MONOTONICITY FOR COMPLETE GRAPHS AND SYMMETRIC COMPLETE BIPARTITE GRAPHS

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ABSTRACT. Given a graph G, let f_k be the number of forests of cardinality k in G. Then the sequence (f_k) has been conjectured to be unimodal for any graph G. In this paper we confirm this conjecture for K_n and $K_{n,n}$ by showing that the sequence for K_n is strictly increasing (when $n \geq 4$) and the sequence for $K_{n,n}$ is strictly increasing except for the very last term. As a corollary we also confirm the conjecture for the complete graphs with multiple edges allowed.

1. INTRODUCTION

Let G be a finite graph with N vertices and let $0 \leq k \leq N-1$. A *spanning forest of cardinality* k in G is a subgraph F with $V(F) = V(G)$ such that each component of F is a tree and the number of edges in F is k. Let f_k(G) (or simply f_k) denote the number of spanning forests of cardinality k in G and we define the f*-sequence* of G, denoted by f_G, to be the sequence $(f_0, f_1, \ldots, f_{N-1})$. In matroid theoretic terms this is the sequence of independent set numbers of the cycle matroid of G. (Refer to [4] for definitions from matroid theory.)

It has been conjectured that f_G for any finite graph G is unimodal. Refer to [3] and [5] for this conjecture and a matroid theoretic generalization of this. From [1], one can deduce that f_G for any graph G with at most 9 vertices will be unimodal. In [2] it was shown that f_G for any planar graph G is unimodal.

In this paper we establish the unimodality of f_G when G is the complete graph K_n or the symmetric complete bipartite graph $K_{n,n}$ as follows. Define f_G to be *(monotone) increasing* and *strictly increasing* if $f_k \leq f_{k+1}$ and $f_k < f_{k+1}$, respectively, for all $0 \leq k \leq N-2$. We will show that f_{K_n} is monotone increasing for $n \geq 1$ and strictly increasing for $n \geq 4$, and that $f_{K_{n,n}}$ is strictly increasing except for the very last term for $n \geq 1$. In particular we will have shown that f_{K_n} and $f_{K_{n,n}}$ are unimodal. As a corollary we will also show via deletion-contraction recursions for f-sequences that allowing multiple edges in K_n will preserve the monotonicity of the f-sequence. Now we will fix some notations and terminology that will be used throughout the paper.

Notations and terminology.
1. For $0 \leq k \leq N-1$, $\mathcal{F}_k(G)$, or $\mathcal{F}_k$ when G is understood, will denote the set of all spanning forests of cardinality k in G. Hence $f_k = f_k(G) = |\mathcal{F}_k(G)|$.
2. For $0 \leq k \leq N-2$, $B_k(G)$ will denote the bipartite graph with the bipartition $\mathcal{F}_k \cup \mathcal{F}_{k+1}$ where $F \in \mathcal{F}_k$ and $F' \in \mathcal{F}_{k+1}$ are adjacent if and only if F is a subgraph of F'.

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3. For each $F \in \mathcal{F}_k$, $\delta(F)$ will denote the degree of F in $B_k(G)$. Then $\delta(F)$ equals the number of edges e in G such that $F \cup e \in \mathcal{F}_{k+1}$. Note that for every $F' \in \mathcal{F}_{k+1}$ the degree of F' in $B_k(G)$ is always $k+1$. It follows that

$$(k+1)f_{k+1} = \sum_{F \in \mathcal{F}_k} \delta(F)\,.$$

2. Monotonicity for complete graphs

Let K_n $(n \geq 1)$ be the complete graph with the vertex set $[n] = \{1, 2, \ldots, n\}$. We see that f_{K_n} is monotone increasing for $n = 1, 2$, and 3 from $f_{K_1} = (1)$, $f_{K_2} = (1, 1)$, and $f_{K_3} = (1, 3, 3)$. The main result of this section will show that f_{K_n} is strictly increasing for $n \geq 4$. Therefore we will have shown that the f_{K_n} is unimodal for all $n \geq 1$. In this section we assume $\mathcal{F}_k = \mathcal{F}_k(K_n)$ for $0 \leq k \leq n-1$. Also for $F \in \mathcal{F}_k$, $\delta(F)$ will be the degree of F in $B_k(K_n)$.

Lemma 1. *Let $0 \leq k \leq n-2$. Then $\delta(F) \geq n-1$ for all $F \in \mathcal{F}_k$.*

Proof. Given $F \in \mathcal{F}_k$, let C be a component of F and let $c = |V(C)|$. Since $k \leq n-2$, F has at least two components and we must have $1 \leq c \leq n-1$. Let e be an arbitrary edge between a vertex in $V(C)$ and a vertex $V(K_n) \setminus V(C)$. Then clearly we have $F \cup e \in \mathcal{F}_{k+1}$ and F is adjacent to $F \cup e$ in $B_k(K_n)$. Therefore $\delta(F)$ is at least $c(n-c)$. Now we note that $c(n-c) \geq (n-1)$ for $1 \leq c \leq n-1$ and the lemma follows. □

Examples. $f_{K_4} = (1, 6, 15, 16)$, $f_{K_5} = (1, 10, 45, 110, 125)$, and $f_{K_6} = (1, 15, 105, 435, 1080, 1296)$.

Theorem 2. *The sequence f_{K_n} is monotone increasing for $1 \leq n \leq 3$ and strictly increasing for $n \geq 4$. In particular, f_{K_n} is unimodal for all $n \geq 1$.*

Proof. We have already checked the cases for $1 \leq n \leq 3$. So assume $n \geq 4$. Clearly we have $1 = f_0 < f_1 = \binom{n}{2}$. So assume $1 \leq k \leq n-2$. Since we have $(k+1)f_{k+1} = \sum_F \delta(F)$, where the sum is over all $F \in \mathcal{F}_k$, it suffices to show $\sum_F \delta(F) > (n-1)f_k$ to prove the theorem. However, we have $\sum_F \delta(F) \geq (n-1)f_k$ by Lemma 1. To obtain strict inequality, we will show that there is some $F \in \mathcal{F}_k$ with $\delta(F) > n-1$. Since $k \leq n-2$, every $F \in \mathcal{F}_k$ has at least two components. Now let $F \in \mathcal{F}_k$ be any forest in which one of the components has exactly two vertices. Then since $n \geq 4$, we have $\delta(F) \geq 2(n-2) > n-1$ by a similar argument as in the proof of Lemma 1. □

Now that the monotonicity of f_{K_n} $(n \geq 1)$ is proved by counting methods, it is natural to ask if there is a combinatorial proof for this. We will answer this question affirmatively by constructing injective mappings $L : \mathcal{F}_k \to \mathcal{F}_{k+1}$ for $0 \leq k \leq n-2$. Given $F \in \mathcal{F}_k$, we define $L(F) \in \mathcal{F}_{k+1}$ as follows (see Figures 1 and 2 below):
Case 1. Suppose the vertices 1 and 2 do not belong to the same component of F. Then define $L(F) = F \cup e$ where e is the edge $\{1, 2\}$. Clearly we have $L(F) \in \mathcal{F}_{k+1}$.
Case 2. If the vertices 1 and 2 belong to the same component C of F, then there will be a unique path P in C from 1 to 2. Let $e_1 = \{1, v\}$ be the unique edge in P that is incident to 1. Since $k \leq n-2$, F has at least two components and $[n] \setminus V(C) \neq \emptyset$. Now let v' be the minimum vertex in $[n] \setminus V(C)$. Then define $L(F) = (F - e_1) \cup (e' \cup e'')$ where $e' = \{1, v'\}$ and $e'' = \{v', v\}$. In other words we *lift the first edge* e_1 to $e' \cup e''$ via the vertex v'. In this case we also have $L(F) \in \mathcal{F}_{k+1}$

because deleting e_1 divides C into two components and adjoining e' and e'' simply connects each of these component to another component of F.

Note that every $F' \in \mathcal{F}_{k+1}$ which is in the image $L(\mathcal{F}_k)$ has vertices 1 and 2 in the same component. Now we check that L is injective. Indeed if $F' = L(F)$ for some $F \in \mathcal{F}_k$, then one can recover the unique preimage F of F' by deleting the edge e in case 1 and that in case 2 by *unlifting the first two edges* of the unique path from 1 to 2 in $L(F)$, i.e. deleting e' and e'' from $L(F)$ and adjoining e_1 back. □

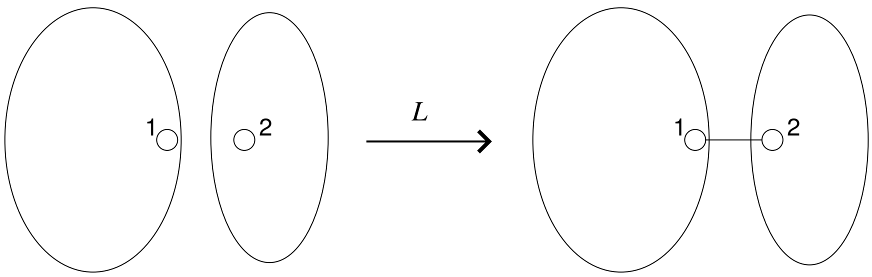

FIGURE 1. Mapping L: case 1 – vertices 1 and 2 are in distinct components

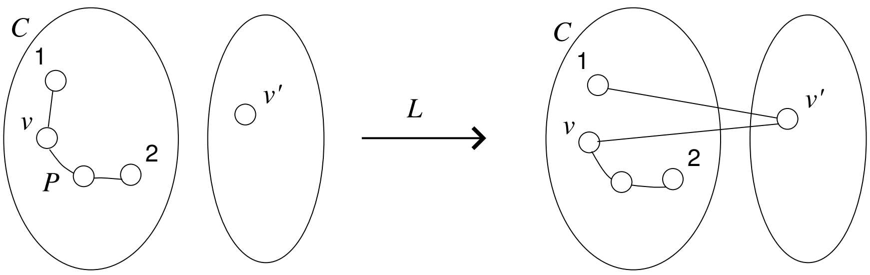

FIGURE 2. Mapping L: case 2 – vertices 1 and 2 are in the same component

We now proceed to extend the result of Theorem 2 to the following classes of graphs. Let $\mathcal{K}_n$ $(n \geq 1)$ denote the set of all finite graphs G with the vertex set $[n]$ that are obtained from K_n by allowing multiple edges, but no loops.

Corollary 3. *Let $G \in \mathcal{K}_n$ $(n \geq 1)$. Then f_G is increasing, hence unimodal.*

Proof. The proof is by double induction on n and the number of edges in G. The result is clear for $n = 1$. Let $G \in \mathcal{K}_n$ for $n > 1$, and assume f_H is increasing for any $H \in \mathcal{K}_r$ $(1 \leq r < n)$. Suppose $|E(G)| = \binom{n}{2}$. Then $G = K_n$ and f_G is increasing by Theorem 2. Now suppose $|E(G)| > \binom{n}{2}$ and let e be an edge in $G \setminus K_n$. Let $G - e$ be the graph obtained by deleting the edge e and G/e the graph obtained by contracting e. Then since e is neither an isthmus nor a loop, we have the following deletion-contraction recursions

$$f_i(G) = f_i(G - e) + f_{i-1}(G/e)$$

for all $0 \le i \le n-1$, where $f_{-1}(G) = 0$ for any graph G. However, we have $G/e \in \mathcal{K}_{n-1}$ and by the induction hypothesis on n, $f_{G/e} = (f_0(G/e), \ldots, f_{n-2}(G/e))$ is increasing. Moreover, by the induction hypothesis on the number of edges, $f_{G-e} = (f_0(G-e), \ldots, f_{n-1}(G-e))$ is also increasing. Therefore f_G, being the "sum" of two increasing sequences, is also increasing and the proof is complete. □

It is also worth noting the monotonicity in the case of rooted forests in K_n. A rooted forest in K_n with t components $(1 \le t \le n)$ is a pair $R = (F, \mathbf{v})$ where the *support* F of R is a forest in $\mathcal{F}_{n-t}$ and the *roots* $\mathbf{v}$ is the set of t vertices, exactly one vertex from each component of F. We denote the set of all such rooted forests by $\mathcal{R}_t$ and $r_t := |\mathcal{R}_t|$. Note that r_t is also the number of spanning trees in K_{n+1} in which the degree of the vertex $n+1$ is t. From this one can show that r_t is the t-th term in the following binomial expansion for the number of spanning trees in K_{n+1}: $(n+1)^{n-1} = \sum_{t=1}^{n} \binom{n-1}{t-1} n^{n-t}$. For example, $r_1 = n^{n-1}$ and $r_n = 1$. It follows that for $n \ge 1$ the sequence of the number of rooted forests in K_n, which we will denote by r_{K_n}, is strictly monotone decreasing: $r_1 > r_2 > \cdots > r_n$.

3. Unimodality for $K_{n,n}$

The main result of this section will show that the sequence $f_0, f_1, \ldots f_{2n-2}$, for $K_{n,n}$ (i.e., the f-sequence of $K_{n,n}$ except the very last term) is strictly increasing. We remark that $f_{2n-2} < f_{2n-1}$ is not true. In fact, we believe $f_{2n-2} \ge f_{2n-1}$ in general. For example, $K_{2,2}$ is the cycle of length 4, and $f_{K_{2,2}} = (1,4,6,4)$. In any case we will see that $f_{K_{n,n}}$ is unimodal for $n \ge 1$. In this section we assume $\mathcal{F}_k = \mathcal{F}_k(K_{n,n})$ for $0 \le k \le 2n-1$. Also for every $F \in \mathcal{F}_k$, $\delta(F)$ denotes the degree of F in $B_k(K_{n,n})$.

Lemma 4. *Let $n \ge 3$ and $0 \le k \le 2n-3$. Then $\delta(F) \ge 2n-2$ for every $F \in \mathcal{F}_k$.*

Proof. Suppose the two partite sets of $K_{n,n}$ are A and B with $|A| = |B| = n$. Let $F \in \mathcal{F}_k$ and let $C_1, C_2, \ldots, C_t$ be the components of F. For all $i \in [t]$, let $A_i = C_i \cap A$ and $B_i = C_i \cap B$, and let $a_i = |A_i|$ and $b_i = |B_i|$. Assume without loss of generality that for $s_i = a_i + b_i$ we have that $1 \le s_1 \le s_2 \le s_3 \le \cdots \le s_t$. Clearly we have $s_1 + \cdots + s_t = 2n$. Furthermore, since $k \le 2n-3$ implies $t \ge 3$, it follows that $s_1, s_2 < n$. Recall that $\delta(F)$ is the number of edges $e \in E(K_{n,n})$ such that $F \cup e \in \mathcal{F}_{k+1}$. Now let $d(F)$ be the number of edges $e = \{v, w\}$ such that $F \cup e \in \mathcal{F}_{k+1}$ and at least one of v and w belongs to $V(C_1) \cup V(C_2)$. Clearly we have $\delta(F) \ge d(F)$ and one can check that

$$d(F) = b_1(n-a_1) + a_1(n-b_1) + b_2(n-a_2) + a_2(n-b_2) - a_1b_2 - b_1a_2\,.$$

We will prove the lemma by showing that $d(F) \ge 2n-2$.

Case 1: $s_1 = s_2 = 1$.

Using $s_i = a_i + b_i$ $(i = 1, 2)$, one can rewrite $d(F) = (s_1 + s_2)n - 2(a_1b_1 + a_2b_2) - (a_1b_2 + a_2b_1)$. Since $s_1 = 1$, we have either $a_1 = 0$ or $b_1 = 0$, hence $a_1b_1 = 0$. Similarly, $s_2 = 1$ implies $a_2b_2 = 0$. Moreover it is easy to check that $a_1b_2 + a_2b_1$ is at most 1 in this case. Therefore $d(F) \ge 2n-1$.

Case 2: $2 \le s_2 \le n-1$.

First we will rewrite $d(F) = D_1 + D_2$, where $D_1 = b_1(n-a_1) + a_1(n-b_1) - a_1b_2 - b_1a_2$ and $D_2 = b_2(n-a_2) + a_2(n-b_2)$. Now, we have $D_1 = b_1(n-(a_1+a_2)) + a_1(n-(b_1+b_2))$, which is easily seen to be non-negative. Therefore, it suffices to show that $D_2 \ge 2n-2$. However we have $D_2 = s_2n - 2a_2b_2 \ge s_2n - s_2^2/2$, where the last inequality follows because the maximum of a_2b_2 is obtained when $a_2 = b_2 = s_2/2$.

Furthermore one checks easily that when $n \geq 3$, $s_2 n - s_2^2/2$ is monotone increasing for $2 \leq s_2 \leq n-1$, hence its minimum is obtained when $s_2 = 2$. Now it follows that $D_2 \geq 2n-2$. □

Theorem 5. *Let $n \geq 2$ and let f_k denote the number of forests with k edges in $K_{n,n}$ for $0 \leq k \leq 2n-1$. Then $f_k < f_{k+1}$ for all $0 \leq k \leq 2n-3$. In particular the sequence $f_{K_{n,n}} = (f_0, f_1, \ldots, f_{2n-1})$ is unimodal.*

Proof: We have already seen the case for $n = 2$. So assume $n \geq 3$. Since we have $(k+1)f_{k+1} = \sum_F \delta(F)$ for $0 \leq k \leq 2n-3$, where the sum is over all $F \in \mathcal{F}_k$, it suffices to show $\sum_F \delta(F) > (2n-2)f_k$. However, we have $\sum_F \delta(F) \geq (2n-2)f_k$ by Lemma 4. Moreover we saw in Case 1 of Lemma 4 that there is a forest $F \in \mathcal{F}_k$ with $\delta(F) \geq d(F) \geq 2n-1$. Therefore we have the strict inequality $\sum_F \delta(F) > (2n-2)f_k$, which completes the proof. □

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