3. The Hahn-Banach separation theorem

Klaus Thomsen matkt@imf.au.dk

Institut for Matematiske Fag
Det Naturvidenskabelige Fakultet
Aarhus Universitet

September 2005

We read in W. Rudin: Functional Analysis
Covering Chapter 3, Theorem 3.4 and its proof. The presentation is based on 'Noter og kommentarer til Rudin'

Theorem

Let X be a topological vector space, and A, B convex non-empty subsets of X. Assume that $A \cap B = \emptyset$.

a) If A is open there is continuous linear functional $\Lambda \in X^*$ and a $\gamma \in \mathbb{R}$ such that

$$\operatorname{Re} \Lambda(a) < \gamma \leq \operatorname{Re} \Lambda(b)$$

for all $a \in A$ and all $b \in B$.

b) If A is compact, B is closed and X is locally convex there is continuous linear functional $\Lambda \in X^*$ and a $\gamma \in \mathbb{R}$ such that

$$\operatorname{Re} \Lambda(a) < \gamma < \operatorname{Re} \Lambda(b)$$

for all $a \in A$ and all $b \in B$.

Proof of the Hahn-Banach separation theorem

Assume first that X is a real vector space.
Let $a_0 \in A$, $b_0 \in B$, and set $x_0 = b_0 - a_0$.
Then $x_0 \neq 0$ since $A \cap B = \emptyset$.
Set $C = A - B + x_0$. Then $0 = a_0 - b_0 + x_0 \in C$ and C is open since A is.
Note that $x_0 \notin C$ since $A \cap B = \emptyset$.
Let $x \in X$. Since C is an open neighborhood of C there is a $t > 0$ such that $t^{-1}x \in C$.
We define $p : X \to [0, \infty[$ such that

$$p(x) = \inf \left\{ t > 0 : \ t^{-1}x \in C \right\}.$$

We want first to prove that $p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$ when $x, y \in X$ and $t \geq 0$.

Proof of the Hahn-Banach separation theorem

Let $a_1, a_2 \in A, b_1, b_2 \in B$, and $t \in [0,1]$. Then

$$\begin{aligned} & t\,(a_1 - b_1 + x_0) + (1-t)\,(a_2 - b_2 + x_0) \\ & = ta_1 + (1-t)a_2 - (tb_1 + (1-t)b_2) + x_0 \in C \end{aligned}$$

proving that C is convex.
It follows that

$$s \geq t > 0,\; t^{-1}x \in C \;\Rightarrow\; s^{-1}x \in C \tag{1}$$

since $s^{-1}x = s^{-1}tt^{-1}x + \left(1 - s^{-1}t\right)0 \in C$.
Let $x, y \in X$ and $a > p(x), b > p(y)$. Then (1) implies that $a^{-1}x, b^{-1}y \in C$. Hence

$$(a+b)^{-1}(x+y) = \frac{a}{a+b}a^{-1}x + \frac{b}{a+b}b^{-1}x \in C.$$

Thus $a + b \geq p(x + y)$, and we conclude that in fact $p(x) + p(y) \geq p(x + y)$.
Let $t > 0$. If $a > p(x)$ we know that $(at)^{-1}tx = a^{-1}x \in C$ and hence $at \geq p(tx)$. It follows that $tp(x) \geq p(tx)$.
But then $p(x) = p\left(t^{-1}tx\right) \leq t^{-1}p(tx) \ \Rightarrow \ tp(x) \leq p(tx)$. It follows that

$$p(tx) = tp(x), \ t \geq 0.$$

In short: p has the properties required in the first version of the Hahn-Banach extension theorem.
Set $M = \mathbb{R}x_0$, and note that we can define a linear map $f : M \to \mathbb{R}$ such that

$$f(tx_0) = t.$$

Since we want to apply the Hahn-Banach extension theorem we must show that $f(x) \leq p(x)$ for all $x \in M$.

Since $x_0 \notin C$, it follows from (1) that $t^{-1}x_0 \notin C$ when $t \in]0,1[$. Thus $p(x_0) \geq 1$ and $f(tx_0) = t \leq tp(x_0) = p(tx_0)$ when $t \geq 0$. When $t < 0$ we find that $f(tx_0) = t < 0 \leq p(tx_0)$, proving want we wanted.

Theorem 3.2 gives us now a linear functional $\Lambda : X \to \mathbb{R}$ such that $\Lambda(y) = f(y), y \in M$, and $\Lambda(x) \leq p(x)$ for all $x \in X$.

We claim now that Λ is continuous. To justify this we must take an open neighborhood W of 0 in $\mathbb{R}$, and find an open neighborhood U of 0 in X such that $\Lambda(U) \subseteq W$.

When $z \in C$, $p(z) \leq 1$, so $\Lambda(z) \leq p(z) \leq 1$. It follows that $\Lambda(-z) = -\Lambda(z) \geq -1$.

Hence $-1 \leq \Lambda(z) \leq 1$ when $z \in C \cap (-C)$.

Since W is an open neighborhood of 0 there is a $\delta > 0$ such that $[-\delta, \delta] \subseteq W$.

Set $U = \delta\left(C \cap (-C)\right)$. Then U is an open neighborhood of 0 in X and we conclude that Λ is continuous since $\Lambda(U) = \delta\Lambda\left(C \cap (-C)\right) \subseteq [-\delta, \delta] \subseteq W$.

Proof of the Hahn-Banach separation theorem

It remains now only to show that Λ separates A and B in the prescribed way.
Let $a \in A, b \in B$. Then $a - b + x_0 \in C$. Since C is open there is an $\epsilon > 0$ such that $(1+\epsilon)(a - b + x_0) \in C$.
It follows that $p((1+\epsilon)(a-b+x_0)) \leq 1$ or
$p((a-b+x_0)) \leq \frac{1}{1+\epsilon}$.
Hence $\Lambda(a) - \Lambda(b) + 1 = \Lambda(a - b + x_0) \leq p((a-b+x_0)) \leq \frac{1}{1+\epsilon}$
and hence
$\Lambda(a) - \Lambda(b) \leq \frac{1}{1+\epsilon} - 1 < 0$.
Set $\gamma = \sup\{\Lambda(a) : a \in A\} \in \mathbb{R}$, and note that

$$\Lambda(a) \leq \gamma \leq \Lambda(b)$$

for all $a \in A, \ b \in B$.
To finish the proof we must show that there is no $a \in A$ with $\Lambda(a) = \gamma$.

Assume therefore that $\gamma = \Lambda(a')$ for some $a' \in A$.
Since $\Lambda \neq 0$ there is a vector $z \in X$ with $\Lambda(z) \neq 0$.
Since A is open there is an $\kappa > 0$ such that $a' + sz \in A$ for all $s \in [-\kappa, \kappa]$.
Then $\gamma \geq \Lambda(a' + sz) = \gamma + s\Lambda(z)$ for all $s \in [-\kappa, \kappa]$, which is absurd.
The proof of a) is complete in the real case.

Proof of the Hahn-Banach separation theorem

Assume now that X is a complex vector space. It follows from the real case that there is a continuous linear *real-valued* functional $l : X \to \mathbb{R}$ and a $\gamma \in \mathbb{R}$ such that

$$l(a) < \gamma \leq l(b)$$

when $a \in A, b \in B$.
Define $\Lambda : X \to \mathbb{C}$ such that

$$\Lambda(x) = l(x) - il(ix).$$

Then $\Lambda \in X^*$ (!!), and $l = \operatorname{Re} \Lambda$.
This completes the proof of a)

Proof of the Hahn-Banach separation theorem - b)

b): By Theorem 1.10 there is a neighborhood V of 0 such that

$$(A + V) \cap B = \emptyset.$$

Since X is locally convex in case b) we may assume that V is convex.
Then $A + V$ is convex, and open.
It follows from a) that there is $\Lambda \in X^*$ and $\gamma_2 \in \mathbb{R}$ such that

$$\operatorname{Re} \Lambda(a) < \gamma_2 \leq \operatorname{Re} \Lambda(b)$$

for all $a \in A, b \in B$.
Since A is compact and $\operatorname{Re} \Lambda$ is continuous,
$\alpha = \sup \{\operatorname{Re} \Lambda(a) : \ a \in A\} < \gamma_2$. (See the following lemma.)
Then $\gamma_1 = \frac{1}{2}(\gamma_2 + \alpha)$ works!

A lemma

Lemma

Let Z be a non-empty compact Hausdorff space and $g : Z \to \mathbb{R}$ a continuous function. There is then an element $z \in Z$ such that

$$g(x) \leq g(z)$$

for all $x \in Z$.

Proof.

Let A be the collection of the real numbers α for which

$$W_\alpha = \{x \in Z : g(x) \geq \alpha\}$$

is not empty. $\square$

Proof.

Note that each W_α is closed in Z and hence compact (!!). Since $\mathbb{R}$ is totally ordered, the intersection $\bigcap_{\alpha \in F} W_\alpha$ is not empty when $F \subseteq A$ is a finite subset. It follows (!!) that $\bigcap_{\alpha \in A} W_\alpha \neq \emptyset$. Let $z \in \bigcap_{\alpha \in A} W_\alpha$. Then $g(x) \leq g(z)$ for all $x \in Z$ because $z \in W_{g(x)}$. $\square$