# Book Review

Solomon Feferman, *In the Light of Logic*, "Logic and Computation in Philosophy" series, Oxford University Press, Oxford and New York, xii–340 pp.

In this book, Feferman collects a number of essays and other articles of historical and philosophical nature, which had previously been scattered in a number of different and (at least in a few cases) not widely accessible publications. This is not — properly speaking — a logic book, but it is a book in which logic occurs at every step to cast some light on some subtle distinction or to bring weight to a philosophical or conceptual point.

While some parts of the book are quite technical, the arguments are framed in such a way as to be accessible to non-specialists, and many parts are indeed adequate for a general audience. Accordingly, the book does not offer complete presentations of any of Feferman's numerous technical contributions. For those, the interested reader will have to follow the references to academic journals and scholarly publications given in the book's extensive bibliography.

Rather, with this book Feferman intends to offer a comprehensive view and a spirited defense of his *predicativist* position in the philosophy of mathematics. Predicativism is the view that one should only allow entities that are definable without making any references to totalities of which they themselves are members. Definitions that violate this requirement are called impredicative: for example, the definition of the maximum of a bounded set of reals is impredicative because it involves quantification over the collection of the upper bounds of the given set. Of course, if we assume that there is a totality of mathematical entities given *sub species aeternitatis*, then impredicative definitions turn out to be perfectly justifiable: after all, from this point of view, each mathematical statement has a perfectly determined truth-value, so that definitions involving unbounded quantification over arbitrary entities are hardly a problem. It follows that predicativism goes hand-in-hand with a rejection of the unbridled power-set principle (the idea that whenever a set $A$ is given, so is the collection of all subsets of $A$), as well as a dynamical conception of mathematical reality, according to which mathematical entities are not given once and for all, but can be conceived as the outcome of generative process.

It is worth noting that, contrary to the grand foundational schemes of the 1920's and 1930's (logicism, intuitionism, formalism), predicativism is not a single philosophical position, but rather a family of positions, each relative to a given conceptual framework taken as basic. Predicativists only allow entities that are non-circularly definable, *given the entities of the basic framework*. For instance, a well-known version of predicativism, first proposed by Hermann Weyl (one of the heroes of Feferman's book) regards unbounded quantification over the natural numbers as legitimate, but not quantification over *sets* of natural numbers. As a result, only sets of natural numbers that are arithmetically definable are allowed. One might think that this is a rather Draconian restriction, but perhaps the main point of the book under review is that most if not all scientifically applicable mathematics can be formalized in a predicatively justified framework inspired by Weyl's approach.

The book comprises 14 chapters, divided into five parts: (1) Foundational problems; (2) Foundational ways; (3) Gödel; (4) Proof theory; and (5) Countably reducible mathematics. In addition, the book also contains a list of defined symbols, an extensive bibliography, and an index of names. Since the various chapters were originally published separately, there is a certain amount of overlap. For instance, chapters 4 and 5 ("Foundational ways" and "Working foundations") are a slimmed down and, respectively, expanded version of an article originally published in the journal *Synthése*. The book is quite pleasantly written: Feferman's prose is clear and incisive, never redundant; altogether this makes the book a great read. It is worth mentioning that in over 300 pages, there appear to be at most a handful of typos, the most bewildering of which are a couple of remnants of un-escaped LaTeX code that found their way into the printed version (pp. 36 and 252).

As already mentioned, the book contains a powerful argument in favor of a predicativistic conception of mathematics; the argument is complemented and supported by a number of expository or historical articles, such as the ones found in parts (1) and (3). Similarly, chapters 11 ("Gödel's *Dialectica* Interpretation and its two-way stretch") and 13 ("Weyl vindicated: *Das Kontinuum* seventy years later"), play a somewhat auxiliary role with respect to the main thrust of the book. It seems useful, therefore, to touch upon these parts first, in order later to present an unobstructed account of the main argument.

Part (1), "Foundational problems," presents two expository pieces, the second of which, "Infinity in mathematics: is Cantor necessary?" is a particularly accessible presentation of a number of foundational issues: after giving a brief account of the origins and early development of set theory with Cantor and Zermelo, Feferman points out questions and problems connected to the underlying Platonism, and presents the "solutions" of Brouwer, Hilbert, and Weyl. While Brouwer's proposal is perceived as too restrictive, and Hilbert's is ultimately failing due to Gödel's incompleteness theorems, Feferman's sympathies go to Weyl's attempt to give a predicatively sound account of modern mathematics which is ultimately reducible to quantification over the natural numbers. The chapter's breadth and accessibility, together with Feferman's remarkably clear prose make it especially suitable as introductory reading in a philosophy of mathematics course.

Part (3), "Gödel," offers an interesting account of Gödel's life and work, emphasizing his philosophical views. As editor-in-chief of Gödel's *Collected Works*, Feferman is in a sense in a unique position to draw upon extensive material from Gödel's *Nachlaß*. Two points are of particular philosophical interest here. In "Kurt Gödel: Conviction and Caution," Feferman points out Gödel's reluctance, while in Vienna, to make explicit his realist views in the philosophy of mathematics (the Vienna environment, dominated by the verificationist philosophy of the *Wiener Kreis*, in turn inspired by Hilbert's formalist program, was probably a factor). This reluctance is likely behind the fact that, although Gödel relied on non-constructive notions such as arithmetical truth when deriving the incompleteness theorems, he took pains to purge any reference to truth from his published work. What is even more surprising perhaps, is that, as Feferman points out, Gödel had discovered the undefinability of arithmetical truth already in 1931, at least a couple of years before Tarski's famous theorem to the same effect.

The other unexpected bit of history contained in Feferman's account is a passage in Gödel's invited address to a meeting of the Mathematical Association of America, held in Cambridge in 1933. Of course, Gödel's long-held Platonist sympathies are well-known, and although as we have seen he might have been reluctant in publicizing them while in Vienna, he unabashedly claimed to have held then all along after coming to the United States in the late 1930's (a time when, coincidentally, the Vienna Circle was undergoing its own diaspora and the fortunes of Hilbert's original program had all but faded). All the more surprising, then, is the passage from the 1933 lecture:

> ...our axioms [i.e, the axioms of Zermelo-Fraenkel set theory], if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent.

Based on the evidence available from Gödel's publications and the *Nachlaß*, this passage can be neither explained nor explained away. The historian's only option is to take it at "face value," and attribute it to a momentary period of doubt; this is where, with Feferman, we leave the matter.

Completing the expository part (although clear-cut distinctions are hard to draw), are the two chapters on Gödel's *Dialectica* interpretation and Weyl's *Das Kontinuum*. These two pieces are exemplary in their level of detail both for their clarity and historical rigor; insofar as they fall within the history of mathematical ideas, these pieces contribute to set a very high standard indeed, and especially the Weyl article is a small classic in its own. The two articles are similar in structure: both move from a historical point — an analysis and reconstruction of Gödel's and Weyl's original proposals — and both follow the contemporary developments that those proposals originated. Both contribute to Feferman's main point, that results, methods, and tools originating from the finitist, constructivist, or predicativist tradition represent a viable alternative to Platonistically inspired mathematics, especially set theory.

We now come to the main philosophical argument of the book, developed in parts (2), (4), and (5). In part (2), "Foundational ways," Feferman presents his view of the nature of the foundations of mathematics. After an appraisal of Lakatos' conception of mathematical practice and a comparison to Pólya's more detailed, but less sweeping account (ch. 3), Feferman proceeds to present his own view in the two already mentioned articles "Foundational ways" and "Working foundations" (chs 4 and 5). Shunning grand proposals purporting to provide foundations for a field that, in the view of most working mathematicians, doesn't need any, Feferman identifies a number of foundational activities that, while less general, provide an arena for logic, philosophy, and mathematics to interact in fruitful ways. In this sense, rather than *philosophy of mathematics*, these activities can be regarded as a form of Russell's *mathematical philosophy*, i.e., that kind of philosophical reflection that is *continuous with* mathematics rather than taking mathematics as object of external investigation.

It is worth spending some time identifying some of these foundational ways. The first is *conceptual clarification*. This takes places once a subject has reached a certain level of maturity: examples from different fields are the concept of continuous function, Tarski's definitions of truth and satisfaction, the Turing-Post notion of computable function. To the philosophical eye these look like as many instances of Carnap's idea of *explication*, i.e., the sharpening of a pre-existing but imprecise concept. In this respect, Feferman notices that an informal concept still in need of an explication is the one of *identity of proofs*: in spite of there being a well-developed proof theory (to which Feferman himself has made fundamental contributions), there is no precise account of what it means to say that two proofs are essentially different or the same.

A second foundational way is the analysis of supposedly problematic concepts and principles through an *interpretation*. For instance, one provides an interpretation of complex numbers as pairs of reals or of the axiom of choice in the constructible universe. Alternatively, instead of providing interpretations for problematic concepts, one can provide *substitutes* for them: this is the case for instance of the usual $\varepsilon$-$\delta$ definition of limit, which dispenses with the problematic notion of *infinitesimal*, or the Bernays-Gödel theory of sets and classes, which allows the replacement of the notion of a category of all categories with the "large" category of all "small" categories.

Axiomatization is perhaps the foundational activity *par excellence*. Coming late in the development of a subject, systematization and axiomatization can sometimes be little more than a bookkeeping exercise. Beside the classical characterizations of the basic number systems, $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, the axiomatization of several different constructive theories is also worth mentioning.

But the most innovative foundational activity, in Feferman's view, is the kind of generalization he refers to as *reflective expansion*. The generalization from the Euclidean space $\mathbb{R}^3$ to arbitrary $n$-dimensional spaces $\mathbb{R}^n$ can be viewed as a reflective expansion, but the most interesting applications of the idea concern the progression from one given body $\mathsf{S}$ of concepts and principles to a further body $\mathsf{S}'$, whose concepts of principles are regarded as "implicit" in those of $\mathsf{S}$. In later work, Feferman will refer to $\mathsf{S}'$ as the *unfolding* of $\mathsf{S}$ (see for instance Feferman's "Gödel's program for new axioms. Why, where, how, and what?" in P. Hajek (ed.), *Gödel '96*, Lecture Notes in Logic 6 (1996), pp. 3–22). The unfolding process can be viewed as the attempt to answer the vaguely normative question, first formulated by Kreisel: "What concepts and principles ought one to accept if one has accepted given concepts and principles?"

An important example of unfolding is the case of *predicativity*, whose analysis is one of Feferman's main contributions. Suppose we accept — with Weyl — the natural number system $\mathbb{N}$ as a completed totality, therefore allowing unrestricted quantification over $\mathbb{N}$ as meaningful. Then, any property of natural numbers expressible by a first-order formula of arbitrary quantifier complexity is well-defined, and thus determines a subset of $\mathbb{N}$. In turn, the collection of subsets one allows determines the strength of the corresponding induction principle. In this manner, one can proceed through a number of stages, where at each stage one allows properties expressible by quantifying over previously determined collections of properties. As independently discovered by Feferman and Schütte, this process gives out at the ordinal $\Gamma_0$, which has since come to be regarded as the ordinal characterizing predicativity.

This idea, together with the variations proposed by Feferman and his collaborators over the years, embodies an alternative view of the foundations of mathematics. Here the contrast is not only to foundational programs of the early 1900's — logicism, formalism and intuitionism as originally formulated have long been abandoned. The contrast, at least implicitly, is also to the dominating trend in logic and foundations to reduce every aspect of mathematical practice to the omni-comprehensive framework of Zermelo-Fraenkel set theory and its extensions by various large cardinal axioms. As Feferman points out, there is room for a point of view that emphasizes ontological frugality over extravagance, and that conservatively tries to identify the resources that are required for carrying out given tasks, with a particular attention to the needs of science.

This ideal is substantiated in the next two parts of the book, "Proof Theory" (4) and "Countably Reducible Mathematics" (5). In the article "What does logic have to tell us about mathematical proofs?" Feferman introduces an analogy that is suggestive of his conception of the role of logic in foundations. A logical theory is comparable to a physical theory, in that logic deals with the structure of mathematics pretty much in the same way in which physics deals with the structure of physical reality. The tools employed are formal systems in one case, and differential equations in the other. Accordingly, logic provides a structural analysis of mathematics that is independent of its traditional subdivision into geometry, algebra, analysis, number theory etc. Orthogonally to these subdivisions, logic tries to characterize the activity of idealized mathematicians as they are engaged in the task of drawing conclusions from given assumptions. In this respect, the tools, ideas, methods, and insights gained from proof theory — arguably the most "philosophical" of all the branches of symbolic logic — play a crucial role.

As argued in "What rests on What? The Proof-Theoretic Analysis of Mathematics," the theory of proofs provides us with a fine-grained metric for the dependency relations among various parts of formalized mathematical theories, determining exactly in what sense one is "reducible" to another. Indeed, there are at least two different notions of reduction that logic affords us. One is the already mentioned *interpretability* of one theory into a stronger one. The other is a notion of *proof-theoretical* (or *foundational*) reduction. This kind of reduction moves in the opposite direction from the other one: a stronger theory is reduced to a weaker one by establishing certain conservativity results. Feferman refers to the task of establishing foundational reductions of this kind as a *relativized Hilbert program*.

The basic idea of a proof-theoretical reduction of a theory $T_1$ to a theory $T_2$, for a class of formulas $\Phi$, is that every proof of a formula $\varphi \in \Phi$ in $T_1$ can be computably translated into a proof of $\varphi$ in $T_2$, and moreover this very fact can be represented in the weaker theory $T_2$ itself. Perhaps the first example of such a reduction, due to Ackermann, is the reducibility of first-order arithmetic with the quantifier-free induction schema to primitive recursive arithmetic. This early result has been strengthened many times in many different directions; particularly interesting is the case in which a second-order theory is reduced to a first-order one. The easiest and most representative result of this kind is the fact that arithmetic with a comprehension principle for arithmetically definable sets and the second-order induction axiom is reducible to the first-order theory of Peano Arithmetic.

The conceptual import of these results is that fairly strong arithmetical systems including various second-order comprehension principles and various forms of induction can be *justified* on the basis of finitistically (in the case of primitive recursive arithmetic) or, in any case, predicativistically (in the case of Peano arithmetic) acceptable frameworks.

This fact can be used to carry out a critical assessment of the the so-called Gödel-doctrine: the idea, first formulated by Gödel in a footnote to his incompleteness paper, that in order to decide more and more arithmetical propositions, one has to ascend the (cumulative) type hierarchy well into the transfinite. The Gödel-doctrine is often invoked as an argument in favor of embracing the "Cantorian transfinite." In particular, a number of results beginning with the famous Paris-Harrington theorem, and culminating with Harvey

Friedman's recent work, have purported to show that the "higher infinite" is necessary for ordinary mathematical practice. The main idea here is that a finite combinatorial statement is produced which is (*a*) of independent interest for the "working mathematician;" and (*b*) not settled by some "standard" theory in which mathematical practice is formalized. In the case of the Paris-Harrington theorem, the statement in question is a finite version of Ramsey's theorem, and it is shown to be independent of Peano arithmetic. But Friedman's recent work, especially on Boolean Relation Theory, has pushed this much further, by producing statements that appear more and more 'natural' from the point of view of the working mathematician ('natural' in the same sense in which Gödel's undecidable sentence might appear contrived and ad hoc), and whose proofs require stronger and stronger principles well beyond the reach of Zermelo-Fraenkel set theory, in particular, large cardinal hypotheses. (The reader interested in keeping up with Friedman's ongoing research in this direction could usefully consult the archive of the "Foundations of Mathematics" (`FoM`) discussion list, at `http://www.math.psu.edu/simpson/fom/`.)

Feferman has two rejoinders to these claims. First, referring among other things to his own work, Feferman shows that (in at least some initial cases of the Paris-Harrington type) ascent of the type hierarchy (objectionable in his own view because it presupposes a well-defined notion of power-set), can be replaced by ascent of another, less ontologically committed kind. This alternative ascent is achieved by adjoining to a give theory either a truth-predicate or appropriate reflection principles. In both cases, one succeeds in going beyond Peano arithmetic without committing to an intrinsically uncountable framework. More importantly, Feferman's second rejoinder questions Friedman's claim that the combinatorial principles are natural and readily understandable — and that therefore the intrinsic necessity of strong systems for everyday finitary mathematics has been established. On closer scrutiny one sees that Friedman's results (important and difficult as they are) establish the equivalence of the combinatorial statements not with the large cardinal hypohteses themselves, but with their 1-consistency, i.e., the claim that all purely existential number-theoretical statements provable from the hypotheses are true. The leap from 1-consistency to truth is perhaps not insormountable, but only if one is already committed to their being a fact of the matter as to the truth of the large cardinal hypotheses. Moreover, contrary to the cases of the Paris-Harrington type, the combinatorial statements themselves cannot be established by ordinary mathematical reasoning, but only under the assumption of the 1-consistency of the large cardinal hypotheses. To turn around and claim that the combinatorial statements provide evidence for those hypotheses is, for Feferman, and evident *petitio principii.*

The Feferman-Friedman debate is perhaps one of the most interesting recent exchanges in the philosophy and foundations of mathematics. The embers of the discussion, although smoldering for some time, have been recently rekindled with the publication of Feferman's expository paper "Does Mathematics Need New Axioms?" (*Amer. Math. Monthly* 106

(1999), pp. 99–111) and again at the 2000 meeting of the ASL in Urbana, IL, where Feferman debated the issues with Harvey Friedman and John Steel (all three contributions will be appearing in the *Bulletin of Symbolic Logic*).

Returning to book's main argument, a potential challenge to Feferman's outlook on the foundations of mathematics is provided by the so-called Quine-Putnam *indispensability arguments* for mathematical realism. The idea is that such realism is justified because the mathematical entities whose existence is asserted — including strongly inaccessible stages in the cumulative hierarchy or various kinds of large cardinals — are necessary for the development of modern science. Such arguments have been variously challenged, perhaps most recently by Penelope Maddy. Feferman is not here concerned with undermining the arguments' validity (although he does share many of Maddy's reservations). Rather, Feferman focuses on the fact that the proponents of these arguments seldom consider exactly how much mathematics is needed for science. Once this assessment is carried out — not a light task, and one that has engaged a number of researchers, including Feferman, for a long time — it will become apparent that a "surprisingly meager (in the proof-theoretical sense) predicatively justified system suffices for the formalization of almost all, if not all, scientifically applicable mathematics" (p. 285). The system in question is Feferman's system W of variable types (with a non-constructive least number search operator), itself inspired by Weyl's attempt to develop a system of "arithmetical analysis." (There are, indeed, bits of mathematics that cannot be represented in W, e.g., those that refer in an essential way to non-measurable objects or non-separable Hilbert spaces. But their use in contemporary physics is controversial, and they might in fact not be needed at all.) The system W is proof-theoretically reducible to Peano arithmetic, of which it is a conservative extension. It follows that "acceptance of W ... does not commit one to a Platonistic ontology ..., although the Platonist is free to understand W in those terms" (p. 296).

The argument is now complete. After having identified a range of appropriate foundational activities, and in particular the notion of the *unfolding* of a given conceptual framework, Feferman proceeds to point out the role of proof-theoretical reductions. In turn, proof-theoretical reduction is applied to justify, in predicativistically acceptable terms, a particular formal system W that is both flexible and powerful enough to represent contemporary scientific mathematics.

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