Suggested Solutions to Midterm 2005

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1 Sargent model and stochastic processes

Consider a simplified version of the Sargent model described by the folllowing equations:

$$\begin{aligned} y_t &= \gamma[p_t - E_{t-1}(p_t)] + u_t \qquad (\textit{Lucas AS curve}) \\ m_t &= p_t \qquad\qquad\qquad\qquad\quad (\textit{LM curve}) \end{aligned}$$

where u_t is of the form:

$$u_t = \alpha u_{t-1} + \epsilon_t$$

and ϵ_t is a White Noise term $\epsilon_t \sim WN(0, \sigma^2)$.

(a) Characterize the (linear) money supply rules that will minimize the variance of output y_t in this economy. (Assume that agents have rational expectations and that the monetary authority does not have any knowledge unavailable to the public.)

Answer:

Since the Lucas AS supply implies that only unexpected changes in the money supply will affect the variance of output, any monetary rule that is a function of variables known at time $t-1$ will minimize the variance of output. That is, the monetary policy that minimizes $Var[y_t]$ is that with no random term.
Note that if the monetary rule has a random term η_t, then:

$$y_t = \gamma[p_t - E_{t-1}(p_t)] + u_t = \gamma[m_t - E_{t-1}(m_t)] + u_t = \gamma\eta_t + u_t$$

and thus

$$Var[y_t] = Var[\gamma\eta_t + u_t] = \gamma^2\sigma_\eta^2 + 2\gamma cov(\eta_t, u_t) + \sigma_u^2.$$

Since the monetary authority has no information on ϵ_t, it is not possible to make $cov(\eta_t, u_t)$ negative and thus the monetary policy that is most stabilizing is that with no random term, i.e. $\eta_t = 0$. Therefore, the money supply rules that will minimize the variance of output are of the form

$$m_t = \sum_i \phi_i x_{i,t-1}$$

where $x_{i,t-1}$ is anything known at time $t-1$.

(b) What will be the minimum variance of output with the monetary rules in your answer to part (a)?

Answer:

The minimum variance of output is

$$Var[y_t] = Var[u_t] = \frac{\sigma^2}{1-\alpha^2}$$

2 Mankiw model (and Shapiro-Stiglitz)

Consider the Mankiw model. A monopolist faces a linear demand. Its marginal costs and fixed costs are both zero.

The demand for the monopolist's product is:

$$D = 10 - p$$

(a) Suppose that there is an increase in demand so that

$$D' = 14 - p$$

Let Z be the "menu cost" of changing the price. What condition must hold for the monopolist to re-optimize? Will the monopolist re-optimize if $Z = 6$?

Answer:

When $D = 10 - p$, the monopolist solves:

$$max(10-p)p \quad \Rightarrow \quad p^* = 5$$

When $D = 14 - p$, if the monopolist re-optimizes she solves:

$$max(14-p)p \quad \Rightarrow \quad p^*_m = 7, \quad q^*_m = 14 - 7 = 7, \quad \pi^*_m = 7 \times 7 = 49$$

Note that if the monopolist does not re-optimize, then

$$p^*_n = 5, \quad q^*_n = 14 - 5 = 9, \quad \pi^*_n = 5 \times 9 = 45$$

The monopolist will change her price in response to the increase in D if

$$\pi^*_m - \pi^*_n > Z \Leftrightarrow 49 - 45 > Z \Leftrightarrow 4 > Z$$

Thus, if $Z = 6$, the monopolist does not change her price.

(b) Now consider a dynamic setting for this model. Suppose that the increase in demand in part (a) is permanent and that the interest rate is $r = 100\%$. Using dynamic programming, as in the Shapiro-Stiglitz model with discrete periods, calculate the monopolist's lifetime value if she re-optimizes, V_m, and her lifetime value if she does not reoptimize, V_n. [Suppose that the monopolist will live forever.]

Given that the menu cost is paid only once (when prices are changed) but the monopolist cares about her lifetime profits, what new condition can you define for the monopolist to re-optimize? If $Z = 6$, will the monopolist change her price? Explain your answer.

Answer:

$$\begin{aligned} V_m &= \pi^*_m + \frac{1}{1+r}V_m = 49 + \frac{1}{2}V_m \quad \Rightarrow \quad V_m = 98 \\ V_n &= \pi^*_n + \frac{1}{1+r}V_n = 45 + \frac{1}{2}V_n \quad \Rightarrow \quad V_n = 90 \end{aligned}$$

The monopolist will change her price if

$$V_m - V_n > Z \Leftrightarrow 98 - 90 > Z \Leftrightarrow 8 > Z$$

Thus, if $Z = 6$, the monopolist will change her price. In this dynamic setting, the monopolist finds it optimal to re-optimize because by doing so she can perceive higher profits in every period while the menu cost is spread over time and thus becomes less significant.

3 Consumption

Consider a consumer who lives for T periods and maximizes

$$U = E_t\left[\sum_{j=0}^{T-t}\left(\frac{1}{1+\delta}\right)^j u(c_{t+j})\right]$$

subject to

$$\sum_{j=0}^{T-t}\left(\frac{1}{1+r}\right)^j c_{t+j} = \sum_{j=0}^{T-t}\left(\frac{1}{1+r}\right)^j y_{t+j}.$$

(a) Write down the Euler equation for maximization of expected utility.

Answer: *The Euler equation is*

$$u'(c_t) = \frac{1+r}{1+\delta}E_t[u'(c_{t+1})]$$

or, more generally,

$$u'(c_t) = \left(\frac{1+r}{1+\delta}\right)^j E_t[u'(c_{t+j})]$$

(b) Name sufficient conditions on the utility function and the parameters that will yield the result that consumption will follow an *exact* random walk. We define c_t as following an exact random walk if:

$$c_t = c_{t-1} + \epsilon_t$$

where $E_{t-1}[\epsilon_t|\Theta_{t-1}] = 0$, and Θ_{t-1} is the information available at time $t-1$.

Answer: *Sufficient conditions are:*

- $\delta = r$ *(so that consumers want a flat consumption path)*
- *the utility function is quadratic (so that marginal utility is linear)*

(c) Prove that in this case consumption will follow an *exact* random walk. (Use the precise definition of random walk given in part (b)).

Answer: *Using the Euler equation and the two conditions defined in part (b):*

$$u'(c_{t-1}) = \frac{1+r}{1+\delta}E_{t-1}[u'(c_t)] \Rightarrow u'(c_{t-1}) = E_{t-1}[u'(c_t)] \Rightarrow u'(c_{t-1}) = u'(E_{t-1}[c_t]) \;\Rightarrow\; c_{t-1} = E_{t-1}[c_t]$$

Multiplying by (-1) *and then adding* c_t *to both sides:*

$$c_t - c_{t-1} = c_t - E_{t-1}[c_t] \;\Rightarrow\; c_t = c_{t-1} + (c_t - E_{t-1}[c_t]) \Rightarrow\; c_t = c_{t-1} + \epsilon_t$$

where $E_{t-1}[\epsilon_t|\Theta_{t-1}] = E_{t-1}[c_t - E_{t-1}[c_t]\;|\Theta_{t-1}] = 0$.