Proof Theory: Part III
Kripke-Platek Set Theory

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Gentzen's result

Gerhard Gentzen showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \mathsf{sup}\{\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots\} \;=\; \mathsf{least}\ \alpha.\ \omega^{\alpha} = \alpha$$

suffices to prove the **consistency** of **Peano Arithmetic**, **PA**.

How natural ordinal representation systems arise

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} \quad = \quad \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle \tag{1}$$

where α is an ordinal, $<_\alpha$ is the ordering of ordinals restricted to elements of α and the f_i are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \longrightarrow \alpha$$

for some natural number k_i.

$$\mathbb{A} \quad = \quad \langle A, g_1, \ldots, g_n, \prec \rangle \tag{2}$$

is a recursive representation of $\mathfrak{A}$ if the following conditions hold:

1. $A \subseteq \mathbb{N}$
2. A is a recursive set.
3. $\prec$ is a recursive total ordering on A.
4. The functions g_i are recursive.
5. $\mathfrak{A} \cong \mathbb{A}$, i.e. the two structures are isomorphic.

Gentzen's ordinal representation system for ε_0 is based on the Cantor normal form, i.e. for any ordinal $0 < \alpha < \varepsilon_0$ there exist uniquely determined ordinals $\alpha_1, \ldots, \alpha_n < \alpha$ such that

- $\alpha_1 \geq \cdots \geq \alpha_n$
- $\alpha = \omega^{\alpha_1} + \cdots \omega^{\alpha_n}$.

To indicate the Cantor normal form we write

$$\alpha =_{CNF} \omega^{\alpha_1} + \cdots \omega^{\alpha_n}.$$

Now define a function

$$\lceil\ .\ \rceil : \varepsilon_0 \longrightarrow \mathbb{N}$$

by

$$\lceil\delta\rceil = \begin{cases} 0 & \text{if } \delta = 0 \\ \langle\lceil\delta_1\rceil, \ldots, \lceil\delta_n\rceil\rangle & \text{if } \delta =_{CNF} \omega^{\delta_1} + \cdots \omega^{\delta_n} \end{cases}$$

where $\langle k_1, \cdots, k_n\rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with p_i being the ith prime number (or any other coding of tuples). Further define

$$\begin{array}{rcl} A_0 & := & \mathbf{ran}(\lceil\ .\ \rceil) \\ \lceil\delta\rceil \prec \lceil\beta\rceil & :\Leftrightarrow & \delta < \beta \\ \lceil\delta\rceil \mathbin{\hat{+}} \lceil\beta\rceil & := & \lceil\delta + \beta\rceil \\ \lceil\delta\rceil \mathbin{\hat{\cdot}} \lceil\beta\rceil & := & \lceil\delta \cdot \beta\rceil \\ \hat{\omega}^{\lceil\delta\rceil} & := & \lceil\omega^\delta\rceil. \end{array}$$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \quad \cong \quad \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec \rangle.$$

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec$ are recursive, in point of fact, they are all elementary recursive.

The **axioms** of **KP** are:

Extensionality: $a = b \rightarrow [F(a) \leftrightarrow F(b)]$ for all formulas F.

Foundation: $\exists x G(x) \rightarrow \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$

Pair: $\exists x\,(x = \{a, b\})$.

Union: $\exists x\,(x = \bigcup a)$.

Infinity: $\exists x\,\left[x \neq \emptyset \;\wedge\; (\forall y \in x)(\exists z \in x)(y \in z)\right]$.

Δ_0 Separation: $\exists x\,\big(x = \{y \in a : F(y)\}\big)$ for all Δ_0–formulas F in which x does not occur free.

Δ_0 Collection: $(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for all Δ_0–formulas G.

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

An ordinal representation system for the Bachmann-Howard ordinal

The **Veblen-function** φ figures prominently in elementary proof theory.

It is defined by transfinite recursion on α by letting $\varphi_0(\xi) := \omega^\xi$ and, for $\alpha > 0$, φ_α be the function that enumerates the class of ordinals

$$\{\gamma : \forall \xi < \alpha \, [\varphi_\xi(\gamma) = \gamma]\}.$$

We shall write $\varphi\alpha\beta$ instead of $\varphi_\alpha(\beta)$.

Let Γ_α be the α^{th} ordinal $\rho > 0$ such that for all $\beta, \gamma < \rho$, $\varphi\beta\gamma < \rho$.

Corollary

1. $\varphi 0 \beta = \omega^\beta$.
2. $\xi, \eta < \varphi\alpha\beta \Longrightarrow \xi + \eta < \varphi\alpha\beta$.
3. $\xi < \zeta \Longrightarrow \varphi\alpha\xi < \varphi\alpha\zeta$.
4. $\alpha < \beta \Longrightarrow \varphi\alpha(\varphi\beta\xi) = \varphi\beta\xi$.

The least ordinal (> 0) closed under the function φ is known as

$$\Gamma_0$$

The proof-theoretic ordinal of **KP**, however, is bigger than Γ_0 and we need another function to obtain a sufficiently large ordinal representation system.

Let Ω be a "big" ordinal. By recursion on α we define sets $C^{\Omega}(\alpha, \beta)$ and the ordinal $\psi_{\Omega}(\alpha)$ as follows:

$$C^{\Omega}(\alpha, \beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \Omega\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \longmapsto \psi_{\Omega}(\xi))_{\xi < \alpha} \end{cases} \tag{3}$$

$$\psi_{\Omega}(\alpha) \simeq \min\{\rho < \Omega : \; C^{\Omega}(\alpha, \rho) \cap \Omega = \rho\}. \tag{4}$$

Note that if $\psi_\Omega(\alpha)$ is defined, then

$$\psi_\Omega(\alpha) < \Omega$$

and

$$[\psi_\Omega(\alpha), \Omega) \cap C^\Omega(\alpha, \psi_\Omega(\alpha)) = \emptyset$$

thus the order-type of the ordinals below Ω which belong to the Skolem hull $C^\Omega(\alpha, \psi_\Omega(\alpha))$ is $\psi_\Omega(\alpha)$.

In more pictorial terms, $\psi_\Omega(\alpha)$ is the α^{th} collapse of Ω.

Lemma $\psi_\Omega(\alpha)$ is always defined; in particular $\psi_\Omega(\alpha) < \Omega$.

Proof: The claim is actually not a definitive statement as I haven't yet said what largeness properties Ω has to satisfy. In the proof below, we assume $\Omega := \aleph_1$, i.e. Ω is the first uncountable cardinal.

Observe first that for a limit ordinal λ,

$$C^{\Omega}(\alpha, \lambda) \;=\; \bigcup_{\xi<\lambda} C^{\Omega}(\alpha, \xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^{\Omega}(\alpha, \lambda)$.

Now define

$$\begin{aligned} \eta_0 &= \sup C^{\Omega}(\alpha, 0) \cap \Omega \\ \eta_{n+1} &= \sup C^{\Omega}(\alpha, \eta_n) \cap \Omega \\ \eta^* &= \sup_{n<\omega} \eta_n. \end{aligned} \tag{5}$$

Since for $\eta < \Omega$ the cardinality of $C^{\Omega}(\alpha, \eta)$ is the same as that of $\max(\eta, \omega)$ and therefore less than Ω, the regularity of Ω implies that $\eta_0 < \Omega$. By repetition of this argument one obtains $\eta_n < \Omega$, and consequently $\eta^* < \Omega$. The definition of η^* then ensures

$$C^{\Omega}(\alpha, \eta^*) \cap \Omega \;=\; \bigcup_n C^{\Omega}(\alpha, \eta_n) \cap \Omega \;=\; \eta^* < \Omega.$$

Therefore, $\psi_{\Omega}(\alpha) < \Omega$. $\square$

Let

$$\varepsilon_{\Omega+1}$$

be the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$.

The next definition singles out a subset

$$\mathcal{T}(\Omega)$$

of

$$C^\Omega(\varepsilon_{\Omega+1}, 0)$$

which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system

$$\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$$

so that

$$\langle \mathcal{T}(\Omega), <, \Re, \psi, \ldots \rangle \;\cong\; \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{6}$$

"..." is supposed to indicate that more structure carries over to the ordinal representation system.

Definition $\mathcal{T}(\Omega)$ is defined inductively as follows:

1. $0, \Omega \in \mathcal{T}(\Omega)$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\Omega)$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\Omega)$.
3. If $\alpha \in \mathcal{T}(\Omega)$ and $\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha))$, then $\psi_{\Omega}(\alpha) \in \mathcal{T}(\Omega)$.

The side condition in the second clause is easily explained by the desire to have unique representations in $\mathcal{T}(\Omega)$.

The requirement

$$\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha))$$

in the third clause also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi_{\Omega}(\alpha)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by

$$\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha)).$$

It can be shown that the foregoing definition of $\mathcal{T}(\Omega)$ is deterministic, that is to say every ordinal in $\mathcal{T}(\Omega)$ is generated by the inductive clauses in exactly one way. As a result, every

$$\gamma \in \mathcal{T}(\Omega)$$

has a unique representation in terms of symbols for

$$0, \Omega$$

and function symbols for

$$+, \alpha \mapsto \omega^{\alpha},\ \alpha \mapsto \psi_{\Omega}(\alpha).$$

The unique representation of will be referred to as the normal form.

Thus, by taking some primitive recursive (injective) coding function $\lceil \cdots \rceil$ on finite sequences of natural numbers, we can code $\mathcal{T}(\Omega)$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \lceil 0, 0 \rceil & \text{if } \alpha = 0 \\ \lceil 1, 0 \rceil & \text{if } \alpha = \Omega \\ \lceil 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \rceil & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \lceil 3, \ell(\beta), \ell(\Omega) \rceil & \text{if } \alpha = \psi_\Omega(\beta), \end{cases}$$

where the distinction by cases refers to the unique representation of ordinals in $\mathcal{T}(\Omega)$. With the aid of ℓ, the ordinal representation system (6) can be defined by letting $\mathcal{OR}$ be the image of ℓ and setting

$$\lhd \; := \; \{(\ell(\gamma), \ell(\delta)) : \gamma < \delta \wedge \delta, \gamma \in \mathcal{T}(\Omega)\}$$

etc. However, a proof that this definition of

$$\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$$

in point of fact furnishes an elementary ordinal representation system is a bit lengthy.

We have seen that in the case of **PA** the addition of an infinitary rule enables us to regain cut elimination.

ω–rule:

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x\, A(x)}.$$

An ordinal analysis for **PA** is then attained as follows:

- Each **PA**–proof can be "unfolded" into a $\mathbf{PA}_\omega$–proof of the same sequent.
- Each such $\mathbf{PA}_\omega$–proof can be transformed into a cut–free $\mathbf{PA}_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like **KP**, we have to work a bit harder. Guided by the ordinal analysis of **PA**, we would like to invent an infinitary rule which, when added to **KP**, enables us to eliminate cuts.

As opposed to the natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe. Here we will use Gödel's constructible universe L. The constructible universe is "made" from the ordinals. It is pretty obvious how to "name" sets in L once we have names for ordinals at our disposal.

Recall that L_α, the αth level of **Gödel's constructible hierarchy** L, is defined by

$$\begin{aligned} L_0 &= \emptyset, \\ L_\lambda &= \bigcup\{L_\beta : \beta < \lambda\} \ \lambda \text{ limit} \\ L_{\beta+1} &= \big\{X : \ X \subseteq L_\beta;\ X \text{ definable over } \ \langle L_\beta, \in\rangle\big\}. \end{aligned}$$

So any element of L of level α is definable from elements of L with levels $< \alpha$ and the parameter L_{α_0} if $\alpha = \alpha_0 + 1$.

The problem of “naming” sets will be solved by erecting a formal constructible hierarchy using the ordinals from $\mathcal{T}(\Omega)$.

Henceforth, we shall restrict ourselves to ordinals from $\mathcal{T}(\Omega)$.

Definition We adopt a language of set theory, $\mathcal{L}$, which has only the predicate symbol $\in$.
The atomic formulae of $\mathcal{L}$ are those of either form $(a \in b)$ or $\neg(a \in b)$.
The $\mathcal{L}$–formulae are obtained from atomic ones by closing off under $\wedge, \vee, (\exists x \in a), (\forall x \in a), \exists x$, and $\forall x$.

Definition The RS_Ω–terms and their levels are generated as follows.

1. For each $\alpha < \Omega$,

$$\mathbb{L}_\alpha$$

 is an RS_Ω–term of level α.

2. The formal expression

$$[x \in \mathbb{L}_\alpha : F(x, \vec{s})^{\mathbb{L}_\alpha}]$$

 is an RS_Ω–term of level α if $F(a, \vec{b})$ is an $\mathcal{L}$–formula (whose free variables are among the indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are RS_Ω–terms with levels $< \alpha$.

 $F(x, \vec{s})^{\mathbb{L}_\alpha}$ results from $F(x, \vec{s})$ by restricting all unbounded quantifiers to $\mathbb{L}_\alpha$.

We shall denote the level of an RS_Ω–term t by $|\, t\,|$;

$t \in \mathcal{T}(\alpha)$ stands for $|\, t\,| < \alpha$ and $t \in \mathcal{T}$ for $t \in \mathcal{T}(\Omega)$.

The RS_Ω–formulae are the expressions of the form

$$F(\vec{s})$$

where $F(\vec{a})$ is an $\mathcal{L}$–formula and $\vec{s} \equiv s_1, \ldots, s_n \in \mathcal{T}$.

For technical convenience, we let $\neg A$ be the formula which arises from A by

(i) putting $\neg$ in front of each atomic formula,

(ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a)$, respectively, and

(iii) dropping double negations.

We use the relation $\equiv$ to mean syntactical identity. For terms s, t with $|\,s\,| < |\,t\,|$ we set

$$s \mathring{\in} t \equiv \begin{cases} B(s) & \text{if} \quad t \equiv [x \in \mathbb{L}_\beta : B(x)] \\ \mathsf{True}_s & \text{if} \quad t \equiv \mathbb{L}_\beta \end{cases}$$

where True_s is a true formula, say $s \notin \mathbb{L}_0$.

Observe that $s \in t$ and $s \mathring{\in} t$ have the same truth value under the standard interpretation in the constructible hierarchy.

The rules of $\mathcal{L}_{RS}$

Having created names for a segment of the constructible universe, we can introduce infinitary rules analogous to the the ω-rule.
Let

$$A, B, C, \ldots, F(t), G(t), \ldots$$

range over RS_Ω–formulae. We denote by upper case Greek letters

$$\Gamma, \Delta, \Lambda, \ldots$$

finite sets of RS_Ω–formulae. The intended meaning of

$$\Gamma = \{A_1, \cdots, A_n\}$$

is the disjunction

$$A_1 \vee \cdots \vee A_n$$

Γ, A stands for $\Gamma \cup \{A\}$ etc.. We also use the abbreviations $r \neq s := \neg(r = s)$ and $r \notin t := \neg(r \in t)$.

The rules of RS_Ω are:

$$(\wedge) \quad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(b\forall) \quad \frac{\cdots \Gamma, s \mathring{\in} t \rightarrow F(s) \cdots (s \in \mathcal{T}(|\, t\,|))}{\Gamma, (\forall x \in t) F(x)}$$

$$(b\exists) \quad \frac{\Gamma, s \mathring{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in \mathcal{T}(|\, t\,|)$$

$$(\forall) \quad \frac{\cdots \Gamma,\ F(s) \cdots (s \in \mathcal{T})}{\Gamma, \forall x\, F(x)}$$

$$(\exists) \quad \frac{\Gamma,\ F(s)}{\Gamma, \exists x\, F(x)} \quad \text{if } s \in \mathcal{T}$$

$$(\notin) \qquad \frac{\cdots \Gamma, s \mathring{\in} t \rightarrow r \neq s \cdots\cdots (s \in \mathcal{T}(|\, t\,|))}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \mathring{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \in \mathcal{T}(|\, t\,|)$$

$$(\mathrm{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$$(\mathrm{Ref}_\Sigma) \qquad \frac{\Gamma, A}{\Gamma, \exists z\, A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

where a formula is said to be in Σ if all its unbounded quantifiers are existential.

A^z results from A by restricting all unbounded quantifiers to z.

$\mathcal{H}$–controlled derivations

If we dropped the rule (Ref_Σ) from RS_Ω, the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules

$$\begin{array}{cc} (\wedge) & (\vee) \\ (\forall) & (\exists) \\ (\notin) & (\in) \end{array}$$

However, partial cut elimination for RS_Ω can be attained by delimiting a collection of derivations of a very uniform kind. Fortunately, Buchholz has provided us with a very elegant and flexible setting for describing uniformity in infinitary proofs, called operator controlled derivations.

Definition Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called operator if $\mathcal{H}$ is a closure operator, i.e monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(ON)$:

1. $0 \in \mathcal{H}(X)$.
2. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then $\alpha \in \mathcal{H}(X) \iff \alpha_1, ..., \alpha_n \in \mathcal{H}(X)$.

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components.

For $Z \in P(ON)$, the operator $\mathcal{H}[Z]$ is defined by

$$\mathcal{H}[Z](X) := \mathcal{H}(Z \cup X).$$

If $\mathfrak{X}$ consists of "syntactic material", i.e. terms, formulae, and possibly elements from $\{0, 1\}$, then let

$$\mathcal{H}[\mathfrak{X}](X) := \mathcal{H}(k(\mathfrak{X}) \cup X)$$

where $k(\mathfrak{X})$ is the set of ordinals needed to build this "material".

Finally, if s is a term, then define $\mathcal{H}[s]$ by $\mathcal{H}[\{s\}]$.

To facilitate the definition of $\mathcal{H}$–controlled derivations, we assign to each RS_Ω–formula A, either a (possibly infinite) disjunction $\bigvee(A_\iota)_{\iota\in I}$ or a conjunction $\bigwedge(A_\iota)_{\iota\in I}$ of RS_Ω–formulae.

This assignment will be indicated by $A \cong \bigvee(A_\iota)_{\iota\in I}$ and $A \cong \bigwedge(A_\iota)_{\iota\in I}$, respectively.

Define:

$$
\begin{array}{rcl}
r\in t & \cong & \bigvee(s\mathring{\in}t \wedge r = s)_{s\in\mathcal{T}_{|t|}} \\
(\exists x\in t)F(x) & \cong & \bigvee(s\mathring{\in}t \wedge F(s))_{s\in\mathcal{T}_{|t|}} \\
\exists x\, F(x) & \cong & \bigvee(F(s))_{s\in\mathcal{T}} \\
A_0 \vee A_1 & \cong & \bigvee(A_\iota)_{\iota\in\{0,1\}} \\
\neg A & \cong & \bigwedge(\neg A_\iota)_{\iota\in I},\ \text{if } A \cong \bigvee(A_\iota)_{\iota\in I}.
\end{array}
$$

Using this representation of formulae, we can define the subformulae of a formula as follows. When $A \cong \bigwedge(A_\iota)_{\iota \in I}$ or $A \cong \bigvee(A_\iota)_{\iota \in I}$, then B is a subformula of A if $B \equiv A$ or, for some $\iota \in I$, B is a subformula of A_ι.

Since one also wants to keep track of the complexity of cuts appearing in derivations, each formula F gets assigned an ordinal rank $rk(F)$ which is roughly the sup of the level of terms in F plus a finite number.

Using the formula representation, in spite of the many rules of RS_Ω, the notion of $\mathcal{H}$–controlled derivability can be defined concisely. We shall use $I{\restriction}\alpha$ to denote the set $\{\iota \in I : |\iota| < \alpha\}$.

Definition Let $\mathcal{H}$ be an operator and let Γ be a finite set of RS_Ω–formulae.

$$\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma$$

is defined by recursion on α. It is always demanded that

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset).$$

The inductive clauses are:

$$(\bigvee) \qquad \frac{\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Lambda, A_{\iota_0}}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Lambda, \bigvee(A_\iota)_{\iota\in I}} \qquad \begin{array}{r} \alpha_0 < \alpha \\ \iota_0 \in I \restriction \alpha \end{array}$$

$$(\bigwedge) \qquad \frac{\mathcal{H}[\iota] \mid\!\frac{\alpha_\iota}{\rho}\, \Lambda, A_\iota \text{ for all } \iota\in I}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Lambda, \bigwedge(A_\iota)_{\iota\in I}} \qquad |\,\iota\,| \leq \alpha_\iota < \alpha$$

$$(\mathrm{Cut}) \qquad \frac{\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Lambda, B \qquad \mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Lambda, \neg B}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Lambda} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(Ref_\Sigma) \qquad \frac{\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Lambda, A}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Lambda, \exists z\, A^z} \qquad \begin{array}{r} \alpha_0, \Omega < \alpha \\ A \in \Sigma \end{array}$$

The specification of the operators needed for an ordinal analysis will, of course, hinge upon the particular theory and ordinal representation system.

To connect **KP** with the infinitary system RS_Ω one has to show that **KP** can be embedded into RS_Ω. Indeed, the finite **KP**-derivations give rise to very uniform infinitary derivations.

Theorem 1 If

$$\mathbf{KP} \vdash B(a_1, \ldots, a_r)$$

then

$$\mathcal{H} \; \Big|\frac{\Omega \cdot m}{\Omega + n} \; B(s_1, \ldots, s_r)$$

holds for some m, n and all set terms $s_1, \ldots, s_r$ and operators $\mathcal{H}$ satisfying

$$\{\xi \, : \, \xi \text{ occurs in } B(\vec{s})\} \cup \{\Omega\} \subseteq \mathcal{H}(\emptyset).$$

m and n depend only on the **KP**-derivation of $B(\vec{a})$.

The usual cut elimination procedure works as long as the cut formulae have not been introduced by an inference Ref_{Σ}. As the principal formula of an inference Ref_{Σ} has rank Ω one gets the following result.

Theorem 2 (Cut elimination I)

$$\mathcal{H} \;\Big|\!\frac{\alpha}{\;\Omega+n+1\;}\, \Gamma \quad \Rightarrow \quad \mathcal{H} \;\Big|\!\frac{\omega_n(\alpha)}{\;\Omega+1\;}\, \Gamma$$

where $\omega_0(\beta) := \beta$ and $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

The obstacle to pushing cut elimination further is exemplified by the following scenario:

$$\dfrac{\dfrac{\mathcal{H} \,\big|\!\frac{\delta}{\Omega}\, \Gamma, A}{\mathcal{H} \,\big|\!\frac{\xi}{\Omega}\, \Gamma, \exists z\, A^z}(\mathrm{Ref}_{\Sigma}) \qquad \dfrac{\cdots \mathcal{H}[s] \,\big|\!\frac{\xi_s}{\Omega}\, \Gamma, \neg A^s \cdots (s \in \mathcal{T})}{\mathcal{H} \,\big|\!\frac{\xi}{\Omega}\, \Gamma, \forall z\, \neg A^z}(\forall)}{\mathcal{H} \,\big|\!\frac{\alpha}{\Omega+1}\, \Gamma}(Cut)$$

Fortunately, it is possible to eliminate cuts in the above situation provided that the side formulae Γ are of complexity Σ. The technique is known as “collapsing” of derivations.

In the course of "collapsing" one makes use of a simple bounding principle.

Lemma. (Boundedness) Let A be a Σ-formula, $\alpha \leq \beta < \Omega$, and $\beta \in \mathcal{H}(\emptyset)$. If

$$\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A$$

then

$$\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A^{\mathbb{L}_\beta}\ .$$

If the length of a derivation of Σ-formulae is $\geq \Omega$, then "collapsing" results in a shorter derivation, however, at the cost of a much more complicated controlling operator.

Theorem 3. (Collapsing Theorem) Let Γ be a set of Σ-formulae. Then we have

$$\mathcal{H}_\eta \ \Big|\frac{\alpha}{\Omega+1}\, \Gamma \quad \Rightarrow \quad \mathcal{H}_{f(\eta,\alpha)} \ \Big|\frac{\psi_\Omega(f(\eta,\alpha))}{\psi_\Omega(f(\eta,\alpha))}\, \Gamma\,,$$

where $\left(\mathcal{H}_\xi\right)_{\xi \in \mathcal{T}(\Omega)}$ is a uniform sequence of ever stronger operators.

From the Bounding Lemma it follows that all instances of Ref_Σ can be removed from derivations of length $< \Omega$.

For derivations without instances of Ref_{Σ} there is a well-known cut-elimination procedure, the so-called predicative cut-elimination. Below this is stated in precise terms.

It should also be mentioned that the φ function can be defined in terms of the functions of $\mathcal{T}(\Omega)$ and that $\varphi\alpha\beta < \Omega$ holds whenever $\alpha, \beta < \Omega$.

Theorem 4. (Predicative cut elimination)

$$\mathcal{H} \mathrel{\vert\!\!\frac{\;\delta\;}{\;\rho\;}} \Gamma \text{ and } \delta, \rho < \Omega \quad \Rightarrow \quad \mathcal{H} \mathrel{\vert\!\!\frac{\;\varphi\rho\delta\;}{\;0\;}} \Gamma\,.$$

The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the Bachmann-Howard ordinal. Combining the previous results of this section, one obtains:

Corollary: If A is a Σ-formula and

$$\mathbf{KP} \vdash A$$

then

$$L_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A.$$

The bound of this Corollary is sharp, that is, $\psi_\Omega(\varepsilon_{\Omega+1})$ is the first ordinal with that property.

Below we list further results that follow from the ordinal analysis of **KP**.

Corollary:

(i) $|\mathbf{KP}| = |\mathbf{KP}|_{\mathsf{sup}} = |\mathbf{KP}|_{\Pi_2} = |\mathbf{KP}|^{E}_{\Pi_2} = \psi_\Omega(\varepsilon_{\Omega+1})$.

(ii) $\mathbf{sp}_{\Sigma_1}(\mathbf{KP}) = \psi_\Omega(\varepsilon_{\Omega+1})$.