We will give an abbreviated discussion of modes in step-index fiber in order to quickly determine the conditions for a single propagating spatial mode.

For details, refer to the following two references, which are on four-hour reserve at the Engineering Library.

- K. Iizuka, *Elements of Photonics, Volume II: For Fiber and Integrated Optics*, Wiley, 2002, Section 11.2. This treatment derives modes without assuming weak guidance, and obtains conventional modes, from which LP modes are formed.


**Electromagnetism background and notation**

**Macroscopic Maxwell’s equations** (in the absence of charge and current)

\[
\nabla \times E = \frac{\partial B}{\partial t} \quad \nabla \times H = \frac{\partial D}{\partial t}
\]

\[
\nabla \cdot D = 0 \quad \nabla \cdot B = 0
\]

**Constitutive relations for nonmagnetic media**

\[
D = \varepsilon_0 E + P
\]

\[
B = \mu_0 H
\]

**Boundary conditions at interfaces between media** (in the absence of charge and current)

- Normal \( D \) and \( B \):

\[
(D_2 - D_1) \cdot \hat{n}_{21} = 0 \quad (B_2 - B_1) \cdot \hat{n}_{21} = 0
\]

- Tangential \( E \) and \( H \):

\[
(E_2 - E_1) \times \hat{n}_{21} = 0 \quad (H_2 - H_1) \times \hat{n}_{21} = 0
\]

**Definition of Fourier transform and inverse Fourier transform**

\[
E(r, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{E}(r, \omega) e^{j\omega t} d\omega
\]

\[
\tilde{E}(r, \omega) = \int_{-\infty}^{\infty} E(r, t) e^{-j\omega t} dt
\]

- We will use similar definitions for all scalar and vector functions of time/frequency.
We will sometimes omit the tilde (−) over the frequency-domain function when confusion is unlikely to arise.

**Dielectric susceptibility relating \( P \) to \( E \) (in a linear, isotropic medium)**

\[
P(r,t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi(r,t,t') E(r,t') dt'
\]

\[
P(r,\omega) = \varepsilon_0 \tilde{\chi}(r,\omega) \cdot \tilde{E}(r,\omega)
\]

**Dielectric constant**

\[
\varepsilon(r,\omega) = \varepsilon_0 \left[ 1 + \tilde{\chi}(r,\omega) \right]
\]

**Refractive index**

\[
\varepsilon(r,\omega) = \left[ n(r,\omega) + i \frac{c}{2\omega} \alpha(r,\omega) \right]^2
\]

- \( n(r,\omega) \) real refractive index
- \( \alpha(r,\omega) \) real absorption coefficient
- \( n(r,\omega) + i \frac{c}{2\omega} \alpha(r,\omega) \) complex refractive index

**Wave equations in step-index optical fibers**

- In computing the modes, we will:
  - Ignore nonlinear effects.
  - Ignore absorption, i.e., let \( \varepsilon(r,\omega) \to n^2(r,\omega) \)
  - Assume the refractive index is constant in space, except at boundaries between media, where boundary conditions are applied.
  - Drop the ~ over \( \tilde{E}(r,\omega) \) and \( \tilde{H}(r,\omega) \).
- Assume a monochromatic wave at frequency \( \omega \).
  Define the free-space wavenumber \( k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda} \).
- From Maxwell’s equations, we can derive the wave equations (see Agrawal Section 2.2):
  \[
  \nabla^2 E(r,\omega) + k_0^2 n^2(\omega) E(r,\omega) = 0
  \]
  \[
  \nabla^2 H(r,\omega) + k_0^2 n^2(\omega) H(r,\omega) = 0
  \]
- Note that although \( E(r,\omega) \) and \( H(r,\omega) \) solve the same wave equation, they are different functions of \( r \).

**Modes in optical fiber**

- There exist both guided modes and radiation modes. We study guided modes only.
• Modes are solutions of the wave equation subject to appropriate boundary conditions. The field distributions \( E(r, \omega) \) and \( H(r, \omega) \) in a mode do not change as the mode propagates along the \( z \) direction, except for an overall multiplicative factor of the form \( e^{i\beta(\omega)z} \).
• \( \beta(\omega) \) is called the \textit{propagation constant} of a mode.
• Different modes propagate with different values of the propagation constant \( \beta(\omega) \), except for degeneracies that may occur. A set of modes having identical (degenerate) \( \beta(\omega) \) forms a \textit{mode group}.
• For each forward-propagating mode \((+\beta)\), there exists a backward-propagating mode \((-\beta)\).
• The set of guided modes form an orthonormal set:
\[
\int_{\text{fiber cross section}} \text{Re}\left( E_i \times H_j^* \right) \cdot dA = \delta_{ij} \frac{\beta}{|\beta|}.
\]
• The radiation modes also form an orthogonal set, and have a similar orthogonality relationship.
• The set of guided and radiation modes form a complete set, which can be used to expand an arbitrary field distribution. Note that the guided modes have a discrete spectrum of \( \beta \), while radiation modes have a continuous spectrum of \( \beta \). Assuming a monochromatic field and suppressing the frequency dependence, we can expand an arbitrary \( E(r) \) as:
\[
E(r) = \sum_{i \text{ guided}} A_i E_i(r) + \int A(\beta) E(r, \beta) d\beta.
\]

**Modes in step-index fiber**

**Approach**
• Initially, we do not assume the \textit{weak guidance condition} \( \Delta << 1 \). Thus we find the \textit{conventional modes}: \( TE, TM, HE, EH \).
• Later, we assume weak guidance \((\Delta << 1)\) and form linear combinations of the conventional modes to obtain \textit{linearly polarized modes} (LP modes).

**Derivation**
• Use cylindrical coordinates \((\rho, \phi, z)\)
• We need to solve the wave equation for six components: \( E_\rho, E_\phi, E_z \) and \( H_\rho, H_\phi, H_z \).
  We will solve the wave equation for the two components: \( E_z \) and \( H_z \), and then use the four Maxwell’s equations to find the remaining four components: \( E_\rho, E_\phi \) and \( H_\rho, H_\phi \).
• The wave equation for \( E_z \) is:
\[
\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + n^2 k_0^2 E_z = 0.
\]
• The refractive index is
\[
n = \begin{cases} n_1 & \rho \leq a \\ n_2 & \rho > a \end{cases}
\]
• We use the technique of separation of variables to convert this partial differential equation into a set of three ordinary differential equations.
  • The solution is written:
    \[ E_z(\rho, \phi, z) = F(\rho)\Phi(\phi)Z(z) \]
  • We define separation constants \( \beta^2 \) and \( m^2 \), which permits us to write:
    \[ \frac{d^2 Z}{dz^2} + \beta^2 Z = 0 \]
    \[ \frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0 \]
    \[ \frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} + \left( n^2 k_0^2 - \frac{m^2}{\rho^2} \right) F = 0 \]
  • The \( z \) and \( \phi \) solutions are of the form:
    \[ Z(z) = e^{i\beta z} \]
    \[ \Phi(\phi) = e^{im\phi} \]
\( \beta \) is the propagation constant, a real number
\( m \) is an integer
• \( F(\rho) \) satisfies Bessel’s equation. Define:
    \[ p^2 = n_1^2 k_0^2 - \beta^2 \]
    \[ q^2 = \beta^2 - n_2^2 k_0^2 \]
(1)
We require \( F(\rho) \) to be finite everywhere and to decay to 0 as \( \rho \to \infty \). Hence:
\[ F(\rho) = \begin{cases} 
  J_m(p\rho) & \rho \leq a \\
  K_m(q\rho) & \rho > a 
\end{cases} \]
\( J_m(\cdot) \): Bessel function of order \( m \) (analogous to sinusoid in cylindrical coordinates)
\( K_m(\cdot) \): modified Bessel function of order \( m \) (analogous to decaying exponential in cyl. coordinates.)
• We write the overall solution for $E_z$ as:

$$E_z(\rho, \phi, z) = \begin{cases} AJ_m(pp)e^{\imath \mu \phi} e^{\imath \beta z} & \rho \leq a \\ CK_m(qp)e^{\imath \mu \phi} e^{\imath \beta z} & \rho > a \end{cases}$$

• Similarly, we write the overall solution for $H_z$ as:

$$H_z(\rho, \phi, z) = \begin{cases} BJ_m(pp)e^{\imath \mu \phi} e^{\imath \beta z} & \rho \leq a \\ DK_m(qp)e^{\imath \mu \phi} e^{\imath \beta z} & \rho > a \end{cases}$$

• Using the four Maxwell’s equations, we obtain the other four components $E_\phi, E_\rho$ and $H_\rho, H_\phi$. In the core, we obtain:

$$E_\rho = \frac{i}{p^2} \left( \beta \frac{\partial E_z}{\partial \rho} + \mu_0 \frac{\omega \partial H_z}{\partial \phi} \right)$$

$$E_\phi = \frac{i}{p^2} \left( \beta \frac{\partial E_z}{\partial \phi} - \mu_0 \omega \frac{\partial H_z}{\partial \rho} \right)$$

$$H_\rho = \frac{i}{p^2} \left( \beta \frac{\partial H_z}{\partial \rho} - \varepsilon_0 n^2 \omega \frac{\partial E_z}{\partial \phi} \right)$$

$$H_\phi = \frac{i}{p^2} \left( \beta \frac{\partial H_z}{\partial \phi} + \varepsilon_0 n^2 \omega \frac{\partial E_z}{\partial \rho} \right)$$

Similar equations can be obtained in the cladding region after replacing $p^2$ by $q^2$.

• We have expressed the fields $E$ and $H$ in the core and cladding in terms of four unknown constants $A$, $B$, $C$, $D$.

• We impose the boundary conditions that the tangential components of $E$ and $H$ be continuous at the core-cladding interface, i.e., $E_z, H_z, E_\phi, H_\phi$ are continuous at $\rho = a$.

• We obtain a set of four homogeneous equations to be satisfied by $A, B, C, D$. Requiring the determinant of the matrix of coefficients to vanish, we obtain the characteristic equation:

$$\begin{vmatrix} J_m'(pa) + K_m'(qa) \frac{K_m'(qa)}{pa J_m(pa)} \frac{K_m'(qa)}{pa J_m(pa)} & J_m'(qa) \frac{K_m'(qa)}{pa J_m(pa)} + n_2^2 \frac{J_m'(pa)}{n_1^2 \frac{pa K_m(pa)}{qa K_m(qa)}} \end{vmatrix} = m^2 \begin{vmatrix} 1 \frac{1}{(pa)^2} + \frac{1}{(qa)^2} \frac{1}{(pa)^2} + \frac{n_2^2}{n_1^2} \frac{1}{(qa)^2} \end{vmatrix} \label{2}$$

• Only values of $pa$ and $qa$ that satisfy the characteristic equation will satisfy the boundary conditions. Since $pa$ and $qa$ are two unknowns, we require one more equation to relate $pa$ and $qa$. We combine the definitions of $p$ and $q$ given in (1) to obtain:

$$v^2 = (pa)^2 + (qa)^2 \label{3},$$

where we have defined

$$v = k_0 a \sqrt{n_1^2 - n_2^2}.$$
• The $V$ number is a parameter describing the fiber, and which governs the number of propagating modes.
• As $V$ increases, the fiber supports more propagating modes. In step-index fiber, as $V \to \infty$, the number of propagating modes per polarization is of the order of $\frac{2V^2}{\pi^2}$.
• $V$ increases for:
  - large $k_0$ (large $\omega$, small $\lambda$)
  - large $\Delta$
  - large $a$

**How to find the modes for a particular fiber**

1. Given $k_0$, $a$, $n_1$, $n_2$, find $V$.

2. For each $m = 0, 1, 2, 3, \ldots$ solve (2) and (3) simultaneously to find the allowed values of $pa$ and $qa$, thus determining the spatial dependence of the fields.

For a given value of $m$, there can exist multiple solutions, which are enumerated as $n = 1, 2, 3, \ldots$

For a given value of $V$, there exist solutions up to some maximum values of $(m, n)$ (propagating modes), beyond which no solutions exist (cutoff modes).

3. For each mode $(m, n)$ found in step 2, substitute $p$ or $q$ in (1) to find the propagation constant $\beta_{mn}$.

**Classification of modes**

- $m = 0$
  - Correspond to meridional rays.
  - $(E_\rho, E_\phi, E_z)$ and $(H_\rho, H_\phi, H_z)$ are independent of $\phi$.
  - The modes are: $TE_{0n}$ ($E_z = 0$)
  - $TM_{0n}$ ($H_z = 0$)

- $m = 1, 2, 3, \ldots$
  - Correspond to skew rays.
  - $(E_\rho, E_\phi, E_z)$ and $(H_\rho, H_\phi, H_z)$ depend on $\phi$.
  - The modes are: $HE_{mn}$ ($H_z$ dominates over $E_z$)
  - $EH_{mn}$ ($E_z$ dominates over $H_z$)

- $m = 0$
  - The right-hand side of (2) vanishes.
  - Note that $J_0( ) = -J_1( )$, $K_0( ) = -K_1( )$.
  - Hence, (2) can be satisfied if either one of two equations is satisfied:
\[
\frac{J_1(pa)}{paJ_0(pa)} + \frac{K_1(qa)}{qaK_0(qa)} = 0 \quad \text{corresponding to } TE_{0n} \text{ modes}
\]
\[
\frac{J_1(pa)}{paJ_0(pa)} + \frac{n_2^2}{n_1^2} \frac{K_1(qa)}{qaK_0(qa)} = 0 \quad \text{corresponding to } TM_{0n} \text{ modes}
\]

- For arbitrary values of \(qa\), the \(TE_{0n}\) and \(TM_{0n}\) solutions yield different curves in the \(pa-qa\) plane.
- The \(TE_{0n}\) and \(TM_{0n}\) solutions have the same asymptotes for \(qa \to 0\), \(qa \to \infty\).

<table>
<thead>
<tr>
<th>Asymptotic Limit</th>
<th>Condition for (TE_{0n}) and (TM_{0n})</th>
<th>Values of (pa) Corresponding to (n = 1, 2, 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(qa \to 0)</td>
<td>(J_0(pa) = 0)</td>
<td>(pa = 2.4, 5.5, 8.7, \ldots)</td>
</tr>
<tr>
<td>(qa \to \infty)</td>
<td>(J_1(pa) = 0)</td>
<td>(pa = 3.8, 7.0, 10.0, \ldots)</td>
</tr>
</tbody>
</table>

- \(m = 1, 2, 3, \ldots\)
  - This case is much more complicated to solve.
  - Assume weak guidance, i.e., \(\Delta \ll 1\) or \(\frac{n_2}{n_1} \approx 1\). Hence, (2) can be rewritten:
    \[
    \frac{J'_m(pa)}{paJ_m(pa)} + \frac{K'_m(qa)}{qaK_m(qa)} = \pm m \left[ \frac{1}{(pa)^2} + \frac{1}{(qa)^2} \right]
    \]
  - This equation has two different sets of solutions:
    \(+m\) corresponding to \(EH_{mn}\) modes
    \(-m\) corresponding to \(HE_{mn}\) modes
  - It can be shown that under the weak guidance condition, the \(HE_{m+1,n}\) and \(EH_{m-1,n}\) modes have identical characteristic equations, and thus identical values of the propagation constant \(\beta\).
  - It can be shown that in the asymptotic limits \(qa \to 0\) and \(qa \to \infty\), the following conditions on \(pa\) can be established.

<table>
<thead>
<tr>
<th>Asymptotic Limit</th>
<th>Condition for (HE_{mn})</th>
<th>Condition for (EH_{mn})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m = 1)</td>
<td>(J_1(pa) = 0) including (pa = 0)</td>
<td>(J_m(pa) = 0) excluding (pa = 0)</td>
</tr>
<tr>
<td>(m \geq 2)</td>
<td>(J_m(pa) = 0) excluding (pa = 0)</td>
<td>(J_m(pa) = 0)</td>
</tr>
<tr>
<td>(qa \to 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(qa \to \infty)</td>
<td>(J_{m-1}(pa) = 0)</td>
<td>(J_{m+1}(pa) = 0)</td>
</tr>
</tbody>
</table>
Graphical solution of (2) and (3) in the pa-qa plane

- A plot of (3) is simply a circle of radius $V$.
- A plot of (2) involves many curves, corresponding to different values of $(m,n)$.

$$(pa)^2 + (qa)^2 = V^2$$

for larger $V$. Four modes propagate.

$$(pa)^2 + (qa)^2 = V^2$$

for small $V$. Only $HE_{11}$ propagates.

- Note that for $V < 2.405$, only the $HE_{11}$ mode can propagate.

Behavior of a mode near cutoff

- Recall that in the cladding:
  $$F(q) = K_m(q) \quad \rho > a.$$  
- One can show that:
  $$K_m(q) \approx \frac{\pi}{2q\rho} \cdot \exp(-q\rho) \quad q\rho >> a.$$  
- As a mode approaches cutoff
  $$q \to 0,$$
  and exponential decay does not occur in the cladding.
Effective index of a mode

- Definition

\[ \tilde{n} = \frac{\beta}{k_0} = c \frac{\beta}{\omega} \]

- For each propagating mode:

\[ n_2 < \tilde{n} < n_1 \]

and as a mode approaches cutoff:

\[ \tilde{n} \to n_2. \]

Normalized propagation constant of a mode

- Definition

\[ b = \frac{\tilde{n} - n_2}{n_1 - n_2} \]

- For each propagating mode:

\[ 0 < b < 1 \]

and as a mode approaches cutoff:

\[ b \to 0. \]

Linearly polarized modes

- LP modes are valid under the weak guidance condition \( \Delta \ll 1 \), which is satisfied by almost all glass fibers.

- Certain sets of conventional modes (TE, TM, HE, EH) have nearly identical \( \beta \) (equivalently, nearly identical \( \tilde{n} \) or \( b \)), and are said to form a (conventional) mode group.

- We can form linear combinations of the conventional modes within a group to form LP modes. A group of LP modes having nearly identical \( \beta \) is said to form an (LP) mode group.
• For each LP mode:

\[ E_z \approx 0, \; H_z \approx 0 \]

• For each LP mode, there are two orthogonal polarization modes that have identical \( \beta \) (ignoring birefringence or polarization-mode dispersion):

\[ E \approx E_x \hat{x}, \; H \approx H_y \hat{y} \]

\[ E \approx E_y \hat{y}, \; H \approx H_x \hat{x} \]

• In each of the two polarization modes

\[ E_x(\rho, \phi), \; H_y(\rho, \phi) \text{ have the same dependence on } (\rho, \phi) \]

\[ E_y(\rho, \phi), \; H_x(\rho, \phi) \text{ have the same dependence on } (\rho, \phi) \]

• An LP mode corresponds to launching light from a linearly polarized source, such as a laser diode.
<table>
<thead>
<tr>
<th>LP Mode Group</th>
<th>Conventional Modes in LP Mode Group</th>
<th>Conventional Modes in LP Mode</th>
<th>Field Lines of LP Mode</th>
<th>Intensity of LP Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LP_{01}$</td>
<td>$HE_{11}$</td>
<td>$HE_{11}$</td>
<td><img src="image" alt="Field Lines" /></td>
<td><img src="image" alt="Intensity" /></td>
</tr>
<tr>
<td>$LP_{11}$</td>
<td>$HE_{21}, TE_{01}$</td>
<td>$HE_{21}, TE_{01}$</td>
<td><img src="image" alt="Field Lines" /></td>
<td><img src="image" alt="Intensity" /></td>
</tr>
<tr>
<td></td>
<td>$HE_{21}, TM_{01}$</td>
<td>$HE_{21}, TM_{01}$</td>
<td><img src="image" alt="Field Lines" /></td>
<td><img src="image" alt="Intensity" /></td>
</tr>
<tr>
<td></td>
<td>$HE_{21}, TM_{01}$</td>
<td>$HE_{21}, TM_{01}$</td>
<td><img src="image" alt="Field Lines" /></td>
<td><img src="image" alt="Intensity" /></td>
</tr>
</tbody>
</table>
Single-mode step-index fiber

- We mainly discuss single-mode step-index fiber hereafter.
- For $V < 2.405$, a step-index fiber supports only one mode: $HE_{11} = LP_{01}$ (in each of two orthogonal polarizations).
- We usually design the fiber so that $V$ is not much smaller than 2.405, so that the mode remains fairly well confined to the core.
- Example: standard telecommunications single-mode fibers
  - $n_1 = 1.45$
  - $\Delta = 3 \times 10^{-3}$
  - $a = 4 \, \mu m$

<table>
<thead>
<tr>
<th>$\lambda$ (nm)</th>
<th>$V$</th>
<th>Approximate Confinement Factor $\Gamma$ of Fundamental Mode</th>
<th>Characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 1175</td>
<td>&gt; 2.405</td>
<td>&gt; 0.84</td>
<td>Multi-mode</td>
</tr>
<tr>
<td>1175</td>
<td>2.405</td>
<td>0.84</td>
<td>Single-mode</td>
</tr>
<tr>
<td>1300</td>
<td>2.171</td>
<td>0.78</td>
<td>Single-mode</td>
</tr>
<tr>
<td>1550</td>
<td>1.821</td>
<td>0.63</td>
<td>Single-mode</td>
</tr>
</tbody>
</table>

Fundamental mode in single-mode step-index fiber

- The $x$- and $y$-polarized $HE_{11} = LP_{01}$ modes have electric fields given respectively by:
  \[
  E_x = E_{0x} \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho \leq a
  \]
  \[
  E_y = E_{0y} \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho > a
  \]
  Note that $E_x$ or $E_y$ is independent of $\phi$.
- Since the $HE_{11}$ mode has $m = 1$, it is a skew mode. In either the $x$ or $y$ polarization, $E_m, E_\phi$ do depend on $\phi$. For the $x$- and $y$-polarized $HE_{11} = LP_{01}$ modes, these components are given respectively by:
  \[
  E_\rho = E_{0x} \cdot \cos(\phi) \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho \leq a
  \]
  \[
  E_\phi = E_{0x} \cdot \sin(\phi) \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho > a
  \]
  \[
  E_\rho = E_{0y} \cdot \sin(\phi) \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho \leq a
  \]
  \[
  E_\phi = E_{0y} \cdot \cos(\phi) \cdot \left(\frac{J_0(pa)}{K_0(qa)}\right) \cdot e^{j\beta z} \quad \rho > a
  \]
- The $z$ component, $E_z$, is much smaller than $E_x$ or $E_y$, and its radial dependence is given in terms of $J_1(pp)$ or $K_1(qp)$. 

Approximate Confinement Factor $\Gamma$ of Fundamental Mode

<table>
<thead>
<tr>
<th>$\lambda$ (nm)</th>
<th>$V$</th>
<th>Characteristic</th>
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<tbody>
<tr>
<td>&lt; 1175</td>
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<td>1175</td>
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</tr>
<tr>
<td>1550</td>
<td>1.821</td>
<td>Single-mode</td>
</tr>
</tbody>
</table>
Gaussian approximation of fundamental mode

- It is common practice to approximate the $HE_{11} = LP_{01}$ mode by a Gaussian function. The $x$- and $y$-polarized modes are approximated respectively as:

$$E_x \approx A_x \cdot \exp \left( -\frac{\rho^2}{w^2} \right) \cdot e^{j\beta z} \quad \forall \rho$$

$$E_y \approx A_y \cdot \exp \left( -\frac{\rho^2}{w^2} \right) \cdot e^{j\beta z} \quad \forall \rho$$

The parameter $w$ is called the spot size or field radius. The figure below shows the best-fit value of $w$ as a function of $V$, and shows how well the Gaussian approximation fits the mode at $V = 2.4$.

Mode confinement factor: $\Gamma$ is the fraction of the energy of a mode confined in the core.

- For example, for an LP mode polarized along the $x$ direction:

$$\Gamma = \frac{P_{\text{core}}}{P_{\text{total}}} = \frac{\frac{a}{0} |E_x|^2 \rho d\rho}{\frac{\infty}{0} |E_x|^2 \rho d\rho}.$$  

- In the Gaussian approximation of the fundamental mode:

$$\Gamma = 1 - \exp \left( -\frac{2a^2}{w^2} \right).$$

Approximate values of the mode confinement factor for standard telecommunications fiber are given in the table on the previous page.