polydisc $\Delta(a,s) \subset \mathbf{C}^n$ such that $\bar\partial u = 0$ in the weak sense. Then there is a holomorphic function h such that $u = h$ almost everywhere on $\Delta(a,s)$.*

PROOF. It is sufficient to prove the result for the unit polydisc $\Delta = \Delta(0,1)$ and as the result is local, it will be convenient to assume u is extended to a neighborhood U of $\bar\Delta$ and satisfies there $\bar\partial u = 0$ weakly. Now form the C^∞ functions

$$u_\epsilon(z) \stackrel{\text{def}}{=} u * \rho_\epsilon(z) = \int_{\mathbf{C}^n} u(\zeta)\rho(z-\zeta)\, dm(\zeta), \quad \epsilon > 0 \tag{2a}$$

where $\{\rho_\epsilon\}$ is the standard C^∞ approximate identity on $\mathbf{C}^n = \mathbf{R}^{2n}$, in particular supp$\rho_\epsilon \subset \bar B(0,\epsilon)$ and ρ_ϵ is radial, cf. Section 3.3. We will take $r < d(\Delta, \partial U)/2$ and ϵ always less than r. Note that if u is a genuine holomorphic function, the mean value property over spheres gives that $u_\epsilon = u$. [By introducing polar coordinates in (2a).] Since for $a \in \Delta$ $\rho_\epsilon(a-\zeta) \in C_0^\infty(U)$ we have

$$\frac{\partial u_\epsilon}{\partial \bar z_j}(a) = \int u(\zeta)\frac{\partial \rho_\epsilon}{\partial \bar z_j}(a-\zeta)\, dm(\zeta) = \langle u, \bar D_j \rho_\epsilon(a-\cdot)\rangle = \langle \bar D_j u, \rho_\epsilon(a-\cdot)\rangle = 0.$$

Hence u_ϵ is holomorphic on Δ. On the other hand one easily shows that

$$u_\epsilon \longrightarrow u \quad \text{in} \quad L^1(\Delta) \quad \text{as} \epsilon \downarrow 0, \tag{2b}$$

cf. the proof of Proposition 11.13. If u is continuous the convergence in (2b) is uniform, hence u is continuous. For the general case there is a clever trick: Form

$$(u * \rho_\delta) * \rho_\epsilon = (u * \rho_\epsilon) * \rho_\delta = u_\epsilon,$$

the first equality by general properties of convolution and the second by the remark after (2a). Now let $\epsilon \to 0$. We find $u * \rho_\delta = u$ almost everywhere, and the proof is complete.

Theorem 11.22. *Suppose that the equation $\bar{\partial} u = v$ on $\Omega \subset \mathbf{C}^n$, with $v = \sum_1^n v_j \, d\bar{z}_j$ of class C^p $(1 \le p \le \infty)$, has a weak (locally integrable) solution u_0 on Ω. Then the equation has C^p solutions f on Ω and u_0 is almost everywhere equal to one of them.*

PROOF. By Section 11.1, $\bar{\partial}_1 v = \bar{\partial}_1 \bar{\partial} u_0 = 0$ in distributional and hence ordinary sense. It follows that our equation has local solutions of class C^p [Proposition 7.58]. Hence every point $a \in \Omega$ belongs to a polydisc $U_\lambda \subset\subset \Omega$ on whose closure the equation $\bar{\partial} u = v$ has a C^p solution f_λ. Every other integrable solution on U_λ is almost everywhere equal to f_λ plus some holomorphic function h_λ. [Apply Proposition 11.21 to the difference with f_λ.] This will in particular be the case for our global weak solution u_0:

$$u_0 = f_\lambda + h_\lambda \quad \text{a.e. on } U_\lambda, h_\lambda \in \mathcal{O}(U_\lambda). \tag{2c}$$

Ω is covered by polydiscs U_λ. On an intersection $U_{\lambda\mu}$,

$$f_\lambda + h_\lambda \overset{\text{a.e.}}{=} u_0 \overset{\text{a.e.}}{=} f_\mu + h_\mu,$$

hence the smooth functions on the left and right must be equal throughout $U_{\lambda\mu}$. Thus we may define a global C^p function f on Ω by setting

$$f \overset{\text{def}}{=} f_\lambda + h_\lambda \quad \text{on} \quad U_\lambda, \quad \forall \lambda.$$

This f will be an ordinary solution of our $\bar{\partial}$ equation on Ω: on each U_λ, $\bar{\partial} f = \bar{\partial} f_\lambda = v$. Finally $u_0 = f$ a.e. on Ω.

11.3 General solvability of $\bar{\partial}$ for $n = 1$. Hörmander's L^2 method is best explained in the simple case of a planar domain Ω. There is no restriction on the open set $\Omega \subset \mathbf{C}$ and we will think of v as a locally square integrable function [rather than a form]. The $\bar{\partial}$ equation then becomes

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y}\right) = v \quad \text{or} \quad \bar{D} u = v \quad \text{on } \Omega \subset \mathbf{C}. \tag{3a}$$

Since there is only one complex variable, there is no integrability condition.

A weak L^2 solution of (3a) is a locally square integrable function u on Ω such that $\langle \bar{D} u, \phi \rangle = \langle v, \phi \rangle$ for all test functions ϕ on Ω. we may express this condition in terms of the inner product $(\ ,\ )_0$ of $L^2(\Omega)$, replacing ϕ by $\bar{\phi}$:

$$\begin{aligned} (v, \phi)_0 &= \int_\Omega v \bar{\phi} \, dm = \langle v, \phi \rangle = \langle \bar{D} u, \bar{\phi} \rangle \\ &= -\langle u, \bar{D} \bar{\phi} \rangle = -(u, D\phi)_0, \qquad D = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right). \end{aligned} \tag{3a$'$}$$

The essential idea is to look for a solution u in an appropriate *weighted* space $L^2_\beta = L^2(\Omega, e^{-\beta})$, where β is a real C^∞ function. Here the inner product is

$$(f, g)_\beta = \int_\Omega f \bar{g} e^{-\beta} \, dm = \int f \bar{g} e^{-\beta}. \tag{3b}$$

We use the same notation $(v, \phi)_\beta$ if v is locally in L^2 and $\phi \in C_0^\infty(\Omega)$. The domain Ω and the Lebesgue measure dm on Ω will usually be omitted from our integrals.

Condition (3a′) must also hold for test functions $e^{-\beta}\phi$ instead of ϕ:

$$
\text{(3c)} \qquad \begin{aligned} (v, \phi)_\beta &= (v, \phi e^{-\beta})_0 = (\bar{D}, \phi e^{-\beta})_0 = -(u, D\{\phi e^{-\beta}\})_0 \\ &= -(u, \{D\phi - D\beta \cdot \phi\} e^{-\beta})_0 = (u, -D\phi + D\beta \cdot \phi)_\beta = (u, \delta\phi)_\beta, \quad \forall \phi. \end{aligned}
$$

Here we have written δ for the *formal adjoint* to $\bar{D}$ relative to the weight $e^{-\beta}$, which as usual is defined by:

$$
\text{(3c}'\text{)} \qquad \delta = \delta_\beta = -D + D\beta \cdot \mathrm{id}, \qquad (\bar{D}u, \phi)_\beta = (\bar{D}, \phi e^{-\beta})_0 = (u, \delta\phi)_\beta \quad \forall \phi.
$$

Observe that (3c) is completely equivalent with (3a′): the product $\phi e^{-\beta}$ runs over all test functions on Ω precisely when ϕ does. We will use (3c) to derive a necessary and sufficient condition for the existence of a weak solution in L^2_β:

Proposition 11.31. *The equation $\bar{D}u = v$ with v locally in L^2, has a weak solution u in $L^2_\beta = L^2(\Omega, e^{-\beta})$ if and only if there is a constant $A = A_v$ independent of ϕ such that*

$$
\text{(3d)} \qquad |(\phi, v)_\beta| = |(v, \phi)_\beta| \le A \|\delta\phi\|_\beta, \quad \forall \phi \in C_0^\infty(\Omega).
$$

Under condition (3d) there is a solution u_0 of minimal norm $\|u_0\|_\beta \le A$; it is orthogonal to all holomorphic functions in L^2_β.

PROOF. (i) If u is a weak solution in L^2_β, then by (3c)

$$
|(v, \phi)_\beta| = |(u, \delta\phi)_\beta| \le \|u\|_\beta \|\delta\phi\|_\beta, \quad \forall \phi.
$$

(ii) Suppose we have (3d). Then the pairing

$$
\text{(3e)} \qquad l : \delta\phi \mapsto (\phi, v)_\beta, \quad \forall \phi
$$

will define a continuous linear functional on the linear subspace W of L^2_β, consisting of all test functions of the form $\delta\phi$. Indeed, l is well defined on W because $\delta\phi_1 = \delta\phi_2$ implies $(\phi_1, v)_\beta = (\phi_2, v)_\beta$, see (3d) with $\phi = \phi_1 - \phi_2$. By the same inequality the linear functional has norm $\le A$. We extend l by continuity to the closure $\bar{W}$ of W in L^2_β: if ψ_k in W tends to ψ in L^2_β, $l(\psi_k)$ tends to a limit [Cauchy criterion] which we call $l(\psi)$. The extended linear functional will still be called l and there is no change in norm.

Applying the *Riesz representation theorem* to l on the Hilbert space $H = \bar{W}$, we conclude that there is a unique element $u_0 \in \bar{W} \subset L^2_\beta$ such that

$$
\text{(3e}'\text{)} \qquad l(w) = (w, u_0)_\beta, \quad \forall w \in \bar{W}
$$

and

$$
\text{(3f)} \qquad \|u_0\|_\beta = \|l\| \le A.
$$

Specializing to $w = \delta\phi$ we obtain the relation

$$(\phi, v)_\beta = l(\delta\phi) = (\delta\phi, u_0)_\beta \quad \text{or} \quad (v, \phi)_\beta = (u_0, \delta\phi)_\beta, \ \forall \phi.$$

By (3c), u_0 is a weak solution of the equation $\bar{D}u = v$ on Ω; by (3f), it satisfies the growth condition $\|u_0\|_\beta \leq A$.

(iii) The solution of equation (3a) in L^2_β is unique up to a solution of the homogeneous equation $\bar{D}u = 0$, that is, up to a holomorphic function h in L^2_β. Thus the general solution has the form $u = u_0 + h$ with u_0 as above. The solution u_0 in $\bar{W}$ will be orthogonal to every holomorphic h in L^2_β. Indeed,

$$0 = (\bar{D}h, \phi)_\beta = (h, \delta\phi)_\beta, \quad \forall \phi,$$

hence $h \perp W$ and therefore $h \perp u_0 \in \bar{W}$. Thus our special solution u_0 has minimal norm in L^2_β: $\|u_0 + h\|^2_\beta = \|u_0\|^2_\beta + \|h\|^2_\beta$.

Derivation of a suitable basic inequality (3d). The starting point is provided by an important a priori inequality for test functions ϕ. When we compute the commutator of $\bar{D}$ and its adjoint $\delta = \delta_\beta$, the Laplacian of β will appear:

$$(\bar{D}\delta - \delta\bar{D})\phi = \bar{D}(-D\phi + D\beta \cdot \phi) - (-D + D\beta \cdot \mathrm{id})\bar{D}\phi = \bar{D}D\beta \cdot \phi.$$

We will set

$$\bar{D}D\beta = \beta_{z\bar{z}} = \frac{1}{4}\Delta\beta = b.$$

The commutator formula shows that

$$(b\phi, \phi)\beta = (\bar{D}D\beta \cdot \phi, \phi) = (\bar{D}\delta\phi, \phi) - (\delta\bar{D}\phi, \phi) = (\delta\phi, \delta\phi)_\beta - (\bar{D}\phi, \bar{D}\phi)_\beta.$$

Thus we arrive at the following

A PRIORI INEQUALITY 11.32:

$$\int_\Omega |\phi|^2 e^{-\beta} b \leq \int_\Omega |\delta\phi|^2 e^{-\beta}, \quad \forall \phi \in C_0^\infty(\Omega).$$

In order to exploit (11.32) we have to impose the condition

$$b = \frac{1}{4}\Delta\beta > 0,$$

that is β must be *strictly subharmonic* on Ω, cf. Section 8.3. We now apply the Schwarz inequality to $(\phi, v)_\beta$. Since we have (11.32) it is natural to estimate in the following way:

$$\begin{aligned} |(\phi, v)_\beta|^2 &= \left|\int \phi(e^{-\beta}b)^{1/2} \cdot \bar{v}(e^{-\beta}b^{-1})^{1/2}\right|^2 \\ &\leq \int |\phi|^2 e^{-\beta} b \int |v|^2 e^{-\beta} b^{-1} \leq \int |v|^2 e^{-\beta} b^{-1} \|\delta\phi\|^2_\beta. \end{aligned} \tag{3g}$$

In words

BASIC INEQUALITY ($n = 1$) 11.33. For every strictly subharmonic function $\beta \in C^\infty(\Omega)$ and every function $v \in L^2_{\beta + \log b} = L^2(\Omega, e^{-\beta}b^{-1})$, there is an inequality (3d) with

$$A = A_v = \|v\|_{\beta + \log b}.$$

Combining (11.33) and Proposition 11.31 and referring to Theorem 11.22 on the existence of smooth solutions, we obtain

First Main Theorem 11.34. (case $n = 1$). *Let $\beta \in C^\infty(\Omega)$ be strictly subharmonic, so that $b = \beta_{z\bar{z}} > 0$. Let v be any function in $L^2(\Omega, e^{-\beta}b^{-1})$. Then there exists a function u in $L^2(\Omega, e^{-\beta})$ which solves the equation $\bar{D}u = v$ in the weak sense on Ω and which satisfies the growth condition*

$$\int_\Omega |u|^2 e^{-\beta}\, dm \le \int_\Omega |v|^2 e^{-\beta} b^{-1}\, dm.$$

If v is of class C^p on Ω, $1 \le p \le \infty$, the solution u can be modified on a set of measure zero so as to become a classical C^p solution.

It is easy to show that for every C^p function v on Ω, there is a strictly subharmonic function β such that v is in $L^2(\Omega, e^{-\beta}b^{-1})$, cf. Lemma 11.63 below. Thus the $\bar{\partial}$ equation is generally solvable on every planar domain. For other applications it is convenient to derive a second main theorem which does not involve derivatives of β (Section 11.7). In the next sections we will extend the first main theorem to pseudoconvex domains in $\mathbf{C}^n$.

11.4 The L^2 method for $\bar{\partial}$ when $n \ge 2$. We will describe how to obtain weak L^2 solutions of the equation $\bar{\partial}u = v$ on domains $\Omega \subset \mathbf{C}^n$. Here v is a (0,1) form $\sum_1^n v_j\, d\bar{z}_j$ that is locally in L^2 [that is $v_j \in L^2_{\text{loc}}(\Omega)$, $\forall j$ and which satisfy the integrability condition $\bar{\partial}_1 v = 0$ (1b). More precisely, our forms v as well as the solutions u will belong to certain weighted spaces $L^2_\beta = L^2(\Omega, e^{-\beta})$, where β is a real C^∞ function. For (0,1) forms the defining inner product is

$$(4a) \qquad (f, g)_\beta = \int_\Omega \left(\sum_1^n f_j \bar{g}_j \right) e^{-\beta}\, dm = \sum_1^n (f_j, g_j)_\beta.$$

We also write $f \cdot \bar{g}$ for $\sum_1^n f_j \bar{g}_j$ and $|g|^2$ for $\sum_1^n |g_j|^2$. The same notations are used if f is only locally in L^2 while G is a (0,1) *test form* ϕ on Ω, that is a form $\sum_1^n \phi_j\, d\bar{z}_j$ whose coefficients are test functions. Analogous definitions will apply to (0,2) forms such as $\bar{\partial}_1 v$ in formula (1a). If the context permits, Ω, dm and the weight index β will be omitted from the formulas.

A *weak solution* of the equation $\bar{\partial}u = v$ on Ω is a locally integrable function U such that

$$(\bar{D}_j, \phi_j)_0 = -(u, D_j \phi_j)_0 = (v_j, \phi_j)_0, \quad j = 1, \ldots, n$$

for all test functions ϕ_j on Ω, cf. Definition 11.17 with $\bar{\phi}_j$ instead of ϕ. Introducing the weight functions $e^{-\beta}$, this requirement may be written in the equivalent form

$$(v_j, \phi_j)_\beta = (u, -D_j \phi_j + D_j \beta \cdot \phi_j)_\beta = (u, \delta_j \phi_j)_\beta, \quad \forall j, \phi_j,$$

cf. (3c) We can summarize those equations by a single condition:

$$(4b) \qquad \begin{aligned} (v, \phi)_\beta = (\bar{\partial}u, \phi)_\beta &= \sum_j (\bar{D}_j u, \phi_j)_\beta = \sum_j (u, \delta_j \phi_j)_\beta \\ &= (u, \delta\phi)_\beta, \qquad \forall \text{ test forms } \phi = \textstyle\sum_1^n \phi_j\, d\bar{z}_j \text{ on } \Omega. \end{aligned}$$

Here we have used the inner product notation (4a) for forms and the corresponding notation (3b) for functions, while

$$(4b') \qquad \delta\phi = \delta_\beta\phi = \sum_1^n \delta_j\phi_j, \qquad \delta_j = -D_j + (D_j\beta)\mathrm{id}.$$

By (4b) $\delta = \delta_\beta$ is the *formal adjoint* to $\bar\partial$ relative to the weight $e^{-\beta}$. [Observe that δ sends (0,1) forms to functions.]

It readily follows from (4b) that the equation $\bar\partial u = v$ has a *weak solution* u in $L^2_\beta(\Omega)$ if and only if there is a *basic inequality*

$$(4c) \qquad |(\phi, v)_\beta| = |(v,\phi)_\beta| \le A\|\delta\phi\|_\beta, \quad \forall \text{ test forms } \phi \text{ on } \Omega,$$

where $A = A_v$ is a constant independent of ϕ. Indeed, Proposition 11.31 immediately extends to the n-dimensional situation; the proof remains virtually unchanged. The next step is to derive a suitable a priori inequality for test forms.

For test *functions* ψ on Ω and the operators $\bar D_j$ and their adjoints δ_j in L^2_β, we have the commutator relations

$$(\bar D_k\delta_j - \delta_j\bar D_k)\psi = \bar D_k(-D_j\psi + D_j\beta\cdot\psi) - (-D_j + (D_j)\beta\cdot\mathrm{id})\bar D_k\psi = D_j\bar D_k\beta\cdot\psi.$$

Thus for all test *forms* ϕ on Ω, taking $\psi = \phi_j$ and using the inner product of the function space L^2_β,

$$(4d) \qquad \begin{aligned} \sum_{j,k=1}^n \left(D_j\bar D_k\beta\cdot\phi_j, \phi_k\right) &= \sum_{j,k}\left(\bar D_k\delta_j\phi_j, \phi_k\right) - \sum_{j,k}\left(\delta_j\bar D_k\phi_j, \phi_k\right) \\ &= \sum_{j,k}(\delta_j\phi_j, \delta_k\phi_k) + \{-\sum_{j,k}\left(\bar D_k\phi_j, \bar D_j\phi_k\right)\}. \end{aligned}$$

The first term on the right is equal to

$$(4d') \qquad \Big(\sum_j \delta_j\phi_j, \sum_k \delta_k\phi_k\Big) = \|\delta\phi\|^2.$$

The last term in (4d) may be rewritten as

$$-\sum_{j,k} = \frac{1}{2}\sum_{j,k}\left(\bar D_k\phi_j - \bar D_j\phi_k, \bar D_k\phi_j - \bar D_j\phi_k\right) - \sum_{j,k}\left(\bar D_k\phi_j, \bar D_k\phi_j\right),$$

hence by the definition of $\bar\partial_1$ in (1a),

$$(4d'') \qquad -\sum_{j,k}\left(\bar D_k\phi_j, \bar D_j\phi_k\right) = \sum_{1\le j<k\le n}\left\|\bar D_k\phi_j - \bar D_j\phi_k\right\|^2 - \sum_{j,k}\left\|\bar D_k\phi_j\right\|^2 \le \|\bar\partial_1\phi\|^2.$$

As to the left-hand side of (4d), writing $b = b(z)$ for the smallest eigenvalue $\lambda_\beta(z)$ of the complex Hessian of $\beta(z)$, one has $\sum (D_j \bar{D}_k \beta)\phi_j \bar{\phi}_k \geq b|\phi|^2$, hence

$$(4d''') \qquad \sum_{j,k} \left(D_j \bar{D}_k \beta \cdot \phi_j, \phi_k\right) \geq \int_\Omega b|\phi|^2 e^{-\beta}.$$

Combining all the relations (4d), we obtain the following

A PRIORI INEQUALITY 11.41 (for test forms)

$$\int_\Omega |\phi|^2 e^{-\beta} b \leq \|\delta\phi\|_\beta^2 + \|\bar{\partial}_1 \phi\|_\beta^2, \quad b = \lambda_\beta,\ \delta = \delta_\beta.$$

For the application of (11.41), we will require that $b(z)$ be > 0 on Ω, in other words that β be *strictly plurisubharmonic.*

Because of the final term $\|\bar{\partial}_1 \phi\|^2$ in (11.41), it is not possible to obtain a basic inequality (4c) for $(v, \phi)_\beta$ by straightforward application of Schwarz's inequality as in the case $n = 1$ (3g). In order to keep the norm $\|\bar{\partial}\phi\|$ small, one has to use the fact that $\bar{\partial}_1 v = 0$. Let us assume for the moment that v is in L^2_β. [If necessary, one can initially replace Ω by a suitable subdomain or adjust β outside suppϕ.] The idea is to split the test form ϕ into two parts, one in the null space N of $\bar{\partial}_1$ in L^2_β and one orthogonal to it:

$$(4e) \qquad \phi = f + g, \quad f \in N,\ g \perp N.$$

We will verify that N is closed, so that the decomposition is possible, and that as a result

$$(4f) \qquad (v, \phi)_\beta = (v, f)_\beta, \quad \bar{\partial}_1 f = 0, \quad \delta_\beta f = \delta_\beta \phi \in C_0^\infty(\Omega).$$

Indeed suppose $f_\nu \to \tilde{f}$ in L^2_β and $\bar{\partial}_1 f_\nu = 0$ for all ν in the sense of distributions, in other words,

$$\langle \bar{D}_j f_{\nu k} - \bar{D}_k f_{\nu j}, \phi_{jk} \rangle = -\langle f_{\nu k}, \bar{D}_j \phi_{jk} \rangle + \langle f_{\nu j}, \bar{D}_k \phi_{jk} \rangle = 0$$

for all test functions ϕ_{jk} and all j, k. Passing to the limit in the second member, one concludes that $\bar{\partial}_1 \tilde{f} = 0$, hence $\tilde{f} \in N$. Thus the orthogonal decomposition (4e) exists and since $v \in N$, one has $(v, g)_\beta = 0$ and the first part of (4f) follows. Finally, note that $\bar{\partial}\psi$ is in N for every test function ψ on Ω: $\bar{\partial}_1 \bar{\partial}\psi = 0$. Thus

$$(4f') \qquad 0 = (g, \bar{\partial}\psi)_\beta = (\delta_\beta g, \psi)_\beta, \quad \forall \psi \in C_0^\infty(\Omega)$$

so that $\delta g = 0$ and $\delta f = \delta\phi$.

With (4f) in hand and aiming for a basic inequality (4c), we would like to proceed as follows, cf. (3g), (11.33):

$$
\begin{aligned}
\|(v,\phi)_\beta\|^2 = \|(v,f)_\beta\|^2 &\le \int |f|^2 e^{-\beta} b \\
&\le \|v\|^2_{\beta+\log b}(\|\delta f\|^2_\beta + \|\bar\partial_1 f\|^2_\beta) = A_v^2\|\delta\phi\|^2_\beta.
\end{aligned}
\tag{4g}
$$

Observe that the central step would require an extension of the a priori inequality (11.41) to more general forms f in L^2_β for which δf and $\bar\partial_1 f$ are also in L^2_β. If Ω is all of $\mathbf{C}^n$, such an extension may be proved by straightforward approximation of f by test forms, cf. the approximation theorem 11.51 below. However, on general pseudoconvex Ω, the approximation 11.51 requires modification of the weight function near the boundary of Ω. It is difficult to see then how one could prove the precise analog to (11.41) for our form f, In Section 11.5 we will carefully select a different weight function $e^{-\gamma}$, where $\gamma \ge \beta$ grows very rapidly towards the boundary of Ω. We then decompose our test form ϕ in L^2_γ to prove the desired

BASIC INEQUALITY 11.42. For psh exhaustible $\Omega \subset \mathbf{C}^n$, strictly psh β on Ω [so that $b = \lambda_\beta > 0$] and every (0,1) form v in $L^2(\Omega, e^{-\beta}b^{-1})$ with $\bar\partial_1 v = 0$, one has [just as for n=1!]

$$|(\phi, v)_\beta| \le \|v\|_{\beta+\log b}\|\delta_\beta\phi\|_\beta \quad \text{for all test forms } \phi \text{ on } \Omega.$$

As in Section 11.3, the existence of L^2 solutions to the $\bar\partial$ equation will now follow from the Riesz representation theorem. For the precise result, see Section 11.6.

11.5. Proof of the basic inequality. Let Ω, β and v be as in the statement of the inequality (11.42) and let ϕ be a given test form on Ω. If we decompose ϕ as in (4e), the question arises whether we can extend the a priori inequality (11.41) to more general forms f with δf and $\bar\partial_1 f$ in L^2_β. The answer is yes if we know that f is in $L^2_{\beta-\sigma}$, where the (continuous) function σ becomes sufficiently large near the boundary of Ω:

$$\sigma(z) \ge 2\log^+ \frac{c}{d(z)} \quad \text{for some constant } c > 0 \text{ and } d(z) = d(z, \partial\Omega). \tag{5a}$$

[If $\Omega = \mathbf{C}^n$ one may simply take $\sigma = 0$.] Such a result may be derived from the following

Approximation Theorem 11.51. *To any given (0,1) form f in $L^2_{\beta-\sigma}$ with δf and $\bar\partial_1 f$ in L^2_β and any number $\epsilon > 0$, there is a test form ψ on Ω such that*

$$\|f-\psi\|_\beta + \|\delta(f-\psi)\|_\beta + \|\bar\partial_1(f-\psi)\|_\beta < \epsilon.$$

Here the adjoint δ may belong to β (4b$'$) or to any other given C^∞ function α on Ω.

PROOF. Let f be as in the theorem.

(i) Suppose first that $f = \sum_1^n f_j\, d\bar z_j$ has compact support $K \subset\subset \Omega$. Then one can use approximating test forms ψ of the type $f * \rho_\epsilon$, where $\{\rho_\epsilon\}$ is the usual standard C^∞ approximate identity on $\mathbf{C}^n$ with $\mathrm{supp}\rho_\epsilon = \bar B(0,\epsilon)$, cf. the proofs of (11.13), (11.14).

Indeed, since f, δf and $\bar\partial_1 f$ are in the weighted L^2 spaces on Ω, we have $f_j \in L^2(K)$, $\forall j$ and

$$\delta f = \delta_\alpha f = -\sum D_j f_j + \sum (D_j\alpha) f_j \in L^2(K) \quad \bar\partial_1 f \in L^2(K) \quad \text{(coefficientwise)};$$

as a consequence, also $\sum D_j f_j \in L^2(K)$. Taking $\epsilon < r < d(K, \partial\Omega)/2$, one finds that $\operatorname{supp}(f * \rho_\epsilon) \subset K_r$, the r-neighborhood of K. As in the proof of Proposition 11.14, we then have the following convergence relations in $L^2(K_r)$ [and hence in $L^2(\Omega, e^{-\beta})$] when $\epsilon \downarrow 0$:

$$\begin{aligned}
&f_j * \rho_\epsilon \to f_j, \qquad \forall j,\\
&\delta(f * \rho_\epsilon) = -\left(\sum D_j f_j\right) * \rho_\epsilon + \sum (D_j\alpha)(f_j * \rho_\epsilon) \to \delta f,\\
&\bar\partial_1 (f * \rho_\epsilon) = (\bar\partial_1 f) * \rho_\epsilon \to \bar\partial_1 f \quad \text{(coefficientwise)}.
\end{aligned}$$

(ii) The general case is reduced the the preceding with the aid of cutoff functions ω, but these have to be chosen with some care. Making use of the standard exhaustion of Ω by compact sets

$$E_s = \{z \in \Omega : \; d(z) \geq 1/s, \; |z| \leq s\}, \quad s = 1, 2 \ldots,$$

we take $\rho_\epsilon(z) = \epsilon^{-2n}\rho_1(z/\epsilon)$ as before and define

$$\omega_s = \chi_s * \rho_r : \quad \chi_s \text{ characteristic function of } E_s, \quad r = 1/2s.$$

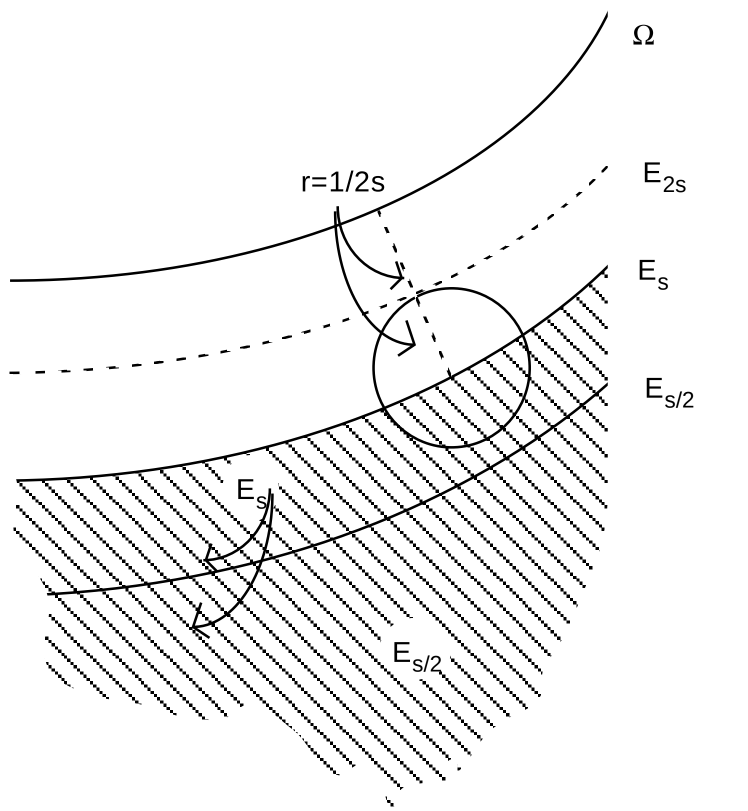
By this definition (cf. fig 11.1), $\operatorname{supp}\omega_s \subset E_{2s}$ and $\omega_s = 1$ on a neighborhood of $E_{s/2}$, so that $\partial\omega_s = D_1\omega_s \, dz_1 + \cdots + D_n\omega_s \, dz_n$ has its support in $E_{2s} - E_{s/2}$. It follows that

$$\begin{aligned}
|D_j\omega_s(z)| = |\bar D_j\omega_s(z)| = |\chi_s * D_j\rho_r(z) &= \left| \int_{B(0,r)} \chi_s(z-\zeta) D_j\rho_r(\zeta)\, dm(\zeta) \right|\\
&\leq \frac{1}{r} \int_{B(0,1)} |D_j\rho_1(w)|\, dm(w) = 2sc_1 < 4c_1/d(z),
\end{aligned} \tag{5b}$$

hence

$$|\delta\omega_s| \leq c_2/d(z)$$

since $d(z) < 2/s$ on $\operatorname{supp}\partial\omega_s$.



Now let $\eta > 0$ be given. We will show that for large s,

$$\|f - \omega_s\| < \eta, \quad \|\delta f - \delta(\omega_s f)\| < \eta, \quad \|\bar\partial_1 f - \bar\partial_1(\omega_s f)\| < \eta. \tag{5c}$$

The first inequality requires only that $f \in L^2_\beta$ [we know more]:

$$\|f - \omega_s\|^2 \le \int_{\Omega \setminus E_{s/2}} |f|^2 e^{-\beta} < \eta^2 \quad \text{for } s > s_1.$$

For the second inequality we observe that

$$\delta(\omega_s f) = -\sum D_j(\omega_s f_j) + \sum (D_j \alpha)\omega_s f_j = \omega_s \delta f - \sum (D_j \omega_s) f_j.$$

Thus

$$\|\delta f - \delta(\omega_s f)\| \le \|\delta f - \omega_s \delta f\| + \|\,|\partial\omega_s|\cdot|f|\,\|;$$

The proof is completed by the estimates

$$\|\delta f - \omega_s\delta f\|^2 \le \int_{\Omega\setminus E_{s/2}} |\delta f|^2 e^{-\beta} < \eta^2/4 \quad \text{for } s > s_2,$$

$$\int |\partial\omega_s|^2 |f|^2 e^{-\beta} \le \int_{E_{2s}\setminus E_{s/2}} c_2^2 |f|^2 e^{-\beta}/d^2$$

$$(c_2^2/c^2)\int_{\Omega\setminus E_{s/2}} |f|^2 e^{-\beta+\sigma} < \eta^2/4 \quad \text{for } s > s_3.$$

In the final step we have used (5b) and inequality (5a): $2\log c/d \le \sigma$; by our hypothesis, f is in $L^2_{\beta-\sigma}$.

The proof of the third inequality (5c) is similar, cf. exercise 11.14. With (5c) established, the proof of Theorem 11.51 is completed by part (i).

New decomposition of ϕ. Returning to the proof of the basic inequality, the difficulty is that in general, the form f in the decomposition (4e) will not be in $L^2_{\beta-\sigma}$. We therefore recommence and do our splitting of ϕ in a space L^2_γ, where γ will be determined later. To begin with, we require that

$$\gamma \ge \beta \text{ on } \Omega, \quad \gamma = \beta \text{ on } \mathrm{supp}\phi, \quad \gamma \ge \beta + \log b \text{ near } \partial\Omega \cup \infty.$$

By the last condition, $v \in L^2_{\beta+\log b}$ will be in L^2_γ. We now split

$$(5d) \qquad \phi = f + g, \quad f \in N(\bar\partial_1) \subset L^2_\gamma, \quad g \perp N.$$

Since $v \in N$, so that $(v,g)_\gamma = 0$ and hence $\delta_\gamma g = 0$ [cf. (4f)],

$$(5d') \qquad (v,\phi)_\beta = (v,\phi)_\gamma = (v,f)_\gamma, \quad \bar\partial_1 f = 0, \quad \delta_\gamma f = \delta_\gamma\phi = \delta_\beta\phi.$$

By Schwarz's inequality,

$$(5e) \qquad |(v,\phi)_\beta|^2 = |(v,f)_\gamma|^2 \le \int |v|^2 e^{-\gamma+\sigma} b^{-1} \int |f|^2 e^{-\gamma-\sigma} b.$$

[The reason for having the factor $\exp(-\gamma-\sigma)$ in the last integral is that we later want to approximate f by test forms, taking the β of the approximation theorem equal to $\gamma+\sigma$.]

It will be necessary to impose suitable additional conditions on σ and γ. The definitive requirements on σ are:

$$(5f) \qquad \sigma \in C^\infty,\ \lambda_\sigma \ge 0, \quad \sigma = 0 \text{ on } K \stackrel{\text{def}}{=} \mathrm{supp}\phi, \quad \sigma(z) \ge 2\log^+ c/d(z).$$

The function γ will be taken C^∞ strictly psh and $\geq \beta+\sigma$, so that the last integral with v in (5e) is finite. The complete set of requirements for γ is listed in (5h) below.

Adjusted a priori inequality for test forms. Our aim is to estimate the final integral in (5e). To that end we first derive an ad hoc inequality for test forms ψ and then we will use approximation. The a priori inequality (11.41) with ϕ replaced by ψ and β by $\gamma+\sigma$ gives

$$(5g)\qquad \int_\Omega |\psi|^2 e^{-\gamma-\sigma}\lambda_{\gamma+\sigma} \leq \|\delta_{\gamma+\sigma}\psi\|^2_{\gamma+\sigma} + \|\bar\partial_1\psi\|^2_{\gamma+\sigma},$$

where $\lambda_{\gamma+\sigma}$ is the smallest eigenvalue of the complex Hessian of $\gamma+\sigma$. We wish to replace $\delta_{\gamma+\sigma}\psi$ by $\delta_\gamma\psi$ since we have information about $\delta_\gamma f$. By (4b$'$),

$$\delta_{\gamma+\sigma}\psi = \delta_\gamma\psi + \sum_1^n (D_j\sigma)\psi_j = \delta_\gamma\psi + \partial\sigma\cdot\psi.$$

Hence by the elementary inequality $|c_1+c_2|^2 \leq (1+\theta)|c_1|^2 + (1+\theta^{-1})|c_2|^2$ with arbitrary $\theta>0$:

$$(5g')\quad \|\delta_{\gamma+\sigma}\psi\|^2 = \|\delta_\gamma\psi + \partial\sigma\cdot\psi\|^2 \leq (1+\theta)\|\delta_\gamma\psi\|^2_{\gamma+\sigma} + (1+\theta^{-1})\int |\partial\sigma|^2|\psi|^2 e^{-\gamma-\sigma}.$$

Combining (5g, g$'$) and noting that $\lambda_{\gamma+\sigma}\geq\lambda_\gamma$, we obtain

$$(5g'')\qquad \int |\psi|^2 e^{-\gamma-\sigma}(\lambda_\gamma - (1+\theta^{-1})|\partial\sigma|^2) \leq (1+\theta)\|\delta_\gamma\psi\|^2_{\gamma+\sigma} + \|\bar\partial_1\psi\|^2_{\gamma+\sigma}.$$

We are thus led to impose the following definitive condition on $\gamma\in C^\infty(\Omega)$:

$$(5h)\qquad \begin{cases} \lambda_\gamma \geq b + (1+\theta^{-1})|\partial\sigma|^2, & \gamma=\beta \quad \text{on } K=\operatorname{supp}\phi,\\ \gamma\geq\beta+\sigma \text{ on } \Omega, & \gamma\geq\beta+\log b \quad \text{near } \partial\Omega\cup\infty.\end{cases}$$

The existence of $\gamma=\gamma_\theta$, after σ has been selected, will be verified by means of Proposition 11.53 below. Inequality (5g$''$) then gives us the desired

AD HOC A PRIORI INEQUALITY (11.52) for test forms ψ on Ω. For $b=\lambda_\beta$, for any constant $\theta>0$ and with σ and $\gamma=\gamma(\beta,\sigma,\theta,K)$ as in (5f), (5h),

$$\int |\psi|^2 e^{-\gamma-\sigma} b \leq (1+\theta)\|\delta_\gamma\psi\|^2_{\gamma+\sigma} + \|\bar\partial_1\psi\|^2_{\gamma+\sigma}.$$

Use of approximation to establish the basic inequality. The above inequality for test forms readily extends to general forms $f\in L^2_\gamma$ with $\delta_\gamma f$ and $\bar\partial_1 f$ in $L^2_{\gamma+\sigma}$. Indeed, let

$E \subset \Omega$ be compact and $\eta > 0$. By the approximation theorem 11.51 with $\gamma+\sigma$ instead of β, there will be a test form ψ such that, using (11.52) in the middle step,

$$
(5i) \qquad \begin{aligned}\int_E |f|^2 e^{-\gamma-\sigma} b &\le \int_E |\psi|^2 e^{-\gamma-\sigma} b + \eta \le (1+\theta)\|\delta_\gamma \psi\|^2_{\gamma+\sigma} + \|\bar\partial_1 \psi\|^2_{\gamma+\sigma} + \eta \\ &\le (1+\theta)\|\delta_\gamma f\|^2_{\gamma+\sigma} + \|\bar\partial_1 f\|^2_{\gamma+\sigma} + 2\eta.\end{aligned}
$$

We now may first let η go to 0 and then let E tend to Ω. Specializing to the form f obtained in (5b), (5b′), we conclude from (5i) that

$$
(5i') \qquad \begin{aligned}\int_\Omega |f|^2 e^{-\gamma-\sigma} b &\le (1+\theta)\|\delta_\gamma f\|^2_{\gamma+\sigma} + \|\bar\partial_1 f\|^2_{\gamma+\sigma} \\ &= (1+\theta)\|\delta_\beta \phi\|^2_\beta \qquad [\gamma+\sigma=\beta \text{ on } K=\text{supp}\phi].\end{aligned}
$$

Hence by (5e), noting that $\gamma-\sigma \ge \beta$ on Ω,

$$
|(v,\phi)_\beta|^2 \le \int |v|^2 e^{-\beta} b^{-1} \cdot (1+\theta)\|\delta_\beta \phi\|^2_\beta.
$$

Since $\gamma=\gamma_\theta$ no longer appears here, we can let θ go to 0 and the basic inequality 11.42 follows.

It remains to verify the existence of σ and γ with the properties listed in (5f), (5h). To that end we prove one final

Proposition 11.53. *Let Ω, β and $b=\lambda_\beta$ be as in the basic inequality 11.42. Then to any compact subset $K \subset \Omega$ and any positive constant A, there exist C^∞ psh functions σ and τ, with $0 \le \sigma \le \tau$ on Ω and $\sigma=\tau=0$ on K, such that*

$$
\begin{aligned}
&\sigma(z) \ge 2\log^+ c/d(z) \quad \textit{for some} \;\; c>0,\\
&\lambda_\tau \ge A|d\sigma|^2 = A\sum_1^n |D_j|\sigma|^2 \;\; \textit{on } \Omega,\\
&\tau(z) \ge \log b(z) \;\; \textit{outside some compact} \;\; K' \subset \Omega.
\end{aligned}
$$

Taking $K=\text{supp}\phi$ and $A=1+\theta^{-1}$, the function σ will satisfy the conditions (5f) and the function $\gamma=\beta+\tau$ will satisfy the conditions (5h) $[\lambda_{\beta+\tau} \ge \lambda_\beta+\lambda_\tau = b+\lambda_\tau]$.

PROOF of the Proposition. The proof is a fairly straightforward application of Theorem 9.21 on the existence of rapidly growing psh C^∞ functions on a psh exhaustible domain Ω. One first observes that there are continuous psh exhaustion functions $\alpha \ge 0$ and $\alpha' \ge 0$ on Ω such that

$$
K \subset Z(\alpha)^0 = \text{int} Z(\alpha), \quad Z(\alpha) \subset Z(\alpha')^0,
$$

where Z stands for "zero set". Starting out with an arbitrary continuous exhaustion function α_0, there will be a constant M with $\alpha_0 - M < 0$ on K and one takes $\alpha = \sup(\alpha_0 - M, 0)$; similarly for α'.

If $\Omega = \mathbf{C}^n$ we choose $\sigma = 0$, otherwise we set

$$2c = d(Z(\alpha'), \partial\Omega), \quad m_1(z) = 2\log^+ c/d(z).$$

The nonnegative function m_1 will vanish for $d(z) \geq c$, so that $m_1 = 0$ on a neighborhood of $Z(\alpha')$. Hence by Theorem 9.21 there is a C^∞ function σ on Ω such that

$$\sigma \geq m_1 \text{ and } \lambda_\sigma \geq 0, \quad \text{while } \sigma = 0 \text{ on a neighborhood of } Z(\alpha).$$

Once σ has been chosen, we set

$$m_2 = \begin{cases} \sigma & \text{on } Z(\alpha') \\ \sup(\sigma, \log b) & \text{on } \Omega \setminus Z(\alpha'), \end{cases} \qquad \mu = A|\partial\sigma|^2 \text{ on } \Omega.$$

Since $m_2 = \mu = 0$ on a neighborhood of $Z(\alpha)$, Theorem 9.21(iii) assures the existence of $\tau \in C^\infty(\Omega)$ such that

$$\tau \geq m_2 \text{ and } \lambda_\tau \geq \mu, \quad \text{while } \tau = 0 \text{ on } K.$$

11.6 General solvability of $\bar\partial$ on pseudoconvex domains. The basic inequality (11.42) and the Riesz representation theorem will give the following result for $\mathbf{C}^n$:

First Main Theorem 11.61. *Let $\Omega \subset \mathbf{C}^n$ be pseudoconvex or plurisubharmonically exhaustible and let $\beta \in C^\infty(\Omega)$ be strictly plurisubharmonic, so that the smallest eigenvalue $b = b(z) = \lambda_{\beta(z)}$ of the complex Hessian $[\frac{\partial^2\beta}{\partial z_i \partial \bar z_j}]$ is strictly positive. Let $v = \sum_1^n v_j \, d\bar z_j$ be a (0,1) form in $L^2(\Omega, e^{-\beta} b^{-1})$ which (distributionally) satisfies the integrability condition $\bar\partial_1 v = 0$. Then there exists a function u in $L^2(\Omega, e^{-\beta})$ which solves the equation $\bar\partial u = v$ in the weak sense on Ω and which satisfies the growth condition*

$$\int_\Omega |u|^2 e^{-\beta} \, dm \leq \int_\Omega |v|^2 e^{-\beta} b^{-1} \, dm.$$

If v is of class $C^p(\Omega)$, $1 \leq p \leq \infty$, such a solution u exists in the classical sense as a C^p function.

For the proof one uses the same method as in Section 11.3: Proposition 11.31 readily extends to $\mathbf{C}^n$, cf. (4c). The basic inequality (11.42) gives the constant $A = A_v = \|v\|\beta + \log b$ which provides the upper bound for $\|u\|_\beta$. If $v \in C^p(\Omega)$ then the solution u may be modified on a set of measure 0 to obtain a C^p solution [Theorem 11.22].

We will now derive

COROLLARY 11.62. On a pseudoconvex domain Ω the equation $\bar\partial u = v$ is globally C^p solvable for every (0,1) form v of class C^p with $\bar\partial_1 v = 0$.

In view of Theorem 11.61 it is enough to prove:

Lemma 11.63. *Let $\Omega \subset \mathbf{C}^n$ be pseudoconvex and let v be a locally square integrable (0,1) form on Ω. Then there exists a strictly psh C^∞ function β on Ω such that $v \in L^2(\Omega, e^{-\beta}\lambda_\beta^{-1})$.*

PROOF. Write $\Omega = \cup_j K_j$, a countable increasing union of compact subsets. Define a locally bounded function m on Ω:

$$m(z) = \log^+\left(j^2 \int_{K_{j+1}\setminus K_j} |v|^2\right) \qquad \text{on } \; K_{j+1}\setminus K_j, \quad j = 1, 2, \ldots,$$

so that

$$\int_{K_{j+1}\setminus K_j} |v|^2 e^{-m} \le 1/j^2.$$

Next set $\mu \equiv 1$. Then $v \in L^2(\Omega, e^{-m}\mu^{-1})$. Now by Theorem 9.21 there exists $\beta \in C^\infty(\Omega)$ with $\beta \ge m$ and $\lambda_\beta \ge \mu$, hence $v \in L^2(\Omega, e^{-\beta}\lambda_\beta^{-1})$.

The case $p = \infty$ of Corollary 11.62 gives the all important final

COROLLARY 11.64. Every pseudoconvex domain is a $\bar\partial$ domain and hence a Cousin-I domain [cf. Section 7.5]. More significant, every pseudoconvex domain is a domain of holomorphy [cf. Section 7.7].

11.7 Another growth estimate for the solution of $\bar\partial$ and interpolation. In the first main theorem 11.61, the factor b^{-1} in the integral involving v is somewhat inconvenient. This factor disappears in the special case $\beta = |z|^2$ for which $b = \lambda_\beta = 1$. [Verify this]. Thus for $v \in L^2(\Omega, e^{-|z|^2})$ one gets a nice symmetric growth estimate [cf. exercise 11.16].

More important, in the general case $v \in L^2(\Omega, e^{-\alpha})$ [with C^∞ psh α] one can also obtain a growth estimate that is free of derivatives of the weight function. Substituting $\beta = \alpha + \gamma$ in the first main theorem, with γ strictly psh so that $\lambda_\gamma > 0$, one has

$$(7a) \qquad e^{-\beta}b^{-1} = e^{-\alpha-\gamma}/\lambda_{\alpha+\gamma} \le e^{-\alpha}e^{-\gamma}/\lambda_\gamma$$

and one would like this to be $\le ce^{-\alpha}$. Thus one requires that

$$(7a') \qquad e^{-\gamma} \le c\lambda_\gamma.$$

Setting $\gamma = g(|z|^2)$ and first taking $n = 1$ so that $\lambda_\gamma = \gamma_{z\bar z}$ one is led to the condition

$$e^{-g(t)} \le c\{tg''(t) + g'(t)\},$$

cf. (8.1). Some experimentation gives the solution $g(t) = 2\log(1 + t)$, $c = 1/2$, which will also work for $n \ge 2$. Theorem 11.61 will now lead to the case $\alpha \in C^\infty$ of the following

Second main Theorem 11.71. *Let $\Omega \subset \mathbf{C}^n$ be pseudoconvex and let v be any (0,1) form of class $C^p(\Omega)$, $1 \le p \le \infty$ such that $\bar\partial_1 v = 0$. Let α be any plurisubharmonic function on Ωsuch that $v \in L^2_\alpha$. Then the equation $\bar\partial u = v$ has a C^p solution u on Ω satisfying the growth condition*

$$\int_\Omega |u|^2 e^{-\alpha}(1+|z|^2)^{-2}\, dm \le \frac{1}{2}\int_\Omega |v|^2 e^{-\alpha}\, dm. \tag{7b}$$

PROOF. (i) In the case $\alpha \in C^\infty$ with $\lambda_\alpha \ge 0$, the result is obtained from Theorem 11.61 by setting

$$\beta = \alpha + 2\log(|z|^2+1).$$

Indeed a short calculation will show that [cf. exercise 11.20]

$$b = \lambda_\beta \ge 2(1+|z|^2)^{-2}, \quad e^{-\beta}b^{-1} \le \frac{1}{2}e^{-\alpha}.$$

Thus if $v \in L^2_\alpha$, then also $v \in L^2_{\beta+\log b}$ and the result follows.

(ii) Since the estimate (7b) with $\alpha \in C^\infty$ contains no derivatives of α, the result can be extended to arbitrary psh functions α on Ω by a suitable limit process.

Let $\{\Omega_k\}$, $k = 1, 2, \ldots$ be an exhaustion of Ω with open pseudoconvex domains as given by (9.1a) or Theorem 9.21, which have compact closure in Ω. Regularizing the given psh function α as in Section 8.4, we can construct C^∞ psh functions α_k defined on Ω_k and such that $\alpha_k \downarrow \alpha$ ($k \ge k_0$) on each compact subset of Ω.

By part (i) there are functions $u_k \in C^p(\Omega_k)$ such that $\bar\partial u_k = v$ on Ω_k and

$$\int_{\Omega_k} |u_k|^2 e^{-\alpha_k}(1+|z|^2)^{-2} \le \frac{1}{2}\int_{\Omega_k} |v|^2 e^{-\alpha_k} \le \frac{1}{2}\int_\Omega |v|^2 e^{-\alpha}, \quad k = 1, 2, \ldots. \tag{7c}$$

As $\alpha_k \le \alpha_j$, it follows that the L^2 norms of the functions u_k on a fixed set Ω_j are uniformly bounded for $k \ge j$. Thus one can choose a subsequence $\{u_\nu\}$, $\nu = \nu_k \to \infty$ which converges weakly in $L^2(\Omega_j)$ for each j to a limit u in $L^2_{loc}(\Omega)$. This convergence is also in distributional sense [“integrate” against a test form].

For such a limit function u, since differentiation is a continuous operation, cf. Section 11.1,

$$\bar\partial u = \lim \bar\partial u_\nu = v \quad \text{in distributional sense on } \Omega.$$

Furthermore, for each j and every $k \ge j$,

$$\begin{aligned}\int_{\Omega_j} |u|^2 e^{-\alpha_k}(1+|z|^2)^{-2} &\le \liminf_\nu \int_{\Omega_j} |u_\nu|^2 e^{-\alpha_k}(1+|z|^2)^{-2} \\ &\le \liminf_\nu \int_{\Omega_j} |u_\nu|^2 e^{-\alpha_\nu}(1+|z|^2)^{-2} \le \frac{1}{2}\int_\Omega |v|^2 e^{-\alpha}.\end{aligned} \tag{7d}$$

Letting $k \to \infty$ the monotone convergence theorem will give that $\int_{\Omega_j} |u|^2 e^{-\alpha}(1+|z|^2)^{-2}$ has the upper bound of (7b) for each j and hence (7b) follows.

Because v is of class C^p, u can finally be changed on a set of measure zero to provide a C^p solution [Theorem 11.22].

The main theorems enable one to obtain solutions to various problems on pseudoconvex domains Ω subject to growth conditions. We mention one:

Interpolation by analytic functions 11.72. Let $\{a_\lambda\}$ be a sequence of pairwise distinct points without limit point in Ω and suppose that α is a psh function on Ω which becomes $-\infty$ in such a way that $e^{-\alpha}$ is non-integrable on every small ball $B_r = B(a_\lambda, r)$:

$$\int_{B_r} e^{-\alpha} dm = +\infty, \quad \forall r \in (0, r_\lambda).$$

Then a continuous function u in $L^2(\Omega, e^{-\alpha}(1+|z|^2)^{-2})$ must vanish at each point a_λ: for small r,

$$\int_{B_r} |u(z)|^2 e^{-\alpha}(1+|z|^2)^{-2} \geq \frac{1}{2}|u(a_\lambda)|^2(1+|a_\lambda|^2)^{-2} \int_{B_r} e^{-\alpha}.$$

This fact can be used to prove the existence of analytic solutions h to interpolation problems

$$(7d) \qquad h(a_\lambda) = b_\lambda, \quad \forall\lambda, \qquad h \in \mathcal{O}(\Omega),\ \Omega \text{ pseudoconvex}$$

which satisfy appropriate growth conditions. One first determines a simple C^2 solution g to the interpolation problem, then subtracts a suitable non-analytic part u to obtain h in the form $g - u$. The condition on u will be

$$(7e) \qquad \bar\partial u = v \overset{\text{def}}{=} \bar\partial g \text{ on } \Omega, \quad u(a_\lambda) = 0,\ \forall\lambda. \quad \text{Here } v \in C^1.$$

One now chooses a psh function α on Ω which is singular on the sequence $\{a_\lambda\}$ in the way indicates above, while $\int |\bar\partial g|^2 e^{-\alpha} < \infty$. [Apparently we had better choose g constant in a suitable neighborhood of $\{a_\lambda\}$ so that $\bar\partial g$ vanishes at the singular points of α.] Then the C^1 solution u of the $\bar\partial$ equation guaranteed by Theorem 11.71 will satisfy the condition (7e) and the difference $h = g - u$ will solve the interpolation problem (7d). The growth of h will be limited by the growth of g and that of u; for the latter one has condition (7b). By the solid mean value theorem for analytic functions on balls cf. [exercise 2.23], an L^2 estimate for h can be transformed into a pointwise estimate.

EXAMPLE 11.74. Determine a holomorphic function h on $\mathbf{C}$ of limited growth such that $h(k) = b_k$, $k \in \mathbf{Z}$, where $\{b_k\}$ is any given bounded sequence of complex numbers.

Thinking of the special case $b_k = 0$, $\forall k$, it is plausible that an interpolating function h will not grow more slowly than $\sin \pi z$. However, it need not grow much faster! Indeed, let ω be a C^2 function on $\mathbf{C}$ such that $\omega(z) = 1$ for $|z| \leq 1/4$, $\omega(z) = 0$ for $|z| \geq 1/2$. Then

$$g(z) = \sum_{-\infty}^{\infty} b_k \omega(z - k)$$

will be a C^2 solution of the interpolation problem. A typical non-integrable function on a neighborhood of 0 in $\mathbf{C}$ is $1/|z|^2$; a function that is non-integrable on every neighborhood of every integer is $1/|\sin^2 \pi z|$. Thus a first candidate for α will be $2\log|\sin \pi z|$. Since $\bar{D}g$ is bounded on $\mathbf{C}$ and vanishes outside the set of annuli $\frac{1}{4} \le |z-k| \le \frac{1}{2}$, while $1/|\sin^2 \pi z|$ is bounded on that set,

$$\int_{\mathbf{C}} |\bar{D}g|^2 \frac{1}{|\sin^2 \pi z|(1+|z|^2)} dm \le \text{const} \int_{|\mathrm{Im} z| \le \frac{1}{2}} \frac{1}{1+|z|^2} dm < \infty.$$

Thus a good subharmonic function α is furnished by

$$\alpha(z) = 2\log|\sin \pi z| + \log(1+|z|^2).$$

The solution of the equation $\bar{\partial} u = \bar{\partial} g$ guaranteed by Theorem 11.71 will satisfy the growth condition

$$\int_{\mathbf{C}} |u|^2 \frac{1}{|\sin^2 \pi z|(1+|z|^2)^3} dm < \infty.$$

Since g is bounded, it will follow that $h = g - u$ is bounded by $c|z|^3|\sin \pi z|$ for $|\mathrm{Im} z| \ge 1$; the bound $c|z|^3$ will also hold for $|\mathrm{Im} z| < 1,\ |z| \ge 1$.

For this particular problem one knows an explicit solution by a classical interpolation series, cf. [Boas]. It is interesting that the general method used here gives a nearly optimal growth result.

Some other applications of Theorem 11.71 are indicated in the exercise 11.21, 11.22, 11.24; cf. also [Hör 1], [Bern], [Sig], [Siu], [Ron]. Further applications are certainly possible.

11.8 "Higher order " $\bar{\partial}$ equations. Up till now we have only discussed the equation

$$\bar{\partial} u = v \quad \text{on} \quad \Omega \subset \mathbf{C}^n \tag{8a}$$

tor the case of (0,1) forms v with $\bar{\partial} v = 0$. More generally, one may think of v as a $(0,q)$ form with locally integrable coefficients and $\bar{\partial} v = \bar{\partial}_q v = 0$. The problem is to determine a $(0,q-1)$ form u on Ω satisfying (8a). On the whole, the treatment in the general case parallels the one for $q=1$. We will discuss the case $q=2$ here, indicating some small differences with the case $q=1$.

For a (0,1) form

$$u = \sum_{k=1}^{n} u_k \, dz_k$$

with locally integrable coefficients we have

$$\bar{\partial} u = \bar{\partial}_1 u \stackrel{\text{def}}{=} \sum_{j,k=1}^{n} \bar{D}_j u_k \cdot d\bar{z}_j \wedge d\bar{z}_k = \sum_{j,k}{}' (\bar{D}_j u_k - \bar{D}_k u_j) \, d\bar{z}_j \wedge d\bar{z}_k. \tag{8b}$$

Here the prime indicates that we only sum over pairs (j,k) with $j<k$; we have used the *anticommutative* relation [cf. Chapter 10]

$$d\bar{z}_j \wedge d\bar{z}_k = -d\bar{z}_k \wedge d\bar{z}_j;$$

the wedge products $d\bar z_j \wedge d\bar z_k$ with $1 \le j < k \le n$ form a basis for the (0,2) forms in $\mathbf{C}^n$. Thus an arbitrary (0,2) form v has a unique representation

$$v = \sum_{j,k}{}' v_{jk} d\bar z_j \wedge d\bar z_k = \frac{1}{2}\sum_{j,k=1}^{n} v_{jk} d\bar z_j \wedge d\bar z_k, \tag{8c}$$

where [as is customary] we have defined the coefficients v_{jk} with $j \ge k$ by antisymmetry: $v_{jk} = -v_{kj}$. For computational purposes it is often convenient to work with the normalized full sums.

A form v on Ω is said to be *of class* $L^2_\beta = L^2(\Omega, e^{-\beta})$ if the coefficients are; the inner product of (0,2) forms is given by

$$(f,g)_\beta = \int_\Omega f\cdot \bar g\, e^{-\beta}, \quad f\cdot\bar g = \sum_{j,k}{}' f_{jk}\bar g_{jk} = \frac{1}{2}\sum_{j,k} f_{jk}\bar g_{jk}. \tag{8d}$$

As before, we will need the formal adjoint $\delta = \delta_\beta$ to $\bar\partial$ in L^2_β. Let ϕ be a (normalized) (0,2) test form, that is, the coefficients are test functions. For our (0,1) form u, using (8b) and the definition of distributional derivatives,

$$\begin{aligned}(\bar\partial u, \phi)_0 = \langle \bar\partial u, \bar\phi\rangle &= \sum_{j<k}{}'\langle \bar D_j u_k - \bar D_k u_j, \phi_{jk}\rangle = \sum_{j,k}\langle \bar D_j u_k, \bar\phi_{jk}\rangle \\ &= -\sum_{j,k}\langle u_k, \bar D_j \bar\phi_{jk}\rangle = -\sum_{j,k}(u_k, D_j\phi_{jk})_0.\end{aligned}$$

Applying this result to $e^{-\beta}\phi$ instead of ϕ with $\beta \in C^\infty$, we obtain

$$(\bar\partial u, \phi)_\beta = \sum_{j,k}(u_k, \delta_j\phi_{jk}) \stackrel{\text{def}}{=} (u, \delta\phi)_\beta, \quad \delta_j = -D_j + D_j\beta\cdot \mathrm{id}.$$

Thus the adjoint $\delta = \delta_\beta$ applied to a (0,2) test form ϕ gives a (0,1) form:

$$\delta\phi = \sum_k\Big(\sum_j \delta_j\phi_{jk}\Big)\, d\bar z_k = \sum_s\Big(\sum_j \delta_j\phi_{js}\Big)\, d\bar z_s. \tag{8e}$$

Using the fact that δ_j and $\bar D_j$ are adjoints in L^2_β and by the commutator relations in Section 8.4. cf. (4d),

$$\begin{aligned}(\delta\phi,\delta\phi)_\beta &= \sum_s\Big(\sum_j \delta_j\phi_{js}, \sum_k \delta_k\phi_{ks}\Big) = \sum_s\sum_{j,k}\left(\bar D_k\delta_j\phi_{js}, \phi_{ks}\right) \\ &= \sum_s\sum_{j,k}\left(D_j\bar D_k\beta\cdot\phi_{js}, \phi_{ks}\right) + \sum_s\sum_{j,k}\left(\bar D_k\phi_{js}, \bar D_j\phi_{ks}\right).\end{aligned} \tag{8f}$$

We also need $(\bar\partial\phi, \bar\partial\phi)_\beta$. The usual definition of $\bar\partial = \bar\partial_2$ gives, cf. (8b),

$$\bar\partial\phi = \sum_s \sum_{j,k}{}' \bar D_s \phi_{jk} \cdot d\bar z_s \wedge d\bar z_j \wedge d\bar z_k = \frac{1}{2}\sum_{s,j,k} \cdots$$
$$= \sum_{i<j<k}{}' \left(\bar D_i\phi_{jk} - \bar D_j\phi_{ik} + \bar D_k\phi_{ij}\right) d\bar z_i \wedge d\bar z_j \wedge d\bar z_k.$$

For the computation of the inner product it is safest to start with the standard representation in the last line, in terms of a basis. Changing over to full sums one then obtains

$$\bar\partial\phi \cdot \partial\bar\phi = \tfrac{1}{4} \sum_{s,j,k} \sum_{t,l,m} \bar D_s\phi_{jk} \cdot D_t\bar\phi_{lm} \cdot \epsilon^{sjk}_{tlm},$$

where the ϵ-factor equals 0 unless (t,l,m) is a permutation of (s,j,k); for an even permutation the value of ϵ is 1, for an odd permutation -1. It follows that

$$(8f') \qquad (\bar\partial\phi, \bar\partial\phi)_\beta = \frac{1}{2}\sum_{s,j,k} \left(\bar D_s\phi_{jk}, \bar D_s\phi_{jk}\right) - \sum_{s,j,k} \left(\bar D_s\phi_{jk}, \bar D_j\phi_{sk}\right).$$

The last sum also occurs at the end of (8f), although with slightly permuted indices. Adding (8f′) to (8f), we obtain

$$(8f'') \qquad \sum_s \sum_{j,k} \left(D_j\bar D_k\beta \cdot \phi_{js}, \phi_{ks}\right) + \frac{1}{2}\sum_{s,j,k} \|\bar D_s\phi_{jk}\|^2 = (\delta\phi, \delta\phi) + (\bar\partial\phi, \bar\partial\phi).$$

Finally introducing the smallest eigenvalue $b = \lambda_\beta$ of $\left[D_j\bar D_k\beta\right]$, we have in view of (8d):

$$\sum_s \sum_{j,k} D_j\bar D_k\beta \cdot \phi_{js}\bar\phi_{ks} \ge \sum_s b \sum_j |\phi_{js}|^2 = 2b\phi\cdot\bar\phi.$$

Combination gives the following *a priori inequality* for (0,2) test forms:

$$(8g) \qquad \int_\Omega |\phi|^2 e^{-\beta} b \le \frac{1}{2}(\delta\phi, \delta\phi)_\beta + \frac{1}{2}(\bar\partial\phi, \bar\partial\phi)_\beta.$$

A weak [locally integrable] solution u of the equation (8a) is characterized by the condition

$$(\delta\phi, u)_\beta = (\phi, \bar\partial u)_\beta = (\phi, v)_\beta, \quad \forall \text{ test forms } \phi.$$

Taking Ω pseudoconvex and β strictly psh, the a priori inequality and suitable approximation arguments may be used to prove the following *basic inequality*, cf. Sections 11.4, 11.5:

$$(8h) \qquad |(\phi, v)_\beta|^2 \le \int |v|^2 e^{-\beta} b^{-1} \cdot \frac{1}{2}\|\delta_\beta\phi\|^2_\beta.$$

As before the Riesz representation theorem then gives

Theorem 11.81. *Let $\Omega \subset \mathbf{C}^n$ be pseudoconvex, let β be a strictly psh C^∞ function on Ω and $b = \lambda_\beta$. Let v be a (0,2) form in $L^2_{\beta+\log b}(\Omega)$ with $\bar\partial v = 0$. Then there is a (0,1) form u in $L^2_\beta(\Omega)$ such that $\bar\partial u = v$ and*

$$\int_\Omega |u|^2 e^{-\beta} \le \frac{1}{2}\int_\Omega |v|^2 e^{-\beta} b^{-1}.$$

In the case $q \ge 2$ it is not true that all the solutions of the equation $\bar\partial u = v$ must be smooth whenever v is, just think of the case $n = q = 2$ and $v = 0$, where the equation becomes $\bar D_1 u_2 - \bar D_2 u_1 = 0$. However, on pseudoconvex Ω, equation (8a) always has a solution which is orthogonal to the nullspace of $\bar\partial_{q-1}$ in L^2_β, cf. Proposition 11.31. Such a solution does have smoothness properties related to those of v, cf. [Hör1]. In particular, for v in C^∞ there always exists a solution u in C^∞.

Exercises

11.1. Show that $f_\nu(x) = \nu^{100} e^{i\nu x} \to f = 0$ distributionally on $\mathbf{R}$ as $\nu \to \infty$.

11.2. Let f_ν, f in $L^2_{\text{loc}}(\Omega)$ be such that for every compact subset $K \subset \Omega$, $f_\nu \to f$ weakly in $L^2(K)$, that is $\int_K f_\nu \bar g \to \int_K f \bar g$, $\forall g \in L^2(K)$. Prove that $f_\nu \to f$ distributionally on Ω.

11.3. Let $\{\rho_\epsilon\}$ be the standard approximate identity on $\mathbf{R}^n$ [Section3.3]. Prove that $\rho_\epsilon \to \delta$ distributionally on every domain $\Omega \subset \mathbf{R}^n$.

11.4. Show that the delta distribution on $\mathbf{R}^n$ is equal to 0 on $\mathbf{R}^n \setminus \{0\}$, so that $\text{supp}\delta = \{0\}$. Deduce that δ can not be equal to a locally integrable function on $\mathbf{R}^n$.

11.5. For a distribution T on $\mathbf{R}^n$ and a test function ϕ, the convolution $T * \phi$ is defined by the formula $T * \phi(x) = \langle T, \phi(\cdot - y)\rangle$. Prove that $\delta * \phi = \phi$ and that this convolution reduces to the ordinary one if T is a locally integrable function.

11.6. Let T be a distribution on $\Omega \subset \mathbf{R}^n$ which is equal to a C^1 function f on $\Omega_0 \subset \Omega$. Prove that $\frac{\partial T}{\partial x_j}$ is distributionally equal to the function $\frac{\partial f}{\partial x_j}$ on Ω_0.

11.7. Let T be a distribution on Ω, $\omega \in C^\infty(\Omega)$. Prove that

$$\frac{\partial}{\partial x_j}(\omega T) = \frac{\partial \omega}{\partial x_j} T + \omega \frac{\partial T}{\partial x_j}.$$

11.8. Let u be a function on $\Omega \subset \mathbf{C}^n$ that depends only on $r = |z| : u(z) = f(r)$. Calculate $\bar\partial u$, assuming that f is piecewise smooth.

11.9. Given that $u_\nu \to u$ distributionally on $\Omega \subset \mathbf{C}^n$, prove that $\bar\partial u_\nu \to \bar\partial u$ distributionally on Ω. [That is the coefficients converge distributionally.]

11.10. Verify that $\bar\partial_1 \bar\partial = 0$ on $\Omega \subset \mathbf{C}^n$ when applied to:

(i) C^∞functions, (ii) distributions.

11.11. Investigate the case of equality in the a priori inequality for test functions (11.32).

11.12. Let u be a locally integrable function on $\Omega \subset \mathbf{C}^n$ such that [each coefficient of] $\bar{\partial} u$ is also locally in L^1. Let $\Omega_0 \subset\subset \Omega$, $\epsilon < d(\Omega_0, \partial\Omega)$. Prove that for our standard C^∞ approximation to the identity ρ_ϵ, $\bar{\partial}(u * \rho_\epsilon) = (\bar{\partial} u) * \rho_\epsilon$ on Ω_0.

11.13. Let v be a (0,1) form in $L^1_{\text{loc}}(\Omega)$ such that $\bar{\partial}_1 v$ is also in $L^1_{\text{loc}}(\Omega)$ and let ω be a C^∞ function on Ω. Calculate the coefficients of $\bar{\partial}_1(\omega v)$. Show that in differential form notation,

$$\bar{\partial}_1(\omega v) = \omega \bar{\partial}_1 v + \bar{\partial}\omega \wedge v.$$

11.14. Let u be an L^2 function on $\mathbf{C}^n$ of bounded support whose distributional derivatives $\frac{\partial u}{\partial \bar{z}_j}$ are in L^2 for $j = 1, \ldots, n$. Prove that all first order partial derivatives of u are in L^2. [Show first that for test functions ϕ, $\|\frac{\partial \phi}{\partial z_j}\| = \|\frac{\partial \phi}{\partial \bar{z}_j}\|$, then use regularization.]

11.15. Let $\Omega \subset \mathbf{C}^n$ be pseudoconvex and let v be a (0,1) form of class C^p on Ω with $\bar{\partial}_1 v = 0$. Prove that the equation $\bar{\partial} u = v$ has a C^p solution on Ω such that

$$\int |u|^2 e^{-|z|^2} \leq \int |v|^2 e^{-|z|^2}.$$

11.16. Describe the steps in the proof of the first main theorem 11.61 for the special case $\Omega = \mathbf{C}^n$.

11.17. Prove that the $\bar{\partial}$ problem considered in the first main theorem 11.61 has a solution orthogonal to all holomorphic functions h in that space. Determine the general solution in L^2_β. Which solution has minimal norm? [Such a minimal solution is sometimes called the Kohn solution.]

11.18. (Behnke Stein theorem) Prove that the limit of an increasing sequence of domains of holomorphy in $\mathbf{C}^n$ is also a domain of holomorphy.

11.19. Show that for $\gamma(z) = 2\log(1 + |z|^2)$ one has $\lambda_\gamma = 2(1 + |z|^2)^{-2}$, so that $e^{-\gamma} = \frac{1}{2}\lambda_\gamma$

11.20. Let $\{a_\lambda\}$ be a sequence of distinct points without limit point in $\Omega \subset \mathbf{C}^n$. Suppose that there is a continuous psh function α on Ω such that $|\alpha(z) - \log|z - \alpha_\lambda|| \leq C_\lambda$ on some small ball $B(a_\lambda, r_\lambda)$ around each point a_λ. Deduce that there is a holomorphic function $h \not\equiv 0$ in Ω which vanishes at the points a_λ and does not grow much faster than $e^{n\alpha}$ towards the boundary of Ω. [Force $h = 1$ at some point $a \in \Omega$ such that $\alpha(z) \geq -C$ on some ball $B(a, r)$.]

11.21. Let u be a psh function on a domain $\Omega \subset \mathbf{C}^n$ and let $c > 0$. Show that the collection of points $z \in \Omega$ such that $\exp -cu$ is not integrable over any neighborhood of z is contained in an analytic variety of dimension $< n$. [Use an idea from the previous exercise].

11.22. (Holomorphic extension from a hyperplane with bounds). Let α be a psh function on $\mathbf{C}^n$ such that for some constant A,

$$|\alpha(z) - \alpha(w)| \leq A \quad \text{whenever } |z - w| < 1.$$

Suppose h is a holomorphic function on a complex hyperplane V such that

$$I(h) = \int_V |h|^2 e^{-\alpha} d\sigma < \infty,$$

where σ denotes Lebesgue measure on V. Prove that there is a holomorphic function g on $\mathbf{C}^n$ such that $g = h$ on V and

$$\int_{\mathbf{C}^n} |g|^2 e^{-\alpha}(1 + |z|^2)^{-3} dm \leq 6\pi e^A I(h).$$

[Let $\omega(t)$ be continuous on $\mathbf{C}$, 1 for $|t| \leq \frac{1}{2}$, 0 for $|t| \geq 1$ and linear in $|t|$ for $\frac{1}{2} \leq |t| \leq 1$. Taking for V the hyperplane $z_n = 0$, set

$$g(z', z_n) = \omega(z_n)h(z') - z_n u(z', z_n)$$

and require $\bar{\partial} g = 0$. Show that $\|\omega h\|_\alpha^2 \leq \pi e^A I(h)$ and $\|\bar{\partial} u\|_\alpha \leq 4\pi e^A I(h)$.]

11.23. Develop a theory of L^2 solutions with growth estimates for the real equation

$$du = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} dx_j = v$$

on appropriate domains Ω in $\mathbf{R}^n$. [Which are the "right" domains?]

11.24. (Research problem) Prove (or disprove) the following: The a priori inequality 11.32 can be extended to all functions f in L^2_β for which δf is also in L^2_β. Cf. Theorem 11.51. If this works, try to extend the a priori inequality 11.41. to all forms f in L^2_β with $\delta f \in L^2_\beta$ and $\bar{\partial}_1 f = 0$.

CHAPTER 12

Divisor problem, Cousin problems and cohomology

Cousin Problems and their history were described in Section 1.10 as well as in Chapter 7 They can be fruitfully described in terms of cohomology of sheaves. From the appropriate cohomology groups the solvability of the Cousin problem can in principle be read off. In this chapter we will formulate the Cousin II problem, introduce sheaves and study cohomology groups.

12.1 The problems. We begin with the "hypersurface problem" for arbitrary open sets $\Omega \subset \mathbf{C}^n$. A subset $V \subset \Omega$ is called a (complex) *analytic hypersurface* (or an analytic set of complex codimension 1, cf. 4.64), if it is locally a zero set. This means that every point $a \in \Omega$ has a neighborhood $U \subset \Omega$ on which there is a holomorphic function f_U, not identically zero [on any component of U], such that

$$V \cap U = \{z \in U : f_U(z) = 0\}.$$

[We don't require that V consist of regular points as in the case of a complex submanifold of codimension 1, cf. Section 5.5.]. The obvious first question is, whether a given analytic hypersurface V in Ω is also *globally* a zero set. In other words, is there a holomorphic function f on Ω such that V, considered as a set, is the same as $Z(f)$?

For closer analysis, we introduce a suitable open covering $\{U_\lambda\}$ of Ω, namely one for which there are functions $f_\lambda \in \mathcal{O}(U_\lambda)$ such that $V \cap U_\lambda = Z(f_\lambda)$, $\forall\lambda$. As long as we ignore multiplicities, we may require that no f_λ be divisible by a square (of a non-unit) on U_λ. This condition will be satisfied if f_λ and, for example, $\frac{\partial f_\lambda}{\partial z_n}$ are relatively prime on U_λ, cf. the proof of Theorem 4.62 on the local form of a zero set. Thus for suitable U_λ and f_λ, all holomorphic functions defining V on U_λ will be multiples of f_λ, see the Nullstellensatz in exercise 4.18. The desired global f also must be a multiple of f_λ on U_λ. On the other hand we don't want f to vanish outside $Z(f_\lambda)$ on U_λ or more strongly than f_λ on $Z(f_\lambda)$, hence we seek f such that

$$(1a) \qquad f = f_\lambda h_\lambda \quad \text{on} \quad U_\lambda, \qquad h_\lambda \in \mathcal{O}^*(U_\lambda), \quad \forall\lambda.$$

Here $\mathcal{O}^*(U) = \{h \in \mathcal{O}(U) : \ h\overline{\nu}0 \text{ on } U\}$, the set of units in $\mathcal{O}(U)$. Note that by our arguments, the given functions f_λ and f_μ will be compatible on every intersection $U_{\lambda\mu} = U_\lambda \cap U_\mu$ in the sense that

$$(1b) \qquad f_\lambda = f_\mu h_{\lambda\mu} \quad \text{on} \quad U_{\lambda\mu} \qquad \text{with } h_{\lambda\mu} \in \mathcal{O}^*(U_{\lambda\mu}), \qquad \forall\lambda,\mu.$$

The following more general problem will lead to precisely the same conditions (1a), (1b). Suppose one start with compatibly given meromorphic functions f_λ on the sets U_λ. Question: Is there a global meromorphic function f on Ω which on each set U_λ has the same zeros and infinities as f_λ, including multiplicities? For a precise formulation we introduce the class $\mathcal{M}^*(U)$ of *invertible* meromorphic functions on U [those that don't vanish on any component of U].

DEFINITION 12.11. Let $\{U_\lambda\}$, $\lambda \in \Lambda$ be a covering of $\Omega \subset \mathbf{C}^n$ by open subsets. A *divisor* on Ω associated with $\{U_\lambda\}$ is a system of data

$$D = \{U_\lambda, f_\lambda\}, \quad \lambda \in \Lambda,$$

involving functions $f_\lambda \in \mathcal{M}^*(U_\lambda)$ that satisfy the compatibility condition (1b). If in addition $f_\lambda \in \mathcal{O}(U_\lambda)$, $\forall\lambda$, one speaks of a *holomorphic* divisor.

If a holomorphic or meromorphic function f on Ω satisfies the conditions (1a) we say that it *has D as a divisor.* A divisor for which there exists an f as in (1a) is called *principal.*

12.12 MEROMORPHIC SECOND COUSIN PROBLEM or DIVISOR PROBLEM: Let D be a divisor on Ω. Is it principal? Or also: Determine a meromorphic function f on Ω which has D as a divisor.

Much the same as in the first Cousin problem, one may take the functions h_λ of (1a) as unknown. By (1a), (1b) they must satisfy the compatibility conditions $h_\mu = h_\lambda h_{\lambda\mu}$ with $h_{\lambda\mu} \in \mathcal{O}^*(U_{\lambda\mu})$, hence

$$h_{\lambda\mu} = 1/h_{\mu\lambda}, \quad h_{\lambda\mu} = h_{\lambda\nu} h_{\nu\mu}$$

on the relevant intersection of the sets U_α. We thus arrive at the so-called

(HOLOMORPHIC) COUSIN-II PROBLEM or MULTIPLICATIVE COUSIN PROBLEM. Let $\{U_\lambda\}$, $\lambda \in \Lambda$ be an open covering of $\Omega \subset \mathbf{C}^n$ and let $\{h_{\lambda\mu}\}$, $\lambda, \mu \in \Lambda$ be a family of zero free holomorphic functions on the (nonempty) intersections $U_\lambda \cap U_\mu$ that satisfy the compatibility conditions

$$(1c) \qquad \begin{cases} h_{\lambda\mu} h_{\mu\lambda} = 1 & \text{on } U_{\lambda\mu} = U_\lambda \cap U_\mu, \quad \forall \lambda, \mu, \\ h_{\lambda\mu} h_{\mu\nu} h_{\nu\lambda} = 1 & \text{on } U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu, \quad \forall \lambda, \mu, \nu. \end{cases}$$

Determine zero free holomorphic functions, $h_\lambda \in \mathcal{O}^*(U_\lambda)$, such that

$$(1d) \qquad h_\mu / h_\lambda = h_{\lambda\mu} \text{ on } U_{\lambda_\mu}, \quad \forall \lambda, \mu.$$

A family of functions $h_{\lambda\mu} \in \mathcal{O}^*(U_{\lambda\mu})$ satisfying (1c) is called a set of *Cousin-II data* on Ω.

Proposition 12.13. *A divisor $D = \{U_\lambda, f_\lambda\}$ on Ω belongs to a meromorphic function F on Ω (in the sense of (1a)) if and only if there is a solution $\{h_\lambda\}$ of the holomorphic Cousin-II problem on Ω with the data $\{U_\lambda, h_{\lambda\mu}\}$ derived from (1b).*

The proof is similar to that of Proposition 7.14 for the first Cousin problem.

The Cousin-II problem is the multiplicative analog of Cousin-I. At first glance it might seem that there is a straightforward reduction of Cousin-II to Cousin-I with the aid of suitable branches of the functions $\log h_{\lambda\mu}$. However, the problem is not that easy: even for simply connected intersections $U_{\lambda\mu}$, it is not clear if one can choose branches $\log h_{\lambda\mu}$ in such a way that, in conformity with (1c),

$$(1d) \qquad \log h_{\lambda\mu} + \log h_{\mu\lambda} = 0, \quad \log h_{\lambda\mu} + \log h_{\mu\nu} + \log h_{\nu\lambda} = 0$$

on all relevant intersections of sets U_α.

Indeed, as was first shown by Gronwall in 1917, the multiplicative Cousin problem may fail to be solvable even on domains of holomorphy. The following nice counterexample is due to Oka.

12.2 Unsolvable and solvable Cousin-II problems. Let Ω be the domain of holomorphy

$$\Omega = A_1 \times A_2, \quad A_j = \{z_j \in \mathbf{C} : 1-\delta < |z_j| < 1+\delta\}, \quad \delta > 0 \text{ small.}$$

We consider the holomorphic function

$$g(z) \stackrel{\text{def}}{=} z_1 - z_2 - 1 \quad \text{on } \Omega.$$

For points (z_1, z_2) of the zero set $Z(g)$ one must have (cf. fig. 12.1):

$$|z_1| \approx 1, \quad |z_1 - 1| = |z_2| \approx 1,$$

hence

$$z_1 \approx e^{\pi i/3},\ z_2 = z_1 - 1 \approx e^{2\pi i/3}, \quad \text{or} \quad z_1 \approx e^{-\pi i/3},\ z_2 \approx e^{-2\pi i/3}.$$

For small δ, the zero set will consist of two components which are a positive distance apart. Setting

$$A_j^+ = A_j \cap \{\mathrm{Im} z_j \geq 0\}, \quad A_j^- = A_j \cap \{\mathrm{Im} z_j \leq 0\},$$

the set $Z(g)$ will have an "upper" part in $A_1^+ \times A_2$ [in fact, in $A_1^+ \times A_2^+$] and a "lower" part in $A_1^- \times A_2$ [in fact, in $A_1^- \times A_2^-$].

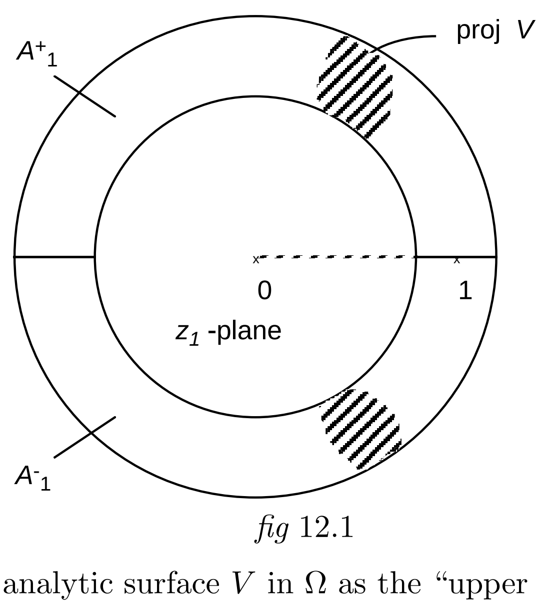


fig 12.1

We now define an analytic surface V in Ω as the "upper part" of $Z(g)$:

$$V = \{z \in A_1^+ \times A_2 :\ z_2 = z_1 - 1\}. \tag{2a}$$

12.21 CLAIM. There is no holomorphic function f on Ω which has V as its exact zero set. In other words, there is no function $f \in \mathcal{O}(\Omega)$ with divisor $D = \{U_j, f_j\}$, $j = 1, 2$ as defined below:

$$\begin{aligned}
&U_1: \quad \text{a "small" } \epsilon\text{-neighborhood of } A_1^+ \times A_2 \text{ in } \Omega, \\
&f_1(z) = z_1 - z_2 - 1 \quad \text{so that } Z(f_1) = V, \\
&U_2: \quad \text{a "small" } \epsilon\text{-neighborhood of } A_1^- \times A_2 \text{ in } \Omega, \\
&f_2(z) = 1 \quad \text{so that } Z(f_2) = \emptyset.
\end{aligned}$$

The corresponding function $h_{12} = f_1/f_2$ on U_{12} is in $\mathcal{O}^*$. It is claimed that the Cousin-II problem for U_1, U_2 and h_{12} is unsolvable: h_{12} can not be written as h_2/h_1 with $h_j \in \mathcal{O}^*(U_j)$.

PROOF. Suppose on the contrary that there exists $f \in \mathcal{O}(\Omega)$ with divisor D as above, or equivalently, that the corresponding Cousin-II problem has a solution $\{h_j\}$, $j = 1, 2$. In both cases we can write

$$f = f_j h_j \text{ on } U_j, \qquad \text{with } h_j \in \mathcal{O}^*(U_j),\ j = 1, 2. \tag{2b}$$

We will obtain a contradiction by comparing the increase of $\arg f(1, w)$ along the unit circle with that of $\arg f(-1, w)$. In fact, computation of the difference in the increases by remaining inside U_1 will differ from what we get by remaining inside U_2. We start with the latter.

Our f would be in $\mathcal{O}^*(U_2)$, hence for fixed $z_1 \in A_1^-$, the function $f(z_1, w)$ is holomorphic and zero free on A_2. There is then a continuous (even holomorphic) branch of $\log f(z_1, w)$ on the open arc $C^1 = C(0, 1) \setminus \{1\}$. With $\Delta_{C^1} g$ denoting the increment of g along C^1, we have

$$\nu_f(z_1) \stackrel{\text{def}}{=} \frac{1}{2\pi} \Delta_{C^1} \arg f(z_1, w) = \frac{1}{2\pi i} \Delta_{C^1} \log f(z_1, w) = \frac{1}{2\pi i} \int_{C(0,1)} \frac{\partial f(z_1, w)/\partial w}{f(z_1, w)} dw.$$

This integer valued function of z_1 is continuous on A_1^-, hence constant. In particular

$$\nu_f(1) - \nu_f(-1) = 0.$$

We will now compute the same difference via the domain U_1. On U_1,

$$f = f_1 h_1 = (z_1 - z_2 - 1) h_1 = gh,$$

say, where $h = h_1 \in \mathcal{O}^*(U_1)$. Thus for h, just as for f before but now remaining inside U_1,

$$\nu_h(1) - \nu_h(-1) = 0.$$

However, for $g(z_1, w) = z_1 - w - 1$ direct calculation gives

$$\nu_g(z_1) = \frac{1}{2\pi i} \int_{C(0,1)} \frac{-1}{z_1 - w - 1} dw = \begin{cases} 1 & \text{if } z_1 = 1, \\ 0 & \text{if } z_1 = -1. \end{cases}$$

Hence, going via U_1, we obtain the answer

$$\nu_f(1) - \nu_f(-1) = \nu_g(1) + \nu_h(1) + \nu_g(-1) - \nu_h(-1) = 1!$$

This contradiction shows that our divisor problem or Cousin-II problem has no solution: there is no $f \in \mathcal{O}(\Omega)$ with $Z(f) = V$.

REMARK 12.22. The method may be adapted to show that the above Cousin-II problem does not even have a continuous solution. That is, there exist no functions $g_j \in C^*(U_j)$ (zero free continuous functions) such that $h_{12} = g_1/g_2$ on U_{12}. [For merely continuous f one can of course not express $\nu_f(z_1)$ by the integral used above.] The non-existence of a continuous solution suggests a *topological obstruction*. In fact, Oka proved a result on the holomorphic divisor problem akin to the following

Theorem 12.23. *Let $\Omega \subset \mathbf{C}^n$ be a Cousin-I domain. The Cousin-II problem on Ω with compatible data $\{U_\lambda, h_{\lambda\mu}\}$ has a holomorphic solution if and only if it has a continuous solution.*

PROOF. We only give an outline since we will prove a more refined result later on.

(i) Just as in the case of Cousin-I the given Cousin-II problem will be solvable if and only if its refinements are solvable (cf. Proposition 7.32).

(ii) Refining our problem, if necessary, we assume that the sets U_λ, and hence the intersections $U_{\lambda\mu}$, are convex. On a convex set a zero free holomorphic (or continuous) function has a holomorphic (or continuous) logarithm, cf. Proposition 12.72 below.

(iii) Supposing now that our original Cousin-II problem has a solution, the same is true for the refined problem. That is, if we denote the (possibly) refined data also by $\{U_\lambda, h_{\lambda\mu}\}$, there exist functions $g_\lambda \in C^*(U\lambda)$ such that $h_{\lambda\mu} = g_\mu/g_\lambda$ on $U_{\lambda\mu}$, for all λ, μ. Choosing continuous logarithms $\log g_\lambda$ on the sets U_λ, we then define

$$\log h_{\lambda\mu} = \log g_\mu - \log g_\lambda \quad \text{on } U_{\lambda\mu}, \quad \forall \lambda, \mu.$$

Since $h_{\lambda\mu}$ is holomorphic and $\log h_{\lambda\mu}$ continuous, $\log h_{\lambda\mu}$ will be holomorphic on $U_{\lambda\mu}$. Indeed, $\log h_{\lambda\mu}$ will have local representations similar to (7a) below.

The present functions $\log h_{\lambda\mu}$ will automatically satisfy the compatibility conditions (1d) for the additive Cousin problem. Thus since Ω is a Cousin-I domain, there exist functions $\varphi_\lambda \in \mathcal{O}(U_\lambda)$ such that

$$\log h_{\lambda\mu} = \varphi_\mu - \varphi_\lambda \text{ on } U_{\lambda\mu}, \quad \forall \lambda, \mu.$$

It follows that

$$h_{\lambda\mu} = e^{\varphi_\mu}/e^{\varphi_\lambda} \text{ on } U_{\lambda\mu},$$

that is, the Cousin-II problem is solved by the functions $h_\lambda = e^{\varphi_\lambda} \in \mathcal{O}^*(U_\lambda)$.

Theorem 12.23 is an example of the heuristic "Oka principle": If a problem on a domain of holomorphy is locally holomorphically solvable and if it has a global continuous solution, then it has a global holomorphic solution.

12.3 Sheaves. Sheaves were introduced and studied by Cartan, Leray and Serre. They were used by Cartan and Grauert in connection with the solution of the Levi-problem. Sheaves have been a highly successful tool in several parts of mathematics, particularly in algebraic geometry. Examples of sheaves are scattered all over this book. It is high time we formally define them.

DEFINITION 12.31. A *sheaf* $\mathcal{F}$ over a space X with *projection* π is a triple $(\mathcal{F}, \pi, X)$ where $\mathcal{F}$ and X are topological spaces and π is a surjective local homeomorphism.

A *section* of $(\mathcal{F}, \pi, X)$ over an open $U \subset X$ is a continuous map $\sigma : U \to \mathcal{F}$ such that $\sigma \circ \pi$ is the identity mapping on U. The sections over U are denoted by $\mathcal{F}(U)$ or $\Gamma(U) = \Gamma(U, \mathcal{F})$. A *stalk* of $(\mathcal{F}, \pi, X)$ is a subset of $\mathcal{F}$ of the form $\pi^{-1}(x)$ where $x \in X$.

A sheaf of *rings*, *(abelian) groups*, etc. is a sheaf $\mathcal{F}$ with the property that the stalks $\mathcal{F}(x)$ have the structure of a ring, respectively, an (abelian) group, etc. of which the algebraic operations like addition or multiplication are continuous. The latter means the following: form the product space $\mathcal{F} \times \mathcal{F}$ with product topology and consider the subset

$$\mathcal{F} \cdot \mathcal{F} = \{(f_1, f_2) \in \mathcal{F} \times \mathcal{F} : \pi(f_1) = \pi(f_2)\}.$$

Now addition (for example) in the stalks of $\mathcal{F}$ gives rise to a map

$$+ : \mathcal{F} \cdot \mathcal{F} \to \mathcal{F}, \quad (f_1, f_2) \mapsto f_1 + f_2,$$

which has to be continuous.

EXAMPLES 12.32. Let D be a domain in $\mathbf{C}^n$.

(i) The constant sheaves $\mathbf{C} \times D$, $\mathbf{Z} \times D$, etc. over D. Projection is ordinary projection on D. Observe that $\mathbf{C}$ (and $\mathbf{Z}$ etc.) need be equipped with the discrete topology.

(ii) The Riemann domains $(\mathcal{R}, \pi, D)$ of Definition 2.12 equipped with the usual topology, that is, defined by the basic neighborhoods $\mathcal{N}(p, V, g)$.

(iii) The sheaf of germs of holomorphic functions on U, denoted by $\mathcal{O}_U$, with projection $\pi : [f]_a \mapsto a$. Here $[f]_a$ denotes the germ of an analytic function f at a point $a \in U$. For $\mathcal{O}_U$ to become a sheaf we have to give it a topology that makes π a local homeomorphism. This can be done in a way similar to example (ii): A base for the topology is given by the sets

$$(3a) \qquad \mathcal{N}(V, f) = \{[f]_a : \ a \in V\} \quad \text{where } f \in \mathcal{O}(V).$$

Sections over V can be identified with holomorphic functions on V: To a holomorphic function f on V we associate the section

$$\sigma_f : a \mapsto [f]_a.$$

It is an easy exercise to check that σ is continuous. The fact that $\mathcal{O}(V)$ indicates both sections over V and holomorphic functions on V reflects this association.

(iv) Let $\mathcal{K}$ denote an algebra of functions on U. Thus $\mathcal{K}$ could be $C^\infty(U)$ or $\wedge^{p,q}(U)$ the (p,q)-forms on U (our functions may well be vector valued!) A germ of a function in $\mathcal{K}$ was defined in Section 2.1. As in the previous example these germs together form a sheaf a base for the topology of which is given similar to (3a). We thus obtain the sheaf C^∞_U of germs of smooth functions on U, the sheaf $\wedge^{p,q}_U$ of germs of smooth p,q forms on U, the sheaf $\mathcal{O}^*_U$ of holomorphic zero free functions on U, the sheaf $\mathcal{M}_U$ of germs of meromorphic functions on U (strictly speaking this one does not consists of germs of *functions*), etc. Again sections and functions can be identified.

It is easily seen that the examples (i, iii, iv) have the property that the stalks are abelian groups or have even more algebraic structure. We leave it to the reader to check that the algebraic operations are continuous.

We need some more definitions.

DEFINITION 12.33. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X with projections $\pi_{\mathcal{F}}$, respectively $\pi_{\mathcal{G}}$. A continuous map $\varphi : \mathcal{F} \to \mathcal{G}$ is called a *sheaf map* if

$$\pi_{\mathcal{F}} = \pi_{\mathcal{G}} \circ \varphi.$$

$\mathcal{G}$ is called a subsheaf of $\mathcal{F}$ if for all $x \in X$ we have $\mathcal{G}_x \subset \mathcal{F}_x$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups then a map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is called a *sheaf homomorphism* if its restriction to each stalk is a group homomorphism. Similarly one defines homomorphisms of sheaves of rings, etc.

Observe that a sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ induces homomorphisms $\varphi^* : \mathcal{F}(U) \to \mathcal{G}(U)$, $\varphi^*(\sigma) = \varphi \circ \sigma$ for $\sigma \in \mathcal{F}(U)$.

12.4 Cohomological formulation of the Cousin-problems. Cousin-I and Cousin-II data $\{h_{\lambda\mu}\}$ associated with an open covering $\{U_\lambda\}$ of $\Omega \subset \mathbf{C}^n$ are examples of so-called cocycles consisting of sections in the sheaves $\mathcal{O}$ and $\mathcal{O}^*$ over Ω. These are sheaves of abelian groups. The group operation will always be denoted by + (it is multiplication of germs in case of $\mathcal{O}^*$!). In this section we will deal with Cousin problems associated to arbitrary sheaves of abelian groups over domains in $\mathbf{C}^n$

12.41. GENERAL COUSIN PROBLEM. Suppose we have a open covering $\mathcal{U} = \{U_\lambda\}$, $\lambda \in \Lambda$ of $\Omega \subset \mathbf{C}^n$. For a sheaf of abelian groups $\mathcal{F}$ over Ω, Cousin data associated with $\mathcal{U}$ consist of sections $f_{\lambda\mu}$ of $\mathcal{F}$, one over each intersection $U_{\lambda\mu}$, $\lambda, \mu \in \Lambda$, such that

$$f_{\lambda\mu} + f_{\mu\lambda} = 0, \tag{4a}$$

$$f_{\lambda\mu} + f_{\mu\nu} + f_{\nu\lambda} = 0 \tag{4b}$$

on the relevant intersections of sets U_α. One tries to determine a family of sections $f_\lambda \in \Gamma(U_\lambda, \mathcal{F})$, $(\lambda \in \lambda)$, such that

$$f_{\lambda\mu} = f_\mu - f_\lambda \quad \text{on } U_{\lambda\mu}, \ \ \forall \lambda, \mu. \tag{4c}$$

We introduce some further terminology. With a covering $\mathcal{U} = \{U_\lambda\}$ of Ω there are associated various cochains "with values in" $\mathcal{F}$.

DEFINITION 12.42. (Cochains for $\mathcal{U}$ with values in $\mathcal{F}$). A *zero-cochain* f^0_- is a family of sections $\{f_\lambda\}$, $f_\lambda \in \Gamma(U_\lambda)$. It is simply a function on Λ assuming specific sections of $\mathcal{F}$ as values:

$$f^0_- : \ \lambda \mapsto f^0_\lambda \in \Gamma(U_\lambda), \quad \lambda \in \Lambda.$$

A *1-cochain* f^1_- is a family of sections $\{f_{\lambda\mu}\}$, $f_{\lambda\mu} \in \Gamma(U_{\lambda\mu})$ with the alternating property (4a). It is an alternating function on Λ^2:

$$f^1_- : \ (\lambda, \mu) \mapsto f^1_{\lambda,\mu} \in \Gamma(U_{\lambda\mu}), \quad \lambda, \mu \in \Lambda.$$

An *s-cochain* f^s_- is an alternating function on Λ^{s+1}:

$$f^s_- : (\lambda_0, \lambda_1, \ldots, \lambda_s) \mapsto f^s_{\lambda_0\lambda_1\ldots\lambda_s} \in \Gamma(U_{\lambda_0\lambda_1\ldots\lambda_s}), \quad \lambda_j \in \Lambda.$$

Here $U_{\lambda_0\lambda_1\ldots\lambda_s} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_s}$, while *alternating* means that for a permutation σ with sign $\epsilon(\sigma)$ we have $f^s_{\sigma(\lambda_0\lambda_1\ldots\lambda_s)} = \epsilon(\sigma) f^s_{\lambda_0\lambda_1\ldots\lambda_s}$.

For s-cochains associated to $\mathcal{U}$ one defines addition as addition of the values of the cochain. Thus one obtains the abelian *group of s-cochains*:

$$C^s(\mathcal{U}) = C^s(\mathcal{U}, \mathcal{F}).$$

Starting with a 0-cochain f^0_- for $\mathcal{U}$, formula (4c) defines a 1-cochain f^1_- which is denoted by δf^0_-. We need a corresponding operator on s-cochains:

$$\delta = \delta_s : \; C^s(\mathcal{U}) \to C^{s+1}(\mathcal{U}).$$

DEFINITION 12.43 (Coboundary operator). For an s-cochain $f^s_- = \{f_{\lambda_0\lambda_1\ldots\lambda_s}\}$ one defines $\delta f^s_- \in C^{s+1}(\mathcal{U})$ by

$$(\delta f^s_-)_{\lambda_0\lambda_1\ldots\lambda_{s+1}} = \sum_{r=0}^{s+1} (-1)^r f_{\lambda_0\ldots\hat{\lambda}_r\ldots\lambda_s} \text{ on } U_{\lambda_0\lambda_1\ldots\lambda_{s+1}}, \quad \forall(\lambda_0, \lambda_1, \ldots \lambda_{s+1}),$$

where $\hat{\lambda}_r$ means that the index λ_r is omitted.

Observe that δ is a homeomorphism.

Illustration. The Cousin data in (12.41) consist of a 1-cochain $f^1_- = \{f_{\lambda\mu}\}$ for $\mathcal{U}$ with values in $\mathcal{F}$ ((4a)), such that $\delta f^1_- = 0$ ((4b)). The Cousin problem is whether there exists a 0-cochain f^0_- for $\mathcal{U}$ with values in $\mathcal{F}$ such that $f^1_- = \delta f^0_-$ ((4c)).

DEFINITION 12.44 (Cocycles and coboundaries for $\mathcal{U}$ and $\mathcal{F}$). An s-cochain f^s_- is called an *s-cocycle* if

$$\delta f^s_- = 0.$$

An s-cochain f^s_- is called a *s-coboundary* if ($s \geq 1$ and)

$$f^s_- = \delta f^{s-1}_-,$$

for some $(s-1)$-cochain f^{s-1}_-.

Because δ is a homomorphism, the s-cocycles form a subgroup

$$Z^s(\mathcal{U}) = Z^s(\mathcal{U}, \mathcal{F}) \subset C^s(\mathcal{U}, \mathcal{F});$$

Similarly the s-coboundaries form a subgroup

$$B^s(\mathcal{U}) = B^s(\mathcal{U}, \mathcal{F}) \subset C^s(\mathcal{U}, \mathcal{F}).$$

Lemma 12.45. *Every s-boundary is an s-cocycle:*

$$\delta^2 = \delta_s\delta_{s-1} = 0, \quad s \geq 1,$$

hence $B^s(\mathcal{U})$ is a subgroup of $Z^s(\mathcal{U})$.

PROOF. This is a verification similar to the one for the $\bar\partial$ operator.

$$(\delta_s\delta_{s-1}f_-^{s-1})_{\lambda_0\lambda_1\ldots\lambda_{s+1}} = \sum_{r=0}^{s+1}(-1)^r(\delta_{s-1}f_-^{s-1})_{\lambda_0\ldots\hat\lambda_r\ldots\lambda_{s+1}}$$
$$= \sum_{r=0}^{s+1}(-1)^r\sum_{k=0}^{r-1}(-1)^k f^{s-1}_{\lambda_0\ldots\hat\lambda_k\ldots\hat\lambda_r\ldots\lambda_{s+1}} + \sum_{r=0}^{s+1}(-1)^r\sum_{k=r+1}^{s+1}(-1)^{k-1}f^{s-1}_{\lambda_0\ldots\hat\lambda_r\ldots\hat\lambda_k\ldots\lambda_{s+1}} = 0,$$

because of cancelation.

The case $s = 0$ is somewhat special: there are no real coboundaries and one defines $B^0 = \{0\}$. For a 0-cocycle $f_-^0 = \{f_\lambda\}$ one has

$$f_\mu - f_\lambda = 0 \text{ on } U_{\lambda\mu}, \quad \forall\lambda,\mu.$$

Apparently a 0-cocycle determines a global section of $\mathcal{F}$: one may define $f \in \Gamma(\Omega,\mathcal{F})$ in a consistent manner by setting

$$f = f_\lambda \text{ on } U_\lambda, \quad \forall\lambda.$$

In this setup the groups $C^s(\mathcal{U},\mathcal{F})$ form what is called a *semi-exact sequence or complex*:

$$\text{(4d)} \qquad \cdots \xrightarrow{\delta} C^{s-1}(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^s(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} C^{s+1}(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \cdots. \quad (\delta\circ\delta = 0)$$

This notion makes sense for sequences of abelian groups connected through homomorphisms with the property that the composition of two consecutive ones is 0. Thus a semi-exact sequence of abelian groups is a sequence

$$\cdots \longrightarrow A_j \xrightarrow{f_j} A_{j+1} \xrightarrow{f_{j+1}} A_{j+2} \longrightarrow \cdots$$

with $f_{j+1}\circ f_j = 0$. If, moreover, the kernel of f_{j+1} equals the image of f_j, the sequence is called exact. The same terminology applies to sequence of sheaves of abelian groups connected through sheaf homomorphisms. Finally, a short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

It follows that here f is injective, while g is surjective.

The important objects are the quotient groups of (4d):

DEFINITION 12.46 ((v)Cech Cohomology groups for $\mathcal{U}$ and $\mathcal{F}$). The quotient group

$$H^s(\mathcal{U},\mathcal{F}) \stackrel{\text{def}}{=} \frac{Z^s(\mathcal{U},\mathcal{F})}{B^s(\mathcal{U},\mathcal{F})} = \frac{s\text{-cocycles}}{s\text{-coboundaries}}$$

is called the s-th cohomology group for the covering $\mathcal{U}$ of Ω with values in $\mathcal{F}$. The elements are equivalence classes of s-cocycles, the cosets of the subgroup of s-coboundaries.

For $s=0$ one has

$$H^0(\mathcal{U},\mathcal{F}) = Z^0(\mathcal{U},\mathcal{F}) = \Gamma(\Omega,\mathcal{F}). \tag{4e}$$

The cohomology groups are zero if and only if (4d) is exact. They measure the "amount of inexactness" of the complex.

ILLUSTRATION The Cousin problem asks if a given 1-cocycle f^1_- for $\mathcal{U}$ and $\mathcal{F}$ is a 1-coboundary. Thus this Cousin problem is always solvable when every 1-cocycle is a 1-coboundary, in other words when

$$H^1(\mathcal{U},\mathcal{F}) = 0.$$

EXAMPLE 12.47. [cf. Example (7.17)]. Take $\mathcal{F}=\mathcal{O}$, $\Omega=\mathbf{C}^2-\{0\}$, $U_j=\{z_j \overline{\nu} 0\}$, $j=1,2$. The associated 0-cochains $h^0_-=\{h_1,h_2\}$ are given by the holomorphic functions

$$h_i(z) = \sum_{\alpha\in\mathbf{Z}^2} a^i_\alpha z^\alpha \text{ on } U_i \quad (i=1,2),$$

with $a^i_\alpha=0$ if $\alpha_{1-i}<0$. The 1-cochains $h^1_-=\{h_{11},h_{12},h_{21},h_{22}\}$ are given by holomorphic functions

$$h_{11}=h_{22}=0, \quad h_{12}(z) = -h_{21}(z) = \sum_{\alpha\in\mathbf{Z}^2} c_\alpha z^\alpha \text{ on } U_{12}. \tag{4f}$$

The relations in (4f) follow from the alternating property of cocycles. The 1-cochains are at the same time 1-cocycles since there are only two different indices:

$$(\delta h^1_-)_{jkl} = h_{kl} - h_{jl} + h_{jk} = 0$$

whenever two indices such as k and l are the same.

The 1-coboundaries are those 1-cocycles, for which h_{12} on U_{12} equals a difference h_2-h_1 of functions $h_j\in\mathcal{O}(U_j)$. That is, (4f) represents a coboundary if and only if

$$c_\alpha = a^1_\alpha - a^2_\alpha, \quad \forall\alpha\in\mathbf{Z}^2.$$

This requirement presents no problem if $\alpha_1\geq 0$ or $\alpha_2\geq 0$ (or both). However if $\alpha_1,\alpha_2<0$ there is no solution unless $c_\alpha=0$ Thus the coboundaries are those cocycles with $c_\alpha=0$,

($\alpha_i < 0$). The cohomology group $H^1(\mathcal{U}, \mathcal{O})$ is isomorphic to the group of holomorphic functions

$$h_{12}(z) = \sum_{\alpha_i < 0} c_\alpha z^\alpha \text{ on } U_{12}.$$

EXAMPLE 12.48. Taking $\mathcal{F}$ and Ω as above, we consider the covering $V_1 = \{0 < |z| < 2\}$, $V_2 = \{1 < |z| < \infty\}$. The associated Cousin problem will be generally solvable. Indeed every holomorphic function h_{12} on $V_{12} = \{1 < |z| < 2\}$ has an analytic continuation to $B(0, 2)$ [by Hartogs' spherical shell theorem, Sections 2.8, 3.4]. Thus such a function is written as $h_{12} = 0 - h_1$ with h_1 the analytic continuation to V_1 of h_{12}. Conclusion: $H^1(\{V_1, V_2\}, \mathcal{O}) = 0$.

12.5 Definition of the domain cohomology groups $H^s(\Omega, \mathcal{F})$. The illustration to (12.46) gives the precise condition $H^1(\mathcal{U}, \mathcal{F}) = 0$ for the general solvability of the Cousin problem for $\mathcal{F}$ and a fixed covering $\mathcal{U}$ of Ω. We would also like to have a condition on Ω which assures the general solvability of the Cousin problem for *every* covering of Ω. [For the sheaf $\mathcal{O}$ such a condition was that Ω be a $\bar\partial$ domain, cf. Chapter 7.] Keeping $\mathcal{F}$ fixed we write

$$H^s(\mathcal{U}, \mathcal{F}) = H^s(\mathcal{U}).$$

By Proposition 7.32 whose proof is valid for general sheaves $\mathcal{F}$, refinement of Cousin data does not affect the solvability of the Cousin problem, Thus if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $H^1(\mathcal{V}) = 0$, so that all Cousin problems for $\mathcal{V}$ are solvable, then in particular all refinements to $\mathcal{V}$ of Cousin problems for $\mathcal{U}$ are solvable, hence all Cousin problems for $\mathcal{U}$ are solvable so that $H^1(\mathcal{U}) = 0$.

What will happen in general to the cohomology groups if we refine the covering $\mathcal{U}$ of Ω to $\mathcal{V}$? We will see that $H^1(\mathcal{U})$ is always (isomorphic to a subgroup of $H^1(\mathcal{V})$. Refinement may lead to large and larger groups $H^1(\mathcal{W})$ which ultimately become constant. The limit group is called $H^1(\Omega, \mathcal{F})$. For $s \geq 2$ the situation is more complicated; in the general case one needs the notion of a direct limit to define $H^s(\Omega, \mathcal{F})$, see below. We need two propositions

Proposition 12.51. *A refinement of the covering $\mathcal{U}$ of Ω to $\mathcal{V}$ via a refinement map σ induces a unique homomorphism $\sigma^* = \sigma(\mathcal{U}, \mathcal{V})$ of $H(\mathcal{U})$ to $H(\mathcal{V})$, that is a sequence of homomorphisms $\sigma_s^*(\mathcal{U}, \mathcal{V}) : H^s(\mathcal{U}) \to H^s(\mathcal{V})$. Uniqueness means here that the homomorphism is independent of the choice of the refinement map.*

PROOF. Let the covering $\mathcal{V} = \{V_j\}$, $j \in J$, of Ω be a refinement of the covering $\mathcal{U} = \{U_\lambda\}$, $\lambda \in \Lambda$ and let $\sigma : J \to \Lambda$ be a refinement map, that is, every set V_j is contained in $U_{\sigma(j)}$. To every cochain $f^s_- \in C^s(\mathcal{U})$ the map assigns a cochain in $C^s(\mathcal{V})$ — denoted by $\sigma(f^s_-)$ — by restriction. Specifically, for $s = 0, 1, \ldots$, we have, with $\sigma = \sigma^s$,

$$\sigma(f^s_-)_{j_0 j_1 \ldots j_s} = f^s_{\sigma(j_0)\sigma(j_1)\ldots\sigma(j_s)} \mid V_{j_0 j_1 \ldots j_s}.$$

The maps σ on cochain groups are clearly homomorphisms. Moreover they commute with the coboundary operator δ:

$$\delta_s \circ \sigma^s = \sigma^{s+1} \circ \delta_s, \quad (C^s(\mathcal{U}) \to C^{s+1}(\mathcal{V})).$$

Thus, the image of a cocycle is again a cocycle and the image of a coboundary is a coboundary, that is, σ^s maps $Z^s(\mathcal{U})$ into $Z^s(\mathcal{V})$ and the subgroups $B^s(\mathcal{U})$ into $B^s(\mathcal{V})$. It follows that σ induces a homomorphisms σ_s^* of the quotient groups by $\sigma_s^* : [f_-^s] \mapsto [\sigma f_-^s]$, in other words, we found a homomorphism

$$\sigma^* : \ H^s(\mathcal{U}) \to H^s(\mathcal{V}).$$

We will now indicate how to show that σ^* depends only on the refinement and not on the refinement map. Here the notion of a *chain homotopy* is useful. Suppose that σ and τ are two (chain) homomorphism from the complexes $C(\mathcal{U})$ to $C(\mathcal{V})$ associated to the refinement mappings σ and τ. A chain homotopy between σ and τ is a (sequence of) map(s)

$$\Theta = \{\Theta_s\}, \ \Theta_s : C^s(\mathcal{U}) \to C^{s-1}(\mathcal{V}), \quad (s = 1, 2, \ldots).$$

with the property that

$$\text{(5}a\text{)} \qquad \Theta_{s+1}\delta_s + \delta_{s-1}\Theta_s = (\sigma^s - \tau^s).$$

Assuming that Θ has been constructed, suppose that f_s is an s-cocycle. Then $\delta f_s = 0$ and (5a) gives $(\sigma^s - \tau^s)f_s = \delta_{s-1}\Theta_s f_s$, which is a coboundary. Thus $\sigma^* = \tau^*$.

Now we have to define Θ:

$$\text{(5}b\text{)} \qquad [\Theta_s f_-^s]_{j_0 j_1 \ldots j_{s-1}} = \sum_{r=0}^{s-1} (-1)^r f_{\tau(j_0)\tau(j_1)\ldots\tau(j_r)\sigma(j_r)\ldots\sigma(s-1)}.$$

Verification of (5b) is a tedious calculation. However, if we can prove that (5b) defines a chain homotopy for those τ and σ which are equal on $J - \{k\}$ for one $k \in J$, then we are done, because we can deform two arbitrary refinement maps to each other by a chain of deformations, changing one $j \in J$ at a time. Now if $\tau(j) = \sigma(j)$ on $J - k$, then there are two possibilities

i. k is not in $j_0, \ldots j_s$. Then $(\sigma^s - \tau^s)(f_{j_0 \ldots j_s}) = 0$ and (5b) equals 0 so we are done.
ii. k is in $j_0, \ldots j_s$. We may assume $k = j_0$. We find

$$(\sigma^s - \tau^s)(f_{j_0 \ldots j_s}) = f_{\sigma(j_0)\ldots\sigma(j_s)} - f_{\tau(j_0)\ldots\tau(j_s)}.$$

To verify (5b) we compute, keeping in mind that $f_{\lambda_0 \ldots \lambda_s} = 0$ if two indices are equal,

$$[\delta\Theta_s f_-^s]_{j_0 j_1 \ldots j_s} = \sum_{l=0}^{s} (-1)^l [\Theta_s f_-^s]_{j_0 j_1 \ldots \hat{j}_l \ldots j_s} = + \sum_{l=1}^{s} (-1)^l f^s_{\tau(j_0)\sigma(j_0)\sigma(j_1)\ldots\hat{j}_l\ldots\sigma(j_{s-1})}$$

and

$$\begin{aligned}[\Theta_{s+1}\delta f_-^s]_{j_0 j_1 \ldots j_s} &= [\delta f]_{\tau(j_0)\sigma(j_0)\ldots j_s} \\ &= (\sigma^s - \tau^s)(f_{j_0 \ldots j_s}) + \sum_{l=1}^{s} (-1)^{l+1} f^s_{\tau(j_0)\sigma(j_0)\sigma(j_1)\ldots\hat{j}_l\ldots\sigma(j_{s-1})}.\end{aligned}$$

Adding yields (5b).

Proposition 12.52. *For $s = 1$ the homomorphism $\sigma^* = \sigma(\mathcal{U}, \mathcal{V})$ in Proposition 12.51 is injective, hence if $\mathcal{V}$ is a refinement of $\mathcal{U}$ then $H^1(\mathcal{U})$ is isomorphic to a subgroup of $H^1(\mathcal{V})$.*

PROOF. Let f^1_- be an arbitrary cocycle in $Z^1(\mathcal{U})$, with cohomology class $[f^1_-] \in H^1(\mathcal{U})$. Supposing that $\sigma^*[f^1_-] = [\sigma f^1_-] = 0$ in $H^1(\mathcal{V})$, we have to show that $[f^1_-] = 0$. But this follows from (the proof of) Proposition 7.32 on refinements of Cousin problems. Indeed if σf^1_- is a coboundary for $\mathcal{V}$, the refined Cousin problem for $\mathcal{V}$ and σf^1_- is solvable, but then the original Cousin problem for $\mathcal{U}$ and f^1_- is also solvable, so that f^1_- is a coboundary for $\mathcal{U}$.

DEFINITION 12.53. The (domain) *cohomology group* $H^s(\Omega) = H^s(\Omega, \mathcal{F})$ is the "direct limit" of the (coverings) groups $H^s(\mathcal{U}) = H^s(\mathcal{U}, \mathcal{F})$ under the mappings $\sigma(\mathcal{U}, \mathcal{V})$, associated with all possible refinements of coverings $\mathcal{U}$ of Ω to coverings $\mathcal{V}$. The direct limit may be defined as the set of equivalence classes of elements in the disjoint union $\cup_{\mathcal{U}} H^s(\mathcal{U})$ over all coverings $\mathcal{U}$ of Ω. Elements $u \in H^s(\mathcal{U})$ and $v \in H^s(\mathcal{V})$ are equivalent if there is a common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that u and v have the same image in $H^s(\mathcal{W})$, that is,

$$\sigma^*(\mathcal{U}, \mathcal{W})u = \sigma^*(\mathcal{V}, \mathcal{W})v.$$

Every element of $[u]$ of $H^s(\Omega)$ has a representative u in some group $H^s(\mathcal{U})$. For any refinement $\mathcal{W}$ of this $\mathcal{U}$ the class $[u]$ will contain the element $\sigma(\mathcal{U}, \mathcal{W})u$ of $H^s(\mathcal{W})$. The sum of two elements $[u]$ and $[v]$ in $H^s(\Omega)$, where $u \in H^s(\mathcal{U})$ and $v \in H^s(\mathcal{V})$ is formed by adding the representatives $\sigma(\mathcal{U}, \mathcal{W})u$ and $\sigma(\mathcal{V}, \mathcal{W})v$ in $H^s(\mathcal{W})$, where $\mathcal{W}$ is a common refinement of $\mathcal{U}$ and $\mathcal{V}$.

By Propositions 12.51 and 12.52 we have the following important

COROLLARY 12.54. The map $u \in H^s(\mathcal{U}) \mapsto [u] \in H^s(\Omega)$ defines a homomorphism of $H^s(\mathcal{U})$ into $H^s(\Omega)$ and for $s = 1$ this homomorphism is injective. In particular $H^1(\mathcal{U})$ is isomorphic to a subgroup of $H^1(\Omega)$ and $H^1(\Omega, \mathcal{F}) = 0$ (The Cousin problem for Ω with values in $\mathcal{F}$ is solvable) if and only if $H^1(\mathcal{U}, \mathcal{F}) = 0$ for every covering $\mathcal{U}$ of Ω.

REMARK. One may also think of the elements of $H^s(\Omega)$ as equivalence classes of cocycles in the disjoint union $\cup_{\mathcal{U}} Z^s(\mathcal{U})$ over all coverings $\mathcal{U}$ of Ω. To this end one extends the notion of cohomologous cocycles to cocycles belonging to different coverings: $f^s_- \in Z^s(\mathcal{U})$ and $\varphi^s_- \in Z^s(\mathcal{V})$ are called equivalent or cohomologous in $\cup_{\mathcal{U}} Z^s(\mathcal{U})$ if they have cohomologous images in $Z^s(\mathcal{W})$ for some common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$. Observe that a common refinement of $\mathcal{U}$ and $\mathcal{W}$ always exists: Take

$$\mathcal{W} = \{W = U \cap V : \; U \in \mathcal{U}, \; V \in \mathcal{V}\}.$$

12.6 Computation of $H(\Omega, \mathcal{F})$ and in particular $H^1(\Omega, \mathcal{O})$. We first prove a general result on the computation of $H^1(\Omega) = H^1(\Omega, \mathcal{F})$.

Theorem 12.61. *Let $\mathcal{U} = \{U_\lambda\}$, $\lambda \in \Lambda$ be any covering of $\Omega \subset \mathbf{C}^n$ by Cousin domains for $\mathcal{F}$: $H^1(U_\lambda, \mathcal{F}) = 0$ for all λ. Then*

$$H^1(\Omega, \mathcal{F}) \cong H^1(\mathcal{U}, \mathcal{F}).$$

PROOF. We have to show that $H^1(\mathcal{W})$ is (isomorphic to) a subgroup of $H^1(\mathcal{U})$ for every covering $\mathcal{W}$ of Ω, so that $H^1(\mathcal{U})$ is maximal and thus equal to $H^1(\Omega)$. Now $H^1(\mathcal{W})$ is a subgroup of $H^1(\mathcal{V})$ for any common refinement $\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{W}$ [Proposition 12.52], hence it is sufficient to show that

$$H^1(\mathcal{V}) \cong H^1(\mathcal{U}) \quad \text{for all refinements } \mathcal{V} \text{ of } \mathcal{U}. \tag{6a}$$

Choose a refinement $\mathcal{V} = \{V_j\}$, $j \in J$ of $\mathcal{U}$ and an associated 1-cocycle $\varphi^1_- = \{\varphi_{jk}\}$. Restriction of φ^1_- to U_λ gives a 1-cocycle (also denoted by φ^1_-) on U_λ for the covering $\{V_j \cap U_\lambda\}$, $j \in J$. By the hypotheses this cocycle is a coboundary: we can choose a 0-cochain $\varphi^0_{-\lambda}$ or φ^λ_- on U_λ such that

$$\varphi_{jk} = \varphi^\lambda_k - \varphi^\lambda_j \quad \text{on} \quad V_{jk} \cap U_\lambda, \quad \forall j, k.$$

We do this for all λ; on U_μ we find

$$\varphi_{jk} = \varphi^\mu_k - \varphi^\mu_j \quad \text{on} \quad V_{jk} \cap U_\mu, \quad \forall j, k.$$

Thus on $U_{\lambda\mu} \cap V_{jk}$, $\varphi^\mu_k - \varphi^\lambda_k = \varphi^\mu_j - \varphi^\lambda_j$, so that we may define $f_{\lambda\mu}$ in a consisted manner on $U_{\lambda\mu}$ by setting

$$f_{\lambda\mu} = \varphi^\mu_j - \varphi^\lambda_j \quad \text{on} \quad U_{\lambda\mu} \cap V_j, \quad \forall j \in J. \tag{6b}$$

One readily verifies that (6b), $\forall \lambda, \mu \in \Lambda$ defines a 1-cocycle f^1_- for the covering $\mathcal{U}$. For given φ^1_-, this cocycle may depend on the choices of the 0-cocycle φ^λ_-. However, if we make different choices ψ^λ_- then $\psi^\lambda_k - \psi^\lambda_j = \varphi^\lambda_k - \varphi^\lambda_j$, hence the differences $\psi^\lambda_j - \varphi^\lambda_j$ define a 0-cochain g_- for $\mathcal{U}$ via $g_\lambda = \psi^\lambda_j - \varphi^\lambda_j$ on $U_\lambda \cap V_j$, $\forall j$. The result is that f^1_- is replaced by the cohomologous cochain $\tilde{f}^1_- = f^1_- + \delta g_-$:

$$\tilde{f}_{\lambda\mu} - f_{\lambda\mu} = \psi^\mu_j - \psi^\lambda_j - (\varphi^\mu_j - \varphi^\lambda_j) = g_\mu - g_\lambda \quad \text{on} \quad U_{\lambda\mu} \cap V_j, \ \forall j.$$

Thus by our process, the cohomology class of f^1_- in $Z^1(\mathcal{U})$ is uniquely determined by φ^1_-. Note also that coboundaries go into coboundaries: for a coboundary φ^1_- we may take the 0-cochains $\varphi^0_{-\lambda}$ equal to φ^0_- independent of λ and we then obtain $f^1_- = 0$! Thus we have a map of cohomology classes belonging to $Z^1(\mathcal{V})$ into cohomology classes belonging to $Z^1(\mathcal{U})$ and this map is a homomorphism. The map is injective too: if f^1_- is a coboundary, $f_{\lambda\mu} = f_\mu - f_\lambda$, (6b) shows that the definition $\chi_j = \varphi^\lambda_j - f_\lambda$, $\forall \lambda$ gives a 0-cochain χ_- for $\mathcal{V}$ and $\varphi^1_- = \delta\chi_-$.

The conclusion is that our process defines an injective homomorphism of $H^1(\mathcal{V})$ into $H^1(\mathcal{U})$, hence $H^1(\mathcal{V})$ is a subgroup of $H^1(\mathcal{U})$. Since conversely $H^1(\mathcal{U})$ is a subgroup of $H^1(\mathcal{V})$ [Proposition 12.52], the proof of (6a) is complete.

EXAMPLE 12.62. For $\Omega = \mathbf{C}^2 - \{0\}$ one may compute $H^1(\Omega, \mathcal{O})$ with the aid of example (12.47).

Theorem 12.61 is a special case of the following theorem of J. Leray, a proof of which can be found in [GuRo], or [GrRe]. Call a covering $\{U_\lambda\}$ of $\Omega \subset \mathbf{C}^n$ *acyclic* if $H^s(U_{\lambda_1 \dots \lambda_j}, \mathcal{F}) = 0$ for all $s \geq 1$ and for all intersections $U_{\lambda_1 \dots \lambda_j}$.

Theorem 12.63 (Leray). *For every acyclic covering $\mathcal{U}$ of Ω:*

$$H^s(\Omega, \mathcal{F}) \cong H^s(\mathcal{U}, \mathcal{F}), \quad s = 0, 1, 2, \dots.$$

In the case of the first Cousin problem, ($\mathcal{F} = \mathcal{O}$), we have general solvability for all coverings $\mathcal{U}$ of $\Omega \subset \mathbf{C}^n$ if and only if $H^1(\Omega, \mathcal{O}) = 0$. By Chapter 7 we also have general solvability if and only if the equation $\bar\partial u = v$ on Ω is generally C^∞ solvable for all (0,1)-forms v (of class C^∞) for which $\bar\partial v = 0$. Recall that (the sheaf of sections of) (p, q) forms on Ω is denoted by $\wedge^{p,q} = \wedge^{p,q}(\Omega)$.

For $p = 0, 1, \dots$ we have an exact sequence of sheaves

$$0 \to \mathcal{O}^{p,0} \xrightarrow{i} \wedge^{p,1} \xrightarrow{\bar\partial} \wedge^{p,2} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \wedge^{p,n} \to 0. \tag{6c}$$

Here $\mathcal{O}^{p,0}$ is the subsheaf of $\wedge^{p,0}$ consisting of germs $(p, 0)$ forms with holomorphic coeffients; $\mathcal{O}^{0,0} = \mathcal{O}$. Exactness follows from the fact that $\bar\partial\bar\partial = 0$ and that *locally*, for example on polydiscs (Section 7.6, Chapter 11), the equation $\bar\partial u = v$, has a solution if $\bar\partial v = 0$. To (6c) is associated a semi-exact sequence of the groups of sections, that is, the groups of smooth differential forms on Ω:

$$0 \to \mathcal{O}^{p,0}(\Omega) \xrightarrow{i} \wedge^{p,1}(\Omega) \xrightarrow{\bar\partial} \wedge^{p,2}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \wedge^{p,n}(\Omega) \to 0. \tag{6d}$$

In general (6d) is not exact, globally the equations $\bar\partial u = v$, may not have a solution even if $\bar\partial v = 0$. Again cohomology groups will measure the amount of inexactness. See Definition 12.68 below.

Before pursuing this any further we will compute some trivial cohomology groups. We have seen in Chapter 7 that it is useful to be able to solve smooth Cousin-I problems, in order to connect them to the Cauchy-Riemann equations. Now we will do something similar in terms of cohomology. First we introduce some terminology.

DEFINITION 12.64. Let $\Omega \subset \mathbf{C}^n$ and let $\mathcal{U}$ be an open covering of Ω and $\mathcal{F}$ a sheaf of abelian groups over Ω. A *partition of unity of $\mathcal{F}$ subordinate to $\mathcal{U}$* is a set of sheaf homomorphisms $\beta_\lambda : \mathcal{F} \to \mathcal{F}$ such that

i. $\sum_\lambda \beta_\lambda = \mathrm{id}$ on $\mathcal{F}$;
ii. $\beta_\lambda([f]_x) = [0]_x$ for all x in some open neighborhood of the complement of $\bar U_\lambda$.

EXAMPLES 12.65. Let $\mathcal{U}$ be a covering of $\Omega \subset \mathbf{C}^n$ and let $\{\tilde\beta_\lambda\}$ be a (usual) partition of unity subordinate to $\mathcal{U}$. Then the $\tilde\beta_\lambda$ give rise to a partition of unity $\{\beta_\lambda\}$ of the sheaf C^∞ on Ω by

$$\beta_\lambda([u]_z) = [\tilde\beta_\lambda u]_z.$$

Similarly $\wedge^{p,q}(\Omega)$ admits a partition of unity subordinate to $\mathcal{U}$.

DEFINITION 12.66. A sheaf of abelian groups $\mathcal{F}$ over Ω is called *fine* if for every (locally finite) covering $\mathcal{U}$ of Ω it admits a partition of unity subordinate to $\mathcal{U}$.

The sheaves in 12.65 are fine sheaves.

Theorem 12.67. *Suppose that $\mathcal{F}$ is a fine sheaf over Ω and that $\mathcal{U}$ is any locally finite covering of Ω. Then $H^p(\Omega,\mathcal{U}) = 0$, $(p \geq 1)$, for every $\mathcal{U}$ and therefore $H^p(\Omega,\mathcal{F}) = 0$ for $p \geq 1$.*

PROOF. Let $\mathcal{U} = \{U_\lambda\}$ be a locally finite covering of Ω and let β_λ be the associated partition of unity of $\mathcal{F}$. It suffices to show that for $p > 0$ every p-cocycle (for $\mathcal{U}$ and $\mathcal{F}$) is a p-coboundary. This is done similarly to the proof of Theorem 7.41. Let $\sigma_- \in Z^p(\mathcal{U},\mathcal{F})$. Put

$$\tau_{\lambda_0\cdots\lambda_{p-1}} = \sum_\mu \beta_\mu(\sigma_{\mu\lambda_0\cdots\lambda_{p-1}}).$$

Notice that $\beta_\mu(\sigma_{\mu\lambda_0\cdots\lambda_{p-1}})$ is at first only defined on $U_\mu \cap U_{\lambda_0\cdots\lambda_{p-1}}$, but extends to $U_{\lambda_0\cdots\lambda_{p-1}}$ because it vanishes in a neighborhood of the boundary of U_μ. Thus τ is a well defined $(p-1)$ cocycle. We compute

$$\begin{aligned}(\delta\tau)_{\lambda_0\cdots\lambda_p} &= \sum_{i=0}^{p}(-1)^i\tau_{\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_p} = \sum_{i=0}^{p}(-1)^i\sum_\mu \beta_\mu(\sigma_{\mu\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_p})\\ &= \sum_\mu \beta_\mu\left(\sum_{i=0}^{p}(-1)^i\sigma_{\mu\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_p}\right) = \sum_\mu \beta_\mu(\sigma_{\lambda_0\cdots\lambda_p}) = \sigma_{\lambda_0\cdots\lambda_p},\end{aligned}$$

where we have used that σ is a cocycle, i.e. $\sum_{i=0}^{p+1}(-1)^i\sigma_{\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_{p+1}} = 0$, for all indices λ_i, in particular with $\lambda_0 = \mu$, and that $\sum_\mu \beta_\mu = \mathrm{id}$.

DEFINITION 12.68. Forms v with $\bar\partial v = 0$ are called $\bar\partial$ *closed*, forms $v = \bar\partial u$ are called $\bar\partial$ *exact*. Let $Z^{p,q}_{\bar\partial}(\Omega)$ denote the group of $\bar\partial$ closed forms in $\wedge^{p,q}(\Omega)$ and let $B^{p,q}_{\bar\partial}(\Omega)$ denote the group of $\bar\partial$ exact forms in $\wedge^{p,q}(\Omega)$. The quotient groups are the *Dolbeault cohomology groups*:

$$H^{p,q}_{\bar\partial}(\Omega) \overset{\text{def}}{=} \frac{Z^{p,q}_{\bar\partial}(\Omega)}{B^{p,q}_{\bar\partial}(\Omega)}.$$

Thus general solvability of the first Cousin problem may also be expressed by the condition $H^{0,1}(\Omega) = 0$. As a consequence $H^1(\Omega,\mathcal{O}) = 0$ if and only if $H^{0,1}_{\bar\partial}(\Omega) = 0$. Much more is true:

Theorem 12.69 (Dolbeault). *Let $\Omega \subset \mathbf{C}^n$ be open and let $\mathcal{U}$ be a locally finite covering of Ω that consists of domains of holomorphy. Then for $p = 0, 1, \ldots, n$*

$$H^q(\Omega,\mathcal{O}) = H^{0,q}_{\bar\partial}(\Omega) = H^q(\mathcal{U},\mathcal{O}).$$

For the proof we need some results from homological algebra. Let

$$0 \to \mathcal{E} \xrightarrow{i} \mathcal{F} \xrightarrow{s} \mathcal{G} \to 0$$

be an exact sequence of sheaves over Ω. For U open in Ω there is an associated exact sequence of groups of sections

$$0 \to \mathcal{E}(U) \xrightarrow{i^*} \mathcal{F}(U) \xrightarrow{s^*} \mathcal{G}_0(U) \to 0. \tag{6e}$$

Here $\mathcal{G}_0(U)$ denotes the image of $\mathcal{F}(U)$ under s^* in $\mathcal{G}(U)$, which need not be all of $\mathcal{G}(U)$. Similarly, if $\mathcal{U}$ is an open covering of Ω, then there is an induced exact sequence of chain groups with $C_0^s(\mathcal{U},\mathcal{G})$ the image of s^* in $C^s(\mathcal{U},\mathcal{G})$:

$$0 \to C^s(\mathcal{U},\mathcal{E}) \xrightarrow{i^*} C^s(\mathcal{U},\mathcal{F}) \xrightarrow{s^*} C_0^s(\mathcal{U},\mathcal{G}) \to 0. \tag{6e$'$}$$

See exercise 12.11.

The coboundary operator δ commutes with the maps i^* and s^*. One thus obtains the following *commutative diagram* with exact columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\delta"] & C^{s-1}(\mathcal{U},\mathcal{E}) \arrow[r, "\delta"] \arrow[d, "\varphi"] & C^{s}(\mathcal{U},\mathcal{E}) \arrow[r, "\delta"] \arrow[d, "\varphi"] & C^{s+1}(\mathcal{U},\mathcal{E}) \arrow[r, "\delta"] \arrow[d, "\varphi"] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{s-1}(\mathcal{U},\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "\psi"] & C^{s}(\mathcal{U},\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "\psi"] & C^{s+1}(\mathcal{U},\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "\psi"] & \cdots . \\
\cdots \arrow[r, "\delta"] & C_0^{s-1}(\mathcal{U},\mathcal{G}) \arrow[r, "\delta"] \arrow[d] & C_0^{s}(\mathcal{U},\mathcal{G}) \arrow[r, "\delta"] \arrow[d] & C_0^{s+1}(\mathcal{U},\mathcal{G}) \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6f}$$

Commutative means, of course, that every two possible compositions of maps originating at the same group and ending in the same group yield the same map. The maps φ and ψ commute with δ and therefore (compare the proof of Proposition 12.51) take cocycles to cocycles, coboundaries to coboundaries. Thus φ and ψ induce homomorphisms φ^*, ψ^*:

$$H^s(\mathcal{U},\mathcal{E}) \xrightarrow{\varphi^*} H^s(\mathcal{U},\mathcal{F}) \xrightarrow{\psi^*} H_0^s(\mathcal{U},\mathcal{G}), \quad s = 0,1,2\ldots \tag{6g}$$

The various sequences (6g) are connected through a homomorphism induced by δ:

Proposition 12.610 (Snake Lemma). *Associated to the commutative diagram (6f) there exist connecting homomorphisms $\delta^* : H_0^s(\mathcal{U},\mathcal{G}) \to H_0^{s+1}(\mathcal{U},\mathcal{E})$ for the sequences (6g) such that the following sequence is exact:*

$$\begin{aligned}
0 \longrightarrow H^0(\mathcal{U},\mathcal{E}) &\xrightarrow{\varphi^*} H^0(\mathcal{U},\mathcal{F}) \xrightarrow{\psi^*} H_0^0(\mathcal{U},\mathcal{G}) \\
\xrightarrow{\delta^*} H^1(\mathcal{U},\mathcal{E}) &\xrightarrow{\varphi^*} H^1(\mathcal{U},\mathcal{F}) \xrightarrow{\psi^*} H_0^1(\mathcal{U},\mathcal{G}) \\
\xrightarrow{\delta^*} H^2(\mathcal{U},\mathcal{E}) &\xrightarrow{\varphi^*} H^2(\mathcal{U},\mathcal{F}) \xrightarrow{\psi^*} H_0^2(\mathcal{U},\mathcal{G}) \xrightarrow{\delta^*} \cdots .
\end{aligned}$$

PROOF. We define δ^* by chasing through the diagram. Take a cocycle $g \in Z^s(\mathcal{U}, \mathcal{G})$. The map ψ is surjective, hence there exists $f \in C^s(\mathcal{U}, \mathcal{F})$ with $\psi f = g$. Observe that $\psi\delta f = \delta\psi f = \delta g = 0$, thus $\delta f \in C^{s+1}(\mathcal{U}, \mathcal{F})$ belongs to the kernel of ψ and as φ is injective, $\exists!\, e = e(f) \in C^{s+1}(\mathcal{U}, \mathcal{E})$ such that $\delta f = \varphi e$. We compute $\varphi\delta e(f) = \delta\delta f = 0$, hence, because φ is injective, $e(f) \in Z^{s+1}(\mathcal{U}, \mathcal{E})$. Now we wish to define $\delta^*[g] = [e(f)]$.

We have to check that this is well defined, that is, independent of the choice of f in the class $[f]$ and, moreover, that if g is a coboundary, e is a coboundary too.

Suppose $\psi(\tilde{f} - f) = 0$. Then $\tilde{f} - f = \varphi e_s$ so that $\delta(\tilde{f} - f) = \varphi\delta e_s$. In other words $e(\tilde{f}) - e(f) = \delta e_s$, that is $[e(\tilde{f})] = [e(f)]$.

Next suppose that $g = \delta g_{s-1}$ is a coboundary. Then $g_{s-1} = \psi f_{s-1}$ for some $f_{s-1} \in C^{s-1}(\mathcal{U}, \mathcal{F})$. Also $g_s = \psi f_s$. Now observe that $\psi(f_s - \delta f_{s-1}) = 0$, so that $f_s - \delta f_{s-1} = \varphi e_s$. We obtain that $\delta f_s = \delta(f_s - \delta f_{s-1}) = \delta\varphi e_s = \varphi\delta e_s$. We conclude that $e(f) = \delta e_s$ a coboundary.

Finally we show exactness of the sequence. This is again done by chasing the diagram (6f).

At $H^p(U, \mathcal{E})$**.** $[e_p] \in \operatorname{im}\delta^* \Leftrightarrow \exists f_{p-1} :\ \delta\psi f_{p-1} = 0$ and $\varphi e_p = \delta f_{p-1} \Leftrightarrow [\varphi e_p] = 0$.

At $H^p(U, \mathcal{F})$**.** $[f_p] \in \ker\psi^* \Leftrightarrow \exists g_{p-1} : \psi f_p = \delta g_{p-1} \Leftrightarrow \exists f_{p-1} :\ \delta\psi f_{p-1} = \psi\delta f_{p-1} = \psi f_p \Leftrightarrow \exists f_{p-1} :\ \psi(f_p - \delta f_{p-1}) = 0 \Leftrightarrow \exists f_{p-1} :\ f_p + \delta f_{p-1} \in \operatorname{im}\varphi \Leftrightarrow [f_p] \in \operatorname{im}\varphi^*$

At $H_0^p(U, \mathcal{G})$**.** $\delta^*[g_p] = 0 \Leftrightarrow \exists f_p :\ \psi f_p = g_p$ and $[\varphi^{-1}\delta f_p] = 0 \Leftrightarrow \exists e_p \varphi^{-1}\delta f_p = \delta e_p \Leftrightarrow \delta(f_p - \varphi e) = 0 \Leftrightarrow f - \varphi e \in Z^p(\mathcal{U}, \mathcal{F})$ and $\psi^*[f - \varphi e] = [\psi f] = [g]$.

Now we wish to pass to the direct limit and also replace C_0 by C in the exact sequence. We need

Lemma 12.611. *Keeping the notation as before, suppose that $g_- \in C^p(\mathcal{U}, \mathcal{G})$. Then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ with refinement map σ such that the refined cochain g_σ is in $C_0^p(\mathcal{V}, \mathcal{G})$.*

PROOF. After refinement if necessary, we may assume that $\mathcal{U}$ is a special open covering in the sense of 7.33 and that there is an open covering $\mathcal{W} = \{W_\lambda\}$ with the property that $\bar{W}_\lambda \subset U_\lambda$. Let $g_{\lambda_0 \cdots \lambda_p}$ be a p cochain in $C^p(\mathcal{U}, \mathcal{G})$. Because the sequence (6e) is exact, there exists for every $z \in \Omega$ and every $\lambda_0 \cdots \lambda_p$ with $z \in U_{\lambda_0 \cdots \lambda_p}$ a neighborhood $V_z \subset U_{\lambda_0 \cdots \lambda_p}$ such that $g_{\lambda_0 \cdots \lambda_p} \mid V_z = s \circ f_{\lambda_0 \cdots \lambda_p} \mid V_z$ for some $f_{\lambda_0 \cdots \lambda_p}$ defined on V_z. For a fixed z there are only finitely many intersections $U_{\lambda_0 \cdots \lambda_p}$ that contain z, because the covering is locally finite. Thus we may choose V_z independent of $\lambda_0 \cdots \lambda_p$. Shrinking V_z if necessary, we may also assume that $V_z \cap W_\lambda \neq \emptyset$ implies that $V_z \in\ U_\lambda$ and $z \in W_\lambda$ implies that $V_z \in W_\lambda$. From $\{V_z\}_{z \in \Omega}$ we select a countable, locally finite subcovering $\{V_i = V_{z_i}\}$ and we define the refinement function σ by choosing $\sigma(i) \in \{\lambda :\ z \in W_\lambda\}$. Suppose that $V_{i_0 \cdots i_p}$ is nonempty. Then for $0 \leq j \leq p$ $V_{i_0} \cap W_{\sigma(i_j)} \neq \emptyset$, hence $V_{i_0} \subset U_{\sigma(i_j)}$. Now the refined cochain $\sigma(g)_{i_0 \cdots\ i_p}$ is the restriction of the function $g_{\sigma(i_0) \cdots \sigma(i_p)}$ defined on $U_{\sigma(i_0) \cdots \sigma(i_p)} \supset V_{i_0}$, and therefore there exists $f = f_{\sigma(i_0) \cdots \sigma(i_p)}$ on V_{i_0} with $s \circ f = g$ on $V_{i_0 \cdots i_p}$.

COROLLARY 12.612 (Snake Lemma). The following sequence is exact

$$\begin{aligned} 0 \longrightarrow H^0(\Omega, \mathcal{E}) &\xrightarrow{\varphi^*} H^0(\Omega, \mathcal{F}) \xrightarrow{\psi^*} H^0(\Omega, \mathcal{G}) \\ \xrightarrow{\delta^*} H^1(\Omega, \mathcal{E}) &\xrightarrow{\varphi^*} H^1(\Omega, \mathcal{F}) \xrightarrow{\psi^*} H^1(\Omega, \mathcal{G}) \\ \xrightarrow{\delta^*} H^2(\Omega, \mathcal{E}) &\xrightarrow{\varphi^*} H^2(\Omega, \mathcal{F}) \xrightarrow{\psi^*} H^2(\Omega, \mathcal{G}) \xrightarrow{\delta^*} \cdots. \end{aligned}$$

PROOF. Lemma 12.611 and the fact that δ commutes with refinement maps imply that every cocycle in $Z^p(\mathcal{V},\mathcal{G})$ may be refined to a cocycle in $Z_0^p(\mathcal{U},\mathcal{G})$. Also a coboundary δg may be refined to a coboundary in $B_0^p(\mathcal{V},\mathcal{G})$ by refining g. We infer that $H_0^2(\Omega,\mathcal{G}) = H^2(\Omega,\mathcal{G})$. Exactness of the sequence in 12.612 is obtained by passing to the direct limit in 12.610.

Proof of Theorem 12.69. Consider the exact sequence

$$0 \longrightarrow \mathcal{L}^{p,q} \xrightarrow{\ i\ } \wedge^{p,q} \xrightarrow{\ \bar\partial\ } \mathcal{L}^{p,q+1} \longrightarrow 0.$$

Here $\mathcal{L}^{p,q}$ stands for the sheaf of germs of $\bar\partial$ closed (p,q) forms (which is of course the same as the sheaf of germs of $\bar\partial$ exact (p,q) forms). The Snake Lemma gives the following exact cohomology sequence

$$(6h)\qquad \cdots \xrightarrow{i^*} H^j(\Omega,\wedge^{p,q}) \xrightarrow{\bar\partial^*} H^j(\Omega,\mathcal{L}^{p,q+1}) \xrightarrow{d^*} H^{j+1}(\Omega,\mathcal{L}^{p,q}) \xrightarrow{i^*} H^{j+1}(\Omega,\wedge^{p,q}) \xrightarrow{\bar\partial^*} \cdots.$$

In view of Theorem 12.67 we obtain

$$0 \xrightarrow{\bar\partial^*} H^j(\Omega,\mathcal{L}^{p,q+1}) \xrightarrow{d^*} H^{j+1}(\Omega,\mathcal{L}^{p,q}) \xrightarrow{i^*} 0.$$

Thus $H^j(\Omega,\mathcal{L}^{p,q+1})$ is isomorphic to $H^{j+1}(\Omega,\mathcal{L}^{p,q})$ and repeating this we find

$$(6i).\qquad H^1(\Omega,\mathcal{L}^{p,q}) \cong H^{q+1}(\Omega,\mathcal{O}^p)$$

Also, from (6h) with $j=0$ we see

$$\Gamma(\Omega,\wedge^{p,q}) \xrightarrow{\bar\partial^*} \Gamma(\Omega,\mathcal{L}^{p,q+1}) \xrightarrow{d^*} H^1(\Omega,\mathcal{L}^{p,q}) \longrightarrow 0$$

is exact, therefore $H^1(\Omega,\mathcal{L}^{p,q}) \cong \Gamma(\Omega,\mathcal{L}^{p,q+1})/\bar\partial^*\Gamma(\Omega,\wedge^{p,q})$. Combining this with (6i) yields

$$H^{q+1}(\Omega,\mathcal{O}^p) \cong \Gamma(\Omega,\mathcal{L}^{p,q+1})/\bar\partial^*\Gamma(\Omega,\wedge^{p,q}),$$

which proves $H^q(\Omega,\mathcal{O}) = H_{\bar\partial}^{0,q}(\Omega)$, by taking $p=0$.

If the covering $\mathcal{U}$ consists of domains of holomorphy, then all $\bar\partial$ equations may be solved on $u \in \mathcal{U}$, hence in (6e, 6e') we have $\mathcal{G}_0(U) = \mathcal{G}(U)$ and $C_0^s(\mathcal{U},\mathcal{G}) = C^s(\mathcal{U},\mathcal{G})$. Thus in the exact sequence of Proposition 12.610 we have $H_0^p(\mathcal{U},\mathcal{G}) = H^p(\mathcal{U},\mathcal{G})$. We conclude that we may repeat the previous proof with Ω replaced by $\mathcal{U}$ and obtain

$$H^{q+1}(\mathcal{U},\mathcal{O}^p) \cong \Gamma(\Omega,\mathcal{L}^{p,q+1})/\bar\partial^*\Gamma(\Omega,\wedge^{p,q}),$$

which proves $H^q(\mathcal{U},\mathcal{O}) = H_{\bar\partial}^{0,q}(\Omega)$.

12.7 The multiplicative Cousin problem revisited. Cousin-II data consist of a covering $\mathcal{U}$ of Ω and an associated 1-cocycle $h^1_- \in Z^1(\mathcal{U}, \mathcal{O}^*)$, cf. (1c, 1d). The group operation in $\mathcal{O}^*$ is multiplication. The question is to determine if h^1_- is a coboundary. The illustration to 12.46 and Corollary 12.54 lead to the following observation.

OBSERVATION 12.71. Let $\Omega \subset \mathbf{C}^n$ be open. The multiplicative Cousin problem is generally solvable for a fixed covering $\mathcal{U}$ of Ω if and only if

$$H^1(\mathcal{U}, \mathcal{O}^*) = 0.$$

It can generally be solved for every covering $\mathcal{U}$ of Ω if and only if

$$H^1(\Omega, \mathcal{O}^*) = 0.$$

One obvious way to try and solve Cousin-II problems is to reduce them to Cousin-I problems by passing to the logarithms of the data. Therefore it is necessary that the functions $h_{\lambda\mu} \in \mathcal{O}^*(U_{\lambda\mu})$ should admit holomorphic logarithms, hence we have to work with appropriate coverings.

Proposition 12.72. *For a domain $V \subset \mathbf{C}^n$, each of the following conditions suffices for the existence of continuous (or holomorphic) logarithms of zero free continuous (or holomorphic) functions g on V:*
(i) V is simply connected: all closed curves in V can be contracted to a point inside V;
(ii) $H^1(V, \mathbf{Z}) = 0$.

PROOF. (i) On a sufficiently small ball $B(c, \delta)$ in V, a continuous (or holomorphic) branch of $\log g$ may be defined by setting

$$\text{(7a)} \qquad \begin{aligned} \log g(z) &= \log g(c) + p.v.\log\left\{1 + \frac{g(z) - g(c)}{g(c)}\right\} \\ &= \log g(c) + \sum_1^\infty \frac{(-1)^{k-1}}{k}\left\{\frac{g(z) - g(c)}{g(c)}\right\}^k. \end{aligned}$$

Here $\log g(c)$ is an arbitrary value of the logarithm; one takes $\delta > 0$ so small that $|g(z) - g(c)| < |g(c)|$ throughout $B(c, \delta)$.

On every Jordan arc from a fixed point a to a point b in V, a continuous branch of $\log g$ may be obtained with the aid of a suitable covering of the arc by small balls. If all arcs from a to b in V are homotopically equivalent (that is, obtainable from each other by continuous deformation within V), then $\log g(b)$ may be defined unambiguously in terms of $\log g(a)$ with the aid of connecting Jordan arcs in V. Thus on simply connected V, a zero free continuous function g has a continuous logarithm. If g is holomorphic, so is the logarithm, as can be seen from (7a) locally.

(ii) Consider the exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z} \stackrel{i}{\longrightarrow} \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0,$$

where exp denotes the map $f \mapsto e^{2\pi i f}$. This gives rise to a long exact cohomology sequence

$$
\begin{aligned}
(7b) \qquad & 0 \longrightarrow H^0(V,\mathbf{Z}) \xrightarrow{i^*} H^0(V,\mathcal{O}) \xrightarrow{\exp^*} H^0(V,\mathcal{O}^*) \\
& \xrightarrow{\delta^*} H^1(V,\mathbf{Z}) \xrightarrow{i^*} H^1(V,\mathcal{O}) \xrightarrow{\exp^*} H^1(V,\mathcal{O}^*) \xrightarrow{\delta^*} H^2(V,\mathbf{Z}) \longrightarrow \cdots
\end{aligned}
$$

Recalling that $H^0(V,\mathcal{F})$ equals the global sections of $\mathcal{F}$, we see that $H^1(V,\mathbf{Z}) = 0$ implies that $\exp^*$ is surjective to $H^0(V,\mathcal{O}^*)$, hence every zero free holomorphic function is of the form $\exp g$ with g holomorphic on V.

There is a similar exact sequence of sheaves for continuous functions

$$0 \longrightarrow \mathbf{Z} \xrightarrow{i} \mathcal{C} \xrightarrow{\exp} \mathcal{C}^* \longrightarrow 0,$$

and the preceding argument gives the result for continuous logarithms.

The exact sequence (7b) gives further insight into the Cousin-II problem:

Theorem 12.73 (Serre) Let $\Omega \subset \mathbf{C}^n$ be a Cousin-I domain. *Then the Cousin-II problem (and by Proposition 12.13, also the divisor problem) is generally solvable on Ω whenever*

$$H^2(\Omega,\mathbf{Z}) = 0.$$

PROOF. We have to prove that $H^1(\Omega,\mathcal{O}^*) = 0$. Looking at the exact sequence (7b) and using that $H^1(\Omega,\mathcal{O}) = H^2(\Omega,\mathbf{Z}) = 0$ we derive from exactness of

$$0 = H^1(\Omega,\mathcal{O}) \xrightarrow{\exp^*} H^1(\Omega,\mathcal{O}^*) \xrightarrow{\delta^*} H^2(\Omega,\mathbf{Z}) = 0,$$

that $H^1(\Omega,\mathcal{O}^*) = 0$.

As an application we obtain an answer to the so-called Poincaré problem: When do meromorphic functions have global representations as quotients of holomorphic functions?

Theorem 12.74. *Let Ω be a Cousin-I domain in $\mathbf{C}^n$ such that $H^2(\Omega,\mathbf{Z}) = 0$. Then every meromorphic function f on Ω has a global representation*

$$f = \frac{g}{h}, \quad g,h \in \mathcal{O}(\Omega)$$

with g and f relatively prime everywhere on Ω.

PROOF. Let f be meromorphic on Ω, that is, every point $a \in \Omega$ has a neighborhood U_a on which f can be represented as a quotient g_a/h_a of holomorphic functions (Section 7.1). It may be assumed that g_a and $h-a$ are relatively prime at a. They are then relatively prime on some neighborhood of a, cf. Section 4.6 and exercise 4.19. Hence there is a covering $\mathcal{U}$ of Ω such that

$$f = \frac{\varphi_\lambda}{\psi_\lambda} \text{ on } U_\lambda, \quad \forall \lambda \in \Lambda$$

with φ_λ and ψ_λ relatively prime everywhere on U_λ.

On $U_{\lambda\mu}$ one has $\varphi_\lambda \psi_\mu = \varphi_\mu \psi_\lambda$. It follows that φ_λ and φ_μ have the same prime factors at every point of $U_{\lambda\mu}$:

$$\frac{\varphi_\lambda}{\varphi_\mu} = h_{\lambda\mu} \in \mathcal{O}^*(U_{\lambda\mu});$$

similarly $\psi_\lambda/\psi_\mu = 1/h_{\lambda\mu}$. The pairs $\{U_\lambda, \varphi_\lambda\}$ will form a holomorphic divisor D_1 on Ω. By Theorem 12.72 the divisor problem for D_1 is solvable: there are holomorphic functions $h_\lambda \in \mathcal{O}^*(U_\lambda)$ such that $h_{\lambda\mu} = h_\mu/h_\lambda$ on $U_{\lambda\mu}$ and the formula

$$g \stackrel{\text{def}}{=} \varphi_\lambda h_\lambda \text{ on } U_\lambda, \quad \forall\lambda$$

defines a holomorphic function g on Ω with divisor D_1. Similarly the formula $h = \psi_\lambda/h_\lambda$ on U_λ, $\forall\lambda$ defines a holomorphic function h on Ω with divisor $\{U_\lambda, \psi_\lambda\}$. Finally, $f = g/h$ on every U_λ and hence on Ω, and the functions g and h are relatively prime everywhere on Ω.

12.8 Cousin-II and Chern classes. It is very reasonable to ask which *individual* Cousin-II problems $\{\mathcal{U}, h^1_-\}$ on $\Omega \subset \mathbf{C}^n$ are solvable. For that question we will take a closer look at the map

$$c: \ H^1(\mathcal{U}, \mathcal{O}^*) \xrightarrow{\delta*} H^2(\mathcal{U}, \mathbf{Z}) \xrightarrow{j} H^2(\Omega, \mathbf{Z}). \tag{8a}$$

DEFINITION 12.81. Let $\mathcal{U}$ be an open covering of $\Omega \subset \mathbf{C}^n$. The *Chern class* of a 1-cocycle $h^1_- \in Z^1(\mathcal{U}, \Omega)$ is the element $c(h^1_-)$ in $H^2(\Omega, \mathbf{Z})$ assigned to it by (8a). The *Chern class of a divisor* $D = \{U_\lambda, f_\lambda\}$ is defined as the Chern class of the corresponding cocycle $h^1_- = \{h_{\lambda\mu} = f_\lambda/f_m\}$ on $U_{\lambda\mu}$, see 12.12

$$c(D) = c(h^1_-).$$

REMARK . From (8a) it is clear that the Chern class $c(h^1_-)$ only depends on the cohomology class $[h^1_-]$.

We now compute the Chern map of a 1-cocycle $h^1_- \in Z^1(\mathcal{U}, \Omega)$, that is, we make the computation of δ^* in (7b) explicit. If necessary we refine the covering $\mathcal{U}$ to $\mathcal{V}$ via a refinement map σ in order to make sure that $\sigma(h^1_-) \in C^1_0(\mathcal{V}, \mathcal{O}^*)$. Pulling back $\sigma(h^1_-)_{ij}$ under exp yields a 1-cochain $\log h_{\sigma(i)\sigma(j)} \in C^1(\mathcal{V}, \mathcal{O})$. Applying δ to the result gives a 2-coboundary

$$g_{ijk} = \log h_{jk} - \log h_{ik} + \log h_{ij} \in C^2(\mathcal{V}, \mathcal{O}), \tag{8b}$$

where we have suppressed σ in the indices. Because h^1_- is a (multiplicative) 1-cocycle, that is $h_{\lambda\mu}h_{\mu\nu}h_{\nu\lambda} = 1$, (8b) gives that $g_{ijk}/2\pi i$ is in fact $\mathbf{Z}$ valued. As the Snake Lemma shows, $g_{ijk}/2\pi i$ is a 2-cocycle in $Z^2(\mathcal{U}, \mathbf{Z})$. Thus $c(h^1_-)_{ijk} = g_{ijk}/2\pi i \in H^2(\Omega, \mathbf{Z})$.

We summarize:

Theorem 12.82. *Suppose $\Omega \subset \mathbf{C}^n$ is a Cousin-I domain, $H^1(\Omega, \mathcal{O}) = 0$. A Cousin-II problem $\{\mathcal{U}, h^1_-\}$ on Ω is solvable if and only if the Chern class $c(h^1_-)$ is zero. A divisor D on Ω is principal if and only if its Chern class $c(D)$ equals zero.*
Suppose moreover that $H^2(\Omega, \mathcal{O}) = 0$. Then the Chern map is an isomorphism,

$$H^1(\Omega, \mathcal{O}) \cong H^2(\Omega, \mathbf{Z}).$$

PROOF. The first part rephrases what we have seen before, the last part follows from the long exact sequence (7b).

Exercises

12.1 Let Ω be a domain in $\mathbf{C}$, $\{a_\lambda\}$ a family of isolated points in Ω and $\{m_\lambda\}$ any corresponding family of positive integers. Construct a continuous function on Ω which for each Λ is equal to $(z - a_\lambda)^{m_\lambda}$ on a suitable disc $\Delta(a_\lambda, \rho_\lambda)$ and which is equal to 1 outside $\cup_\lambda \Delta(a_\lambda, 2\rho_\lambda)$.

12.2 (Continuation) Prove that there is a holomorphic function f on Ω which vanishes of precise order m_λ in a_λ, $\forall \lambda$ but which has no other zeros on Ω.

12.3 What does an arbitrary divisor on $\Omega \subset \mathbf{C}$ look like? Split it into a positive and a negative part. Prove that every divisor problem on $\Omega \subset \mathbf{C}$ is solvable.

12.4 Prove Proposition (12.13) on the equivalence of the divisor problem and the corresponding Cousin-II problem.

12.5 Is every Cousin-II problem on a domain $\Omega \subset \mathbf{C}$ solvable?

12.6 Let Ω be a Cousin-II domain in $\mathbf{C}^n$, M an $(n-1)$-dimensional complex submanifold of Ω. Prove that there is a global holomorphic defining function f for M, that is,

$$M = \{z \in \Omega : \; f(z) = 0\},$$

while f is nowhere divisible by the square of a non-unit. [By the last condition, every holomorphic function on a neighborhood U of $a \in \Omega$ which vanishes on $M \cap U$ must equal a multiple of f around a.]

12.7 (Continuation). Let h be a holomorphic function on M. Prove that there is a holomorphic function g on Ω such that $g \mid M = h$. [If $\frac{\partial f}{\partial z_n}\overline{v}0$ at $a \in M$, then M is locally given by $z_n = \varphi(z')$ and $h(z', \varphi(z'))$ will be holomorphic on a neighborhood of a', hence one may interpret h as a holomorphic function on a neighborhood of a which is independent of z_n. Now look at the proof of Theorem (7.21), but divide by f instead of z_n.]

12.8 Calculate $H^s(\mathcal{U}, \mathcal{F})$ for the trivial covering $\mathcal{U} = \{\Omega\}$ of Ω.

12.9 Prove that Ω is a Cousin-I domain if and only if $H^1(\Omega, \mathcal{O}) = 0$. Compute $H^1(\Omega, \mathcal{O})$ for $\Omega = \mathbf{C}^3 - \{0\}$.

12.10 Check that refinement commutes with the coboundary operator.

12.11 Prove that an exact sequence of sheaves induces exact sequences of (chain) groups (6e,6e')

12.12 Let $\Omega \subset \mathbf{C}^n$ be a simply connected domain in the usual sense. Prove that $H^1(\Omega, \mathbf{Z}) = 0$.

12.13 Compute $H^1(A, \mathbf{Z})$ for the annulus $A = \{z \in \mathbf{C} : \ 1 < |z| < 2\}$.

12.14 Let Ω and Ω' in $\mathbf{C}^n$ be biholomorphically equivalent (or at least homeomorphic). Prove that $H^2(\Omega', \mathbf{Z}) = 0$ if and only if $H^2(\Omega, \mathbf{Z}) = 0$. Can you prove, more generally, that $H^p(\Omega', \mathbf{Z}) \cong H^p(\Omega, \mathbf{Z})$?

12.15 Show that all convex domains in $\mathbf{C}^n$ are Cousin-II domains.

12.16 Let Ω be a domain in $\mathbf{C}^2$. Prove that Ω is a Cousin-II domain

(i) if $\Omega = D_1 \times D_2$ where $D_1 \subset \mathbf{C}$ and $D_2 \subset \mathbf{C}$ are simply connected;

(ii) if $\Omega = D_1 \times D_2$ where $D_2 \subset \mathbf{C}$ is simply connected.

12.17 Give an example of an exact sequence of sheaves such that for some covering $\mathcal{U}$ of Ω and some s, $C^s(\mathcal{U}, \mathcal{G}) \overline{\nu} C_0^s(\mathcal{U}, \mathcal{G})$. (Cp. (6e))

12.18 (**Sheaf of divisors**) The quotient sheaf $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$ of germs of invertible meromorphic functions modulo invertible holomorphic functions over the points of Ω is called the *sheaf of divisors* of Ω.

(i) Show that a divisor $D = \{U_\lambda, f_\lambda\}$ belonging to a covering $\mathcal{U}$ of Ω is a global section of $\mathcal{D}$ over Ω;

(ii) Show that the divisor problem for given D may be formulated as follows: Is there a section of f of $\mathcal{M}^*$ over Ω which is mapped onto the given section D under the quotient map $q : \ \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}^*$?

(iii) Show that the following sequence of sheaves over Ω is exact:

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{q} \mathcal{D} \longrightarrow 0.$$

(iv) Show that the divisor problem for D is solvable if and only if $D \subset \operatorname{kernel} \varphi$ where φ is the map $\Gamma(\Omega, \mathcal{D}) \longrightarrow H^1(\Omega, \mathcal{O}^*)$ in the long exact cohomology sequence generated by the sequence in (iii).

12.19 Let Ω be the domain $\mathbf{C}^n - \{0\}$, $n \geq 3$. Show that $H^1(\Omega, \mathcal{O}) = 0$. Next show $H^2(\Omega, \mathbf{Z}) = 0$. Conclude that the Poincaré problem for Ω is solvable and observe that the proof of Theorem 5.73 is completed.

12.20 Let $\Omega = \mathbf{C}^n - \{z : \ z_1 = z_2 = \cdots = z_k = 0\}$. Prove that if $k \leq n - 2$, then $H^1(\Omega, \mathcal{O}) = 0$.

12.21 **De Rham cohomology** Dolbeault cohomology is modeled on the (easier) De Rham cohomology: Consider a domain $\Omega \subset \mathbf{R}^n$, and its sheaf of germs of s-forms $\wedge^s$.

(i) Define a linear operator d from $\mathcal{C}^\infty$ to Λ^1 by

$$df = \sum_1^n \frac{\partial f}{\partial x_j} dx_j$$

and from Λ^s to Λ^{s+1} by

$$d(f(x)\,dx_{i_1} \wedge \ldots \wedge dx_{i_s}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_s}$$

and linearity. Show that $d^2 = 0$.

(ii) A p-form u is called closed if $du = 0$, exact if it is of the form dv. Prove that on a ball every closed p-form is exact. Conclude that

$$0 \longrightarrow \mathbf{C} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \ldots$$

is an exact sequence of sheaves.

(iii) Introduce de Rham cohomology groups $H^p_d(\Omega)$ as closed p-forms modulo exact p-forms. Copy the proof of the Dolbeault Theorem to show that

$$H^p(\Omega, \mathbf{C}) \cong H^p_d(\Omega).$$