Topological Orders in Rigid States*

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ABSTRACT: We study a new kind of ordering – topological order – in rigid states (the states with no local gapless excitations). We concentrate on characterization of the different topological orders. As an example we discuss in detail chiral spin states of 2+1 dimensional spin systems. Chiral spin states are described by the topological Chern-Simons theories in the continuum limit. We show that the topological orders can be characterized by a non-Abelian gauge structure over the moduli space which parametrizes a family of the model Hamiltonians supporting topologically ordered ground states. In 2+1 dimensions, the non-Abelian gauge structure determines possible fractional statistics of the quasi-particle excitations over the topologically ordered ground states. The dynamics of the low lying global excitations is shown to be independent of random spatial dependent perturbations. The ground state degeneracy and the non-Abelian gauge structures discussed in this paper are very robust, even against those perturbations that break translation symmetry. We also discuss the symmetry properties of the degenerate ground states of chiral spin states. We find that some degenerate ground states of chiral spin states on torus carry non-trivial quantum numbers of the 90° rotation.

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Characterization of ground state of a condensed matter system is one of the most important problems in understanding the low temperature properties of the system. The concept of order parameters and the related broken symmetries give us deeper insight about the properties of ground state and phase transition between different states. However, for some systems, the ground state is not completely characterized by the order parameters (related to broken symmetries). The ground state may contain some sorts of topological orders.[1] In this paper we are going to discuss possible topological orders in the rigid ground states. A rigid ground state is defined as a state in which all local quasi-particle excitations have finite energy gaps. We will call the systems with rigid ground states rigid systems.

From the renormalization group point of view one may naively expect that a rigid system is trivial in the infrared limit because there are no local excitations at low energies. However, in Ref. 1 an example is given to demonstrate that a rigid system may not be trivial even in the infrared limit. Although local excitations are not allowed at low energies, the system supports global excitations, which appear in the form of ground state degeneracy if the space is compactified. The number of global excitations (the ground state degeneracy) is shown to depend on the topology of the compactified space. This dependence of the ground state degeneracy on the topology of the space is a sign of the topological orders. The example suggests that a rigid system may have nontrivial infrared fixed points. A rigid ground state is not only characterized by its symmetry properties, but also characterized by its topological properties.

Recently, Witten[2] discovered a new class of field theories – topological theories – which contain no scales (and no dimensional parameters). (See also Ref. 3.) Because the theory has no scales, all excitations have zero energy. The dimension of the Hilbert space of the topological theory is found to be finite and is just the vacuum degeneracy. From our definition of rigid states, we see that the infrared fixed point (and the topological order) of a rigid system is classified by the topological theories.

We would like to emphasize that the vacuum degeneracy discussed above (and in Ref. 1) is completely due to (or, protected by) the topological ordering present in the ground state, and has nothing to do with the symmetries of the Hamiltonian. The vacuum degeneracy is robust against small perturbations of the Hamiltonian. Thus the vacuum degeneracy (or more precisely the topological order) characterizes different phases of the system.

Although measuring vacuum degeneracy is the simplest way to probe topological ordering in a system, it is not the most effective and complete one. The vacuum degeneracy may not contain all information of the topological order in the ground state. In order to obtain more complete characterization of the topological orders, we are going to study the relation between the ground states of a family of rigid systems. As an example, we will study frustrated spin models supporting chiral spin states.[4] We will concentrate on the non-Abelian gauge structure[5] induced by continuous deformation of the Hamiltonian. It turns out that the non-Abelian gauge structure contains much richer information about topological order. Knowing the non-Abelian gauge structure of a topologically ordered state, we can determine the possible statistics of the quasi-particle excitations in that state.

One of the most important questions in the theories of the high T_c superconductors is how to characterize spin liquid states. The results obtained in this paper and in Ref. 1 indicate that the rigid spin liquid states are characterized by topological orders. Thus it is very important to work out the physical properties linked to the topological orders, and try to determine experimentally what kind of the topological orders (trivial or non-trivial) are realized by the spin liquid state in high T_c superconductors.

The paper is arranged as follows: In Section 2 we will review and extend the relevant results in Ref. 1. In Section 3 we study the non-Abelian gauge structure on the moduli space which parametrizes a family of chiral spin states, using the continuum effective theory. In Section 4 we discuss how to realize the results obtained in Section 3 in lattice models. In Section 5 we study some applications of the non-Abelian gauge structure. In Section 6 we discuss the symmetry properties of the degenerate ground states. In Section 7 we summarize the results we obtained.

II. GROUND STATE WAVE FUNCTIONS OF CHIRAL SPIN STATE

The rigid states we are going to study in this paper come from studies of high T_c superconductors.[6,4] In Ref. 4, it is shown that frustrated spin models may support a T (time reversal symmetry) and P (parity) breaking vacuum state – chiral spin state. All quasi-particle excitations (*e.g.,* spinons) in chiral spin states have finite energy gap, and thus chiral spin state are rigid. Furthermore, it is shown in Ref. 1 that chiral spin states contain non-trivial topological order. In this section we are going to study the ground wave functions of chiral spin states on torus using the effective action of chiral spin states.

Let us consider a frustrated spin model (*e.g.,* frustrated Heisenberg model) defined on finite square lattice with periodic boundary condition. Assume the spin model supports a chiral spin state. The low energy effective Lagrangian for the chiral spin state is given by[4]

$$S_{\text{eff}} = \int d^3x \left[\frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} f_{\mu\nu} f_{\alpha\beta} \right] \tag{2.1}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength of the $U(1)$ gauge field and k is an integer. $g^{\mu\nu}$ in (2.1) takes a general form

$$(g^{\mu\nu}) = \begin{pmatrix} g^{00} & \\ & -g^{ij} \end{pmatrix}. \tag{2.2}$$

g^{ij} is a 2×2 matrix with positive eigenvalue and is determined by the coupling constants (*e.g.,* the spin-spin coupling J_{ij}) in the spin model (we will come back to this in Section 4). At the moment, we assume $g_{\mu\nu}$ are constants and the model respects the translation symmetry. On the torus we may separate the global excitations and the local excitations by writing a^i as

$$a^i(x) = \frac{\theta_i}{L_i} + \tilde{a}^i(x) \tag{2.3}$$

where L_1 (L_2) is the length of the torus in x^1 (x^2) direction and $\tilde{a}^i$ satisfies

$$\int d^2x \; \tilde{a}^i = 0. \tag{2.5}$$

Because only $e^{i \oint \vec{a} \cdot d\vec{x}} = e^{i\theta_i}$ is physically observable, thus θ_i and $\theta_i + 2\pi$ should be identified. (2.1) can be quantized in the gauge $a_0 = 0$. The equation of motion for a_0 becomes a constraint

$$
\begin{aligned}
0 = \frac{\delta S_{\text{eff}}}{\delta a_0} &= \frac{k}{2\pi}\epsilon^{0ij} f_{ij} + g^{ij}\partial_i f_{0j} \\
&= \frac{k}{2\pi}\epsilon^{0ij}\tilde{f}_{ij} + g^{ij}\partial_i \tilde{f}_{0j}.
\end{aligned}
\tag{2.6}
$$

Note the constraint only affects the local excitations $\tilde{a}^i$. Now S_{eff} in (2.1) can be written as

$$
\begin{aligned}
S_{\text{eff}} = &\int dt \left[\frac{k}{4\pi}(\dot{\theta}_1\theta_2 - \dot{\theta}_2\theta_1) + \frac{1}{2}m_{ij}\dot{\theta}_i\dot{\theta}_j\right] \\
&+ \int d^3x \left[\frac{k}{4\pi}\tilde{a}_i\partial_0\tilde{a}_j\epsilon^{i0j} + \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\tilde{f}_{\mu\alpha}\tilde{f}_{\nu\beta}\right]
\end{aligned}
\tag{2.7}
$$

where $m_{ij} = g^{ij}g^{00}$.

From (2.6) and (2.7) we find that the global excitations θ_i and the local excitations $\tilde{a}^i$ decouple. Therefore the ground state wave functionals of chiral spin state take the form

$$
\Phi[a^i] = \psi(\theta_i)\cdot\tilde{\Phi}[\tilde{a}^i]. \tag{2.8}
$$

In the rest of this paper we will concentrate on the wave functions of the global excitations $\psi(\theta_i)$ which contain the information about the topological structure of chiral spin state.

The dynamics of the wave function $\psi(\theta_i)$ of the global excitations is governed by the Hamiltonian

$$
H = -\frac{1}{2}(m^{-1})_{ij}\left(\frac{\partial}{\partial\theta_i} - iA_i^\theta\right)\left(\frac{\partial}{\partial\theta_j} - iA_j^\theta\right), \tag{2.9}
$$

which describes a particle moving on a torus parametrized by (θ_1, θ_2). Assume $(m^{-1})_{ij}$ in (2.9) takes the form (by properly choosing the coupling constants in the model, see Section 4)

$$
m^{-1} = \begin{pmatrix} 1 + \left(\frac{\text{Re }\tau}{\text{Im }\tau}\right)^2, & -\frac{\text{Re }\tau}{(\text{Im }\tau)^2} \\ -\frac{\text{Re }\tau}{(\text{Im }\tau)^2}, & \frac{1}{(\text{Im }\tau)^2} \end{pmatrix}\frac{1}{m_0}, \tag{2.10}
$$

where τ is a complex number with Im $\tau > 0$. Then H in (2.9) can be written in a diagonal form if we choose a new coordinate (x, y)

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2\pi}\begin{pmatrix} 1, & \text{Re }\tau \\ 0, & \text{Im }\tau \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.
$$

In the new coordinate H becomes

$$
H = -\frac{1}{2m_0}\left[\left(\frac{\partial}{\partial x} - iA_x\right)^2 + \left(\frac{\partial}{\partial y} - iA_y\right)^2\right] \tag{2.11}
$$

where the "magnetic" field $B = \partial_x A_y - \partial_y A_x = \frac{2\pi k}{\text{Im }\tau}$ corresponds to total flux $\Phi = 2\pi k$ going through the torus parametrized by (x, y). In the new coordinate z and $z+1$, z and $z+\tau$ are identical points where $z = x + iy$ (Fig. 1).

Choosing the gauge

$$A_x = -yB \quad , \quad A_y = 0 \tag{2.12}$$

the ground state wave functions of (2.11) take the form

$$\psi(x,y) = f(z)e^{-\frac{1}{2}By^2} = f(z)e^{-\frac{\pi k}{\mathrm{Im}\tau}y^2} \tag{2.13}$$

where $f(z)$ is a holomorphic function satisfying the following boundary condition[7]

$$\begin{aligned}\frac{f(z+1)}{f(z)} &= 1 \\ \frac{f(z+\tau)}{f(z)} &= e^{-i\pi k(2z+\tau)}.\end{aligned} \tag{2.14}$$

(2.14) is not the most general boundary condition. However, we will show in the appendix that (2.14) is the boundary condition induced by chiral spin state (in which case k is even). The most general function $f(z)$ satisfying (2.14) is spanned by[7]

$$f_m(z|\tau) = \prod_{a=1}^{k} \theta_1(z - z_a|\tau) e^{i\left[\pi(2m-k)z - \pi m\tau\left(\frac{2m}{k}+1\right)\right]} \tag{2.15}$$

where θ_1 is the odd elliptic theta function and m an integer. z_a in (2.15) satisfies the following condition

$$e^{2\pi i \sum_a z_a} = e^{-i\pi(2m-k)\tau}(-)^k. \tag{2.16}$$

The ground state of (2.11) is k fold degenerate. The k orthogonal ground state wave functions $\psi_m(x,y)$, $m = 1, 2, \ldots, k$ correspond to choosing z_a to be

$$\begin{aligned} z_a^{(m)} &= \frac{a}{k} - \frac{m}{k}\tau + z_0 \quad , \quad a = 1, \ldots, k \\ z_0 &= \frac{1}{2}\left(\tau - \frac{1}{k}\right).\end{aligned} \tag{2.17}$$

The phase factor of $f_m(z|\tau)$ is chosen such that

$$f_m(z|\tau) = f_{m+k}(z|\tau).$$

To have a better understanding about the ground states, let us introduce the magnetic translation operator

$$T(R_x + iR_y) = e^{\left[\vec{R}\cdot(\nabla - i\vec{A}) - i\frac{2\pi k}{\mathrm{Im}\tau}\vec{R}\times\vec{r}\right]} \tag{2.18}$$

which commutes with the Hamiltonian (2.9) and preserves the boundary condition (2.14) when $R_x + iR_y$ takes the form $\frac{m}{k} + \frac{n}{k}\tau$. Thus $T_1 \equiv T\left(\frac{1}{k}\right)$ and $T_2 \equiv T\left(\frac{\tau}{k}\right)$ act on the ground state wave functions ψ_m and transform them into each other. After some calculation it can be shown that

$$\begin{aligned} T_1\psi_m &= -e^{i\frac{2\pi}{k}m}\psi_m \\ T_2\psi_m &= \psi_{m+1}\end{aligned} \tag{2.19}$$

and

$$T_1 T_2 = T_2 T_1 e^{-i\frac{2\pi}{k}}. \tag{2.20}$$

Thus T_1 and T_2 generate the Heisenberg group and the ground states form a k dimensional representation of the Heisenberg group.

In terms of θ^i variables, the ground state wave functions become

$$\begin{aligned}\psi_m(\theta_i|\tau) &= e^{i\frac{k\tau}{4\pi}(\theta_2)^2} f_m(\frac{\theta_1+\tau\theta_2}{2\pi}\ |\tau) \\ &= \psi_{m+k}(\theta_i|\tau)\end{aligned} \tag{2.21}$$

if we choose the gauge in (2.9) to be

$$A_1^\theta = -\frac{k}{2\pi}\theta_2 \quad , \quad A_2^\theta = 0. \tag{2.22}$$

Note the wave function ψ_m is not normalized to unit norm. We have

$$\int d\theta_1 d\theta_2 \psi_m^*(\theta_i|\tau)\psi_n(\theta_i|\tau) = g_{mn} = g_m\delta_{mn}. \tag{2.23}$$

However, $g_m = g_0(\tau,\tau^*)$ are independent of m.

It is useful to write down the magnetic translation operator in θ-space

$$T(\vec{\alpha}) = e^{\alpha_i\left(\frac{\partial}{\partial\theta_i} - iA_i^\theta\right) - i\frac{k}{2\pi}\epsilon_{ij}\alpha_i\theta_j}. \tag{2.24}$$

For the gauge condition (2.22) we have

$$\begin{aligned} T_1 &= T(\vec{\alpha} = (\frac{2\pi}{k}, 0)) = e^{\frac{2\pi}{k}\frac{\partial}{\partial\theta_1}} \\ T_2 &= T(\vec{\alpha} = (0, \frac{2\pi}{k})) = e^{i\theta_1} e^{\frac{2\pi}{k}\frac{\partial}{\partial\theta_2}}. \end{aligned} \tag{2.25}$$

T_1 and T_2 satisfy

$$(T_1)^k = (T_2)^k = 1 \tag{2.26}$$

since ψ_m satisfy the boundary conditions

$$\begin{aligned} \psi_m(\theta_1 + 2\pi, \theta_2|\tau) &= \psi_m(\theta_1, \theta_2|\tau) \\ \psi_m(\theta_1, \theta_2 + 2\pi|\tau) &= e^{-ik\theta_1}\psi_m(\theta_1, \theta_2|\tau). \end{aligned} \tag{2.27}$$

Note that the gauge condition (2.22) and the boundary condition (2.27) are independent of τ. Therefore $\psi(\theta_i|\tau)$ and $\psi(\theta_i|\tau')$ belong to the same Hilbert space and their inner product is well defined:

$$\big(\psi(\theta_i|\tau), \psi(\theta_i|\tau')\big) = \int d\theta_1 d\theta_2 \psi^*(\theta_i|\tau)\psi(\theta_i|\tau') \tag{2.28}$$

This fact is crucial to the calculation of the non-Abelian Barry phase associated with the family of the ground states, $\psi_n(\theta_i|\tau)$.

We would like to emphasize that the parameter τ here is determined by the coupling constants in the lattice Hamiltonian. As we change the coupling constants, the Hamiltonian changes. However, the Hilbert space remain the same. The ground states for different τ are just the different states in the same Hilbert space. Their inner product is well defined.

In the standard topological theory, the quantization of the theory depend on the complex structure (on the torus) which is labeled by a complex number τ^c. In this construction the Hilbert spaces formally depend on the complex structures. In order to define the non-Abelian Barry phases, one needs to define the inner product between states in the different Hilbert spaces. In this paper by viewing g_{ij} as a coupling constant instead of the space-time metrics we effectively define (see (2.28)) the inner product between states in the different Hilbert spaces of different complex structures. This definition is consistent with our physical problem of calculating the non-Abelain Barry phases associated with the ground states of a family of lattice Hamiltonians.

The τ parameter used in this paper and the complex structure τ^c of a torus although has the same mathematical structure, their physical meanings are different. τ just represents a collection of coupling constants and has nothing to do with the space-time metrics. As τ changes, the Hilbert space remain the same despite the Hamiltonian changes with τ. While τ^c comes from the space-time metrics. The Hilbert spaces built on different space-time metrics are different.

III. NON-ABELIAN GAUGE STRUCTURES ON MODULI SPACE

In Ref. 1 we discussed the vacuum degeneracy of chiral spin states on generic Riemann surfaces. We found that the vacuum degeneracy depends on the topology of the compactified space, which is a sign of the topological orders in chiral spin states. However, the vacuum degeneracy of a rigid state may not contain all information about the topological order present in that state. In order to obtain a more complete characterization of the topological order, we are going study the relation between ground states of a family of rigid systems. Let us call the parameters that label the rigid systems in the family, moduli. *The moduli space considered in this paper is a subspace in the total coupling constant space.* In this section we are going to study non-Abelian gauge structure on the moduli space. The non-Abelian gauge structure is induced by the degenerate ground states.[5]

Reader may find the discussions in this section are mathematically similar to the discussions of the non-Abelian gauge structure on the moduli spaces of the Riemann surfaces considered in the string theories[8] and in the topological theories[2,3]. However the two gauge structures are phsycally different. The non-Abelian gauge structure considered here lives on the coupling constant space, while the non-Abelian gauge structure considered in the string theories and the topological theories lives on the space of the complex structures of Riemann surfaces. I feel it is necessary to present a detailed calculations of the non-Abelian gauge structure on the coupling constant space . Because it is not obvious that our definition of the inner product (2.28) leads to the same results as that in the string theory and in the topological theories.

In this paper we are interested in a family of frustrated spin models parametrized by a complex number τ. The models under consideration are assumed to support a chiral spin state described by the effective action (2.1), and g^{ij} in (2.2) (and hence m^{ij} in (2.7) and

(2.9)) takes a form

$$(g^{00}g^{ij})^{-1} = \left(m_{ij}\right)^{-1} = \frac{1}{m_0}\begin{pmatrix} 1+\left(\frac{\text{Re } \tau}{\text{Im } \tau}\right)^2, & -\frac{\text{Re } \tau}{(\text{Im } \tau)^2} \\ -\frac{\text{Re } \tau}{(\text{Im } \tau)^2}, & \frac{1}{(\text{Im } \tau)^2} \end{pmatrix}. \tag{3.1}$$

From the previous section we see that the ground states of each spin model in the family are k fold degenerate (ignoring the 2 fold degeneracy from T and P breaking). The ground state wave functions take a form

$$\Phi_m[a^i|\tau] = \psi_m(\theta^i|\tau)\tilde{\Phi}[\tilde{a}^i|\tau] \tag{3.2}$$

where $\psi_m(\theta^i|\tau)$ is given in (2.21). It has been pointed out in Ref. 5 that a family of degenerate ground states induces a non-Abelian gauge structure in the moduli space. In our case the non-Abelian gauge potential in the moduli space is given by

$$\begin{aligned} (A_\tau)_{mn} &= \langle \Phi_m(\tau,\tau^*)|\ i\ \frac{\partial}{\partial \tau}\ |\Phi_n(\tau,\tau^*)\rangle \\ (A_{\tau^*})_{mn} &= \langle \Phi_m(\tau,\tau^*)|\ i\ \frac{\partial}{\partial \tau^*}\ |\Phi_n(\tau,\tau^*)\rangle \end{aligned} \tag{3.3}$$

where $|\Phi_m\rangle$ are normalized ground state wave functions.

A_τ and A_{τ^*} in (3.3) are $k\times k$ Hermitian matrices and represent a $U(1)\times SU(k) = U(k)$ non-Abelian connection:

$$\begin{aligned} (A_\tau)_{mn} &= A_\tau^{U(1)}\delta_{mn} + (A_\tau^{SU(k)})_{mn} \\ (A_{\tau^*})_{mn} &= A_{\tau^*}^{U(1)}\delta_{mn} + (A_{\tau^*}^{SU(k)})_{mn} \end{aligned} \tag{3.4}$$

where $A_\tau^{U(1)}$ is a $U(1)$ connection and $A_\tau^{SU(k)}(A_{\tau^*}^{SU(k)})$ satisfying Tr $A_\tau^{SU(k)} =$ Tr $A_{\tau^*}^{SU(k)} = 0$ is a $SU(k)$ connection. From (3.2) one finds A_τ and A_{τ^*} can be written as

$$\begin{aligned} (A_\tau)_{mn} &= (\mathcal{A}_\tau)_{mn} + \tilde{A}_\tau\delta_{mn} \\ (A_{\tau^*})_{mn} &= (\mathcal{A}_{\tau^*})_{mn} + \tilde{A}_{\tau^*}\delta_{mn} \end{aligned} \tag{3.5}$$

where

$$(\mathcal{A}_\tau)_{mn} = i\langle \psi_m(\tau,\tau^*)|\ \frac{\partial}{\partial \tau}\ |\psi_n(\tau,\tau^*)\rangle \tag{3.6}$$

and

$$\tilde{A}_\tau = i\langle \tilde{\Phi}(\tau,\tau^*)|\ \frac{\partial}{\partial \tau}\ |\tilde{\Phi}(\tau,\tau^*)\rangle \tag{3.7}$$

together with a similar expression for $\mathcal{A}_{\tau^*}$ and $\tilde{A}_{\tau^*}$. It is clear that the $SU(k)$ connection $A_\tau^{SU(k)}$ is completely determined by $\mathcal{A}_\tau$ which in turn can be calculated from the wave functions (2.21). Thus, in the following we will concentrate on the $SU(k)$ connection $A_\tau^{SU(k)}$.

From (3.6) and (2.23) we have

$$(\mathcal{A}_\tau)_{mn} = i\int d^2\theta \frac{1}{\sqrt{g_m}}\psi_m^*(\theta_i|\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{\sqrt{g_n}}\psi_n(\theta_i|\tau)\right]$$
$$= i\sqrt{g_m}\ \frac{\partial}{\partial\tau}\frac{1}{\sqrt{g_n}}\delta_{mn} + i\frac{1}{g_m}\int d^2\theta\psi_m^*\frac{\partial}{\partial\tau}\psi_n.$$

Since ψ_n is holomorphic in τ, the above can be rewritten as

$$(\mathcal{A}_\tau)_{mn} = i\left[-\frac{1}{2}\frac{\partial}{\partial\tau}\ln g_m + \frac{1}{g_m}\frac{\partial}{\partial\tau}g_m\right]\delta_{mn}$$
$$= i\delta_{mn}\frac{1}{2}\frac{\partial}{\partial\tau}\ln g_0. \tag{3.8a}$$

Similarly we find that

$$(\mathcal{A}_{\tau^*})_{mn} = i\delta_{mn}\left(-\frac{1}{2}\right)\frac{\partial}{\partial\tau^*}\ln g_0. \tag{3.8b}$$

The gauge field strength $\mathcal{F}_{\tau\tau^*}$ is given by

$$(\mathcal{F}_{\tau\tau^*})_{mn} = \left(\frac{\partial}{\partial\tau}\mathcal{A}_{\tau^*} - \frac{\partial}{\partial\tau^*}\mathcal{A}_\tau + i[\mathcal{A}_\tau, \mathcal{A}_{\tau^*}]\right)_{mn}$$
$$= \left(-i\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau^*}\ln g_0\right)\delta_{mn}. \tag{3.9}$$

From (3.9) we see that the degenerate ground states induce a *flat* $SU(k)$ connection (and a non-trivial $U(1)$ connection). The induced $SU(k)$ gauge structure is locally trivial. However, this does not imply that the global $SU(k)$ gauge structures are trivial.

First we notice that the systems labeled by τ and $\tau+1$ are identical. This is because under coordinate transformation

$$\begin{aligned} x_1 &\to x_1' = x_1 - x_2 \\ x_2 &\to x_2' = x_2 \end{aligned} \tag{3.10}$$

the coupling constant matrix in (2.1) transforms as

$$g^{ij}(\tau) \to g'^{ij}(\tau) = g^{ij}(\tau+1).$$

Therefore the ground states of the systems labeled by τ and $\tau+1$ span the same subspace in the Hilbert space and are related by a unitary transformation

$$|\Phi_m(x;\tau+1)\rangle = \tilde{U}_{mn}|\Phi_n(x';\tau)\rangle \tag{3.11}$$

if we make the identification (3.10).

We know a non-Abelian gauge structure is determined by parallel transportations along various loops. In our case there are two kinds of loops, we call them small loops and large loops. The small loops start and end at the same point τ. The parallel transportation along a small loop is given by

$$W(\tau,\tau) = \mathrm{P}\ e^{-i\int_\tau^\tau (A_\tau d\tau + A_{\tau^*}d\tau^*)}$$

where P denotes a path ordered product. The parallel transportations along small loops define a local gauge structure. W is an element in $U(1)\times SU(k)$ and can be written as

$$W = e^{i\varphi} W^{SU(k)} \quad , \quad W^{SU(k)} \in SU(k).$$

From (3.5) and (3.8) we find that $W(\tau,\tau)$ is always a pure phase factor

$$(W(\tau,\tau))_{mn} = e^{i\theta}\delta_{mn}.$$

Therefore the local $SU(k)$ gauge structure is trivial.

The large loops start and end at different but equivalent points, *e.g.,* start at τ and end at $\tau+1$. The parallel transportation along a large loop from τ to $\tau+1$ is given by

$$W(\tau+1,\tau) \equiv \tilde{U}\left[\mathrm{P}e^{-i\int_{\tau}^{\tau+1}(A_{\tau}\cdot d\tau + A_{\tau^*}d\tau^*)}\right].$$

In general $W(\tau+1,\tau)$ depends on the path connecting τ and $\tau+1$ which we choose to integrate along. However, for our choice of basis, $\mathcal{A}_\tau$ and $\mathcal{A}_{\tau^*}$ take simple forms in (3.8). It is not hard to see that the path ordered product

$$\mathrm{P}e^{i\int_{\tau}^{\tau+1} A_\tau d\tau + A_{\tau^*}d\tau^*} = e^{i\theta}$$

is always a $U(1)$ phase factor. Although $e^{i\theta}$ depends on the choice of the path connecting τ and $\tau+1$, it only affects the $U(1)$ part of $W(\tau+1,\tau)$. The $SU(k)$ part of $W(\tau+1,\tau)$ (*i.e.,* $W^{SU(k)}$) is path independent and coincides with the $SU(k)$ part of $\tilde{U}$. Thus the $SU(k)$ gauge structure induced by ground states can be obtained by calculating $\tilde{U}$.

In order to compare the ground states between the two systems labeled by τ and $\tau+1$, let us first consider the Hamiltonian for the global excitations (2.9) with mass matrix $m(\tau)$:

$$H = -\frac{1}{2}(m^{-1}(\tau))_{ij}\left(\frac{\partial}{\partial\theta^i} - iA_i^\theta\right)\left(\frac{\partial}{\partial\theta^j} - iA_j^\theta\right) \tag{3.12}$$

in the gauge

$$A_1^\theta = -\frac{k}{2\pi}\theta_2 \quad , \quad A_2^\theta = 0 \tag{3.13}$$

and boundary conditions

$$\frac{\psi(\theta_1+2\pi,\theta_2|\tau)}{\psi(\theta_1,\theta_2)|\tau)} = 1 \qquad , \qquad \frac{\psi(\theta_1,\theta_2+2\pi|\tau)}{\psi(\theta_1,\theta_2|\tau)} = e^{-ik\theta_1}. \tag{3.14}$$

If k is even, then under transformation

$$\begin{aligned} \theta_1 &\to \theta_1' = \theta_1 - \theta_2 \\ \theta_2 &\to \theta_2' = \theta_2 \end{aligned} \tag{3.15}$$

the H in (3.12) becomes

$$H' = -\frac{1}{2}(m'^{-1}(\tau))_{ij}\left(\frac{\partial}{\partial\theta_i'} - iA_i^{\theta'}\right)\left(\frac{\partial}{\partial\theta_j'} - iA_j^{\theta'}\right) \tag{3.16}$$

where

$$m'^{-1}(\tau) = m^{-1}(\tau+1).$$

The gauge and the boundary conditions (3.13) and (3.14) keep the same form

$$A_1^{\theta'} = -\frac{k}{2\pi}\theta'_2 \quad , \quad A_2^{\theta'} = 0 \tag{3.17}$$

$$\frac{\psi(\theta'_1+2\pi,\theta'_2|\tau)}{\psi(\theta'_1,\theta'_2|\tau)} = 1 \quad , \quad \frac{\psi(\theta'_1,\theta'_2+2\pi|\tau)}{\psi(\theta'_1,\theta'_2|\tau)} = e^{-ik\theta'_1} \tag{3.18}$$

after a suitable gauge transformation. This is because the parallel transportations along OA, OC and OB are all equal to unity (Fig. 2).

However, when k is odd, the parallel transportation along OB is equal to -1. Thus, although the transformation (3.15) can still transform H into H', it does not transform the gauge and the boundary conditions (3.13) and (3.14) into (3.17) and (3.18). Noticing that the parallel transportations along $O'D$ and $O'E$ are equal to unity, we find that the following transformation

$$\begin{aligned} \theta_1 &\to \theta'_1 = \theta_1 - \theta_2 - \pi \\ \theta_2 &\to \theta'_2 = \theta_2 \end{aligned} \tag{3.19}$$

transforms the Hamiltonian system (3.12)–(3.14) into (3.16)–(3.18).

Thus the two Hamiltonian systems for mass matrix $m^{-1}(\tau)$ and $m^{-1}(\tau+1)$ are equivalent and the two sets of the ground state wave functions $|\psi_m(\theta_i|\tau)\rangle$ and $|\psi_m(\theta'_i|\tau+1)\rangle$ of the two systems span the same Hilbert space, provided that θ and θ' are related by (3.15) when k is even and are related by (3.19) when k is odd. Thus there is a unitary transformation between those two basis of the Hilbert space:

$$|\psi_m(\tau+1)\rangle = U_{mn}|\psi_n(\tau)\rangle. \tag{3.20}$$

From (3.2) we find that U and $\tilde{U}$ only differ by a phase factor, $U = e^{i\theta}\tilde{U}$. Thus the $SU(k)$ part of $W(\tau+1,\tau)$ is also given by the $SU(k)$ part of U. The non-trivial unitary matrix U gives rise to a non-trivial global $SU(k)$ gauge structure in the moduli space.

To calculate U, let us first concentrate on the case of even k. The simplest way[3] to obtain the unitary matrix U is to notice that $|\psi_m(\tau+1)\rangle$ forms a representation of magnetic translation

$$\begin{aligned} T'_1 = T(\vec{\alpha} = (\frac{2\pi}{k},0)): &\quad \theta'_1 \to \theta'_1 + \frac{2\pi}{k} \\ T'_2 = T(\vec{\alpha} = (\frac{2\pi}{k},\frac{2\pi}{k})): &\quad \theta'_2 \to \theta'_2 + \frac{2\pi}{k}. \end{aligned} \tag{3.23}$$

Under T'_1 and T'_2 we have

$$\begin{aligned} T'_1|\psi_m(\tau+1)\rangle &= -e^{i\frac{2\pi m}{k}}|\psi_m(\tau+1)\rangle \\ T'_2|\psi_m(\tau+1)\rangle &= |\psi_{m+1}(\tau+1)\rangle. \end{aligned} \tag{3.24}$$

Similarly $|\psi_m(\tau)\rangle$ forms a representation of T_1 and T_2 (see (2.19)). From (3.23) we see that

$$\begin{aligned} T_1' &= T_1 \\ T_2' &= e^{-i\frac{\pi}{k}} T_1 T_2. \end{aligned} \tag{3.25}$$

This determines that

$$U = \eta \begin{pmatrix} e^{i\frac{\pi}{k}1^2} & & & \\ & e^{i\frac{\pi}{k}2^2} & & \\ & & \ddots & \\ & & & e^{i\frac{\pi}{k}k^2} \end{pmatrix} \tag{3.26}$$

where η is a phase factor $|\eta| = 1$, which is only related to the $U(1)$ gauge structure.

When k is odd, if we redefine T_1' and T_2' as

$$\begin{aligned} T_1' &= T(\vec{\alpha} = (\frac{2\pi}{k}, 0)) \\ T_2' &= T^{-1}(\vec{\alpha} = (\pi, 0)) T(\alpha = (\frac{2\pi}{k}, \frac{2\pi}{k})) T(\vec{\alpha} = (\pi, 0)) \\ &= -T(\alpha = (\frac{2\pi}{k}, \frac{2\pi}{k})) \end{aligned} \tag{3.27}$$

we find that (3.24) still holds. Thus from

$$\begin{aligned} T_1' &= T_1 \\ T_2' &= -e^{-i\frac{\pi}{k}} T_1 T_2 \end{aligned} \tag{3.28}$$

we determine that

$$U = \eta \begin{pmatrix} -e^{i\frac{\pi}{k}1^2} & & & \\ & +e^{i\frac{\pi}{k}2^2} & & \\ & & \ddots & \\ & & & (-)^k e^{i\frac{\pi}{k}k^2} \end{pmatrix}. \tag{3.29}$$

In the above we discuss the relation between two models labeled by τ and $\tau + 1$ in the family. Similarly, the two models labeled by τ and $-\frac{1}{\tau}$ are also identical. Under transformation

$$\begin{aligned} \theta_1 &\to \theta_1' = \theta_2 \\ \theta_1 &\to \theta_2' = -\theta_1 \end{aligned} \tag{3.30}$$

(3.12)–(3.14) transform into (3.16)–(3.18) with

$$m'^{-1}(\tau) = m^{-1}(-\frac{1}{\tau}). \tag{3.31}$$

Now we have

$$\begin{aligned} T_1' &= T_2 \\ T_2' &= T_1^{-1} \end{aligned} \tag{3.32}$$

We find that

$$|\psi_m(-\frac{1}{\tau})\rangle = \eta' \frac{1}{\sqrt{k}} \sum_n e^{-i\frac{2\pi mn}{k}} |\psi_n(\tau)\rangle. \tag{3.33}$$

Assuming $|\psi_m\left(-\frac{1}{\tau}\right)\rangle$ and $|\psi_m(\tau)\rangle$ are related by S

$$|\psi_m(-\frac{1}{\tau})\rangle = S_{mn}|\psi_n(\tau)\rangle \tag{3.34}$$

(3.33) implies that

$$S_{mn} = \eta' \frac{1}{\sqrt{k}} e^{-i\frac{2\pi mn}{k}} \tag{3.35}$$

where η' is again an undetermined phase factor.

Two transformations $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ generate general moduli transformations

$$\begin{aligned} \tau &\to \frac{a\tau + b}{c\tau + d} \\ \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} &\to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \end{aligned} \tag{3.36}$$

where $a, b, c, d, \in Z$ are integers and $ad - bc = +1$, *i.e.*,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z). \tag{3.37}$$

The pair $\{U, S\}$ associated with the generators of $SL(2, Z)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, generates a k dimensional projective representation of $SL(2, Z)$.

From previous discussions, the $SU(k)$ part of $W(\tau + 1, \tau)$ and $W(-\frac{1}{\tau}, \tau)$ are given by the $SU(k)$ part of U and S defined as

$$\begin{aligned} u &= U/(\det U)^{\frac{1}{k}} \\ s &= S/(\det S)^{\frac{1}{k}} \end{aligned} \tag{3.38}$$

where $\det u = \det s = 1$. We would like to remark that u and s as $k \times k$ matrices are defined only up to a factor of kth root of unity. The kth root of unity generates the center Z_k of $SU(k)$. Thus we can only say $\{s, u\} \subset SU(k)/Z_k$. Also we can not separate the $SU(k)$ part of W without ambiguity. The $SU(k)$ part of W is given by

$$W^{SU(k)} = W/(\det W)^{\frac{1}{k}} \tag{3.39}$$

which is again defined only up to a factor of kth root of unity. We can only unambiguously separate the $SU(k)/Z_k$ part of W which is given by u or s for transformation $\tau \to \tau + 1$ or $\tau \to -\frac{1}{\tau}$.

The projective representation generated by U and S is not an irreducible representation of $SL(2,Z)$. Notice that S^2 is the rotation of 180° and represents $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $SL(2,Z)$. In Section 6 we will show that

$$S^2 = R_{180^\circ} = \left(\delta_{m,-n}\right) \tag{3.40}$$

which has $k_+ = [\frac{k}{2}] + 1$ eigenvalues equal to 1 and $k_- = [\frac{k+1}{2}] - 1$ eigenvalues equal to -1. ($[x]$ is the integer part of x.) (3.40) also implies that $\eta' = \pm 1$. Because $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ commute with all elements in $SL(2,Z)$, one can easily check R_{180° commute with U and S and hence all the matrices generated by U and S. In the basis that R_{180° takes the form

$$R_{180^\circ} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \tag{3.41}$$

U and S are simultaneously block diagonalized

$$\begin{aligned} U &= \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \\ S &= \begin{pmatrix} S_1 & \\ & S_2 \end{pmatrix} \end{aligned} \tag{3.42}$$

Since S_1^2 is an unity operator up to a phase, U_1 and S_1 generate a k_+ dimensional projective representation of $SL(2,Z)/Z_2$ where Z_2 is the center of $SL(2,Z)$. Similarly, U_2 and S_2 generate a k_- dimensional projective representation of $SL(2,Z)/Z_2$. U_1 and S_1 (U_2 and S_2) act on the subspace of $R_{180^\circ} = 1$ ($R_{180^\circ} = -1$).

In terms of more rigorous mathematical language, the distinct models in the family are labeled by τ's in the fundamental region of transformations $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ (which generate the group $SL(2,Z)/Z_2$) (Fig. 3). The fundamental region is called the moduli space D. (Note here the moduli space D is *not* the moduli space that labels the complex structures of the Riemann surfaces. Our moduli space D is just a special subspace in the coupling constant space.) The boundary AC of the fundamental region D is identified with BD and OA is identified with OB. At each point in the moduli space, we have k fold degenerate ground states. Among them k_+ states have $R_{180^\circ} = 1$ and k_- states have $R_{180^\circ} = -1$. The R_{180° even states at each point of the moduli space form a k_+ dimensional complex vector bundle over D. Ignoring the $U(1)$ factor, the k_+ dimensional vector bundle defines a k_+ dimensional projective vector bundle over the moduli space.[8,2] (3.9) implies that the projective vector bundle is a flat bundle. Its structure is determined by the projective representation of $SL(2,Z)/Z_2$ generated by $\{U_1, S_1\}$. Similarly, the R_{180° odd states form a k_- dimensional flat projective vector bundle over D. The structure of the bundle is given by the projective representation generated by $\{U_2, S_2\}$.

The appearance of flat non-Abelian connections on the moduli space of the Riemann surfaces was pointed out in Ref. 8 for the string theories and was pointed out in Ref. 2

(also see Ref. 3) for the topological Chern-Simons theories. The moduli space in Ref. 8 and Ref. 2 is introduced as a collection of different complex structures on the torus. In this paper we study a different physical problem (or a similar mathematical problem with different physical interpretation). We study the non-Abelian gauge structure on the coupling constant space. We want to use the gauge structure to characterize the topological orders in a lattice model.

We would like to emphasize that the gauge structure on the coupling constant space parametrized by τ and the the gauge structure on the moduli space of torus parametrized by τ^c are essentially different physical objects and have very different physical meanings. The gauge structure on the coupling constant space can be measured in practical computer calculations of lattice spin models. In the topological theories and in the string theories, the non-Abelian gauge connection on the moduli space of Riemann surfaces must be flat. While the gauge connection on the coupling constant space (induced from the degenerate ground states) may not be flat. It is quite possible that a consistent theory may induces a non flat connection on the coupling constant space, although the particular non-Abelian gauge structure considered in this section happen to be flat.

In the effective theory the coupling constant is given by the coefficient $g_{\mu\nu}$ in front of the Maxwell term in (2.1). Although the Maxwell term in (2.1) appears to be an irrelevant operator, the low energy physics does depend on the coupling constant $g_{\mu\nu}$ because a non-trivial gauge structure is induced in the coupling constant space.

The above result is very important. We would like to discuss it in more detail. We know that the gauge boson in (2.1) has a mass of order $1/g^{\mu\nu}g^{\mu\nu}$. When $g^{\mu\nu}\to 0$ the gauge boson mass goes to infinity. In this case the only low lying excitations are degenerate ground states. Naively one expect that the low lying global excitations do not depend on $g^{\mu\nu}$ in $g^{\mu\nu}\to 0$ limit because the local excitations become infinite massive in this limit. However from the calculations in this section, we see that the above naive speculation is not correct. Some information about $g^{\mu\nu}$ do survive at low energies, and the low energy global excitations do depend on some structures in $g^{\mu\nu}$.

In the following we are going to argue that the vacuum degeneracy and the non-Abelian gauge structure of the topologically ordered state are vary robust. The results obtained n this paper are valid even when the translation symmetry is broken, *e.g.,* when the spin-spin coupling J_{ij} have a spatial dependent. The following discussions also shed light on the question what kind of structure in $g_{\mu\nu}$ may survive at low energies.

The effects of the broken translation symmetry can be included in the effective theory by assuming the coupling $g_{\mu\nu}$ in (2.1) to have a spatial dependent. The coefficient of the Chern-Simons term must be constant because of the gauge symmetry. We may still separate the global and the local excitations according to the following equations:

$$\begin{aligned} a^i(x) =& \frac{\theta_i}{L_i} + \tilde a^i(x) \\ a^0(x) =& \tilde a^0(x) \end{aligned} \tag{3.43}$$

where $\tilde a^\mu$ satisfies

$$\int d^2x\ \tilde a^\mu = 0. \tag{3.44}$$

However, because $g_{\mu\nu}$ depends on the spatial coordinates, the global and the local excitations no longer separate. But since the local excitations have finite energy gap, we can still integrate out the local excitations to obtain an effective Lagrangian for the global excitations:

$$e^{iS_{eff}(\theta_i)} = e^{i\int dt \frac{k}{4\pi}(\dot\theta_1\theta_2 - \dot\theta_2\theta_1)} \int D\tilde a^\mu \; e^{i\int d^3x [\frac{k}{4\pi}\tilde a_\mu \partial_\nu \tilde a_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} f_{\mu\alpha} f_{\nu\beta}]} \tag{3.45}$$

It is not difficult to see that the path integral in (3.45) only depend on $\dot\theta_i$. Therefore the contribution to the effective action from the path integral takes a form $\delta S_{eff}(\dot\theta_i)$. Because the path integral is invariant under $\theta_i \to -\theta_i$ and $\tilde a_\mu \to -\tilde a_\mu$, $\delta S_{eff}(\dot\theta_i)$ must be an even function of $\dot\theta_i$. Therefore the path integral can only contribute an mass term and the total effective action takes the form

$$S_{eff} = \int dt \left[\frac{k}{4\pi}(\dot\theta_1\theta_2 - \dot\theta_2\theta_1) + \frac{1}{2} m_{ij}\dot\theta_i\dot\theta_j \right] \tag{3.46}$$

m_{ij} still takes the form in (3.1). However, the relation between m_{ij} and $g_{\mu\nu}$ becomes complicated. From the effective action (3.46) we can derive all the results we obtained before. Thus the properties of the chiral spin states discussed in Section 2 and 3 are very robust, even against perturbations that break translation symmetry. We also notice that the global excitations depend on the local coupling constant $g_{\mu\nu}$ only through a complex number τ. A lots of information of $g_{\mu\nu}$ is lost at low energies. However the Maxwell term is not completely irrelevent because some structures (parametrized by τ) of the coupling constant $g_{\mu\nu}$ do survive at low energies. When $g_{\mu\nu}$ takes the form in (2.2) the structures that survive at low energies are just the complex structures of the Riemann surfaces if we regard g_{ij} as a metrics on the Riemann surfaces.

It is easy to see that the above discussion remains to be valid if the Maxwell term is replaced by an arbitrary even function of $f_{\mu\nu}$, $G(f_{\mu\nu})$. As long as the interactions are weak, the ground state degeneracy and the non-Abelian gauge structure remain unchanged.

IV. LATTICE CONSIDERATIONS

In the above we have discussed the non-Abelian gauge structure over a family of chiral spin states. The above calculations are based on the effective Lagrangian of chiral spin states in the continuum limit. In this section we will discuss how to understand the above results in terms of the original lattice model. Especially, we will describe how to measure U and S (which determines the non-Abelian gauge structure in the moduli space) in an actual numerical calculation on lattices.

One can show that[9] the spin singlet chiral spin states must have even levels k. Thus, in this section we will assume k in (2.1) to be even.

As was discussed in Ref. 1, the degenerate ground states studied in Section 2 only appear as the ground states of the model on the unfrustrated lattice. Consider a finite square lattice with $L_x \times L_y$ sites and periodic boundary conditions in both x and y directions. For a chiral spin state with $\frac{2\pi p}{q}$ flux per plaquette (q is even),[4] the $L_x \times L_y$ lattice is unfrustrated if L_xL_y is a multiple of q. For such a chiral spin state k is found to be equal to q. In order to define the non-Abelian gauge structure, we also need to compare two sets of ground states under the transformation (3.30). This corresponds to the comparison of two sets of ground states with x direction and y direction of the lattice interchanged. In order for such a comparison to be possible, we have to require $L_x = L_y$. Thus the non-Abelian gauge structure studied in Section 3 should appear in the unfrustrated lattices with $L_x = L_y = nk$ with n a large integer. The results obtained in the last section are correct only in the thermodynamic limit.

In Section 3 we studied a family of effective Lagrangians parametrized by a complex number. Such a family of the effective Lagrangians can be induced by the following lattice Hamiltonians

$$H_{\tilde{\tau}} = \sum_{ij} J_{ij}(\tilde{\tau}) \vec{S}_i \cdot \vec{S}_j \tag{4.1}$$

of $S = \frac{1}{2}$ spins (assuming $H_{\tilde{\tau}}$ in (4.1) supports a chiral spin state[10] for any complex number $\tilde{\tau}$ with Im $\tilde{\tau} > 0$). $J_{ij}(\tilde{\tau})$ in (4.1) is given by

$$J_{ij}(\tilde{\tau}) = f(R_{ij}^T \eta(\tilde{\tau}) R_{ij}) \tag{4.2}$$

where $R_{ij}^T \equiv (i_x - j_x, i_y - j_y)$, $f(x)$ is a properly chosen smooth function with compact support (*i.e.*, satisfying $f(x) = 0$ for $x > x_c$), and $\eta(\tilde{\tau})$ is a 2×2 matrix given by

$$\eta(\tilde{\tau}) = \frac{1}{(\text{Im } \tilde{\tau})^2} \begin{pmatrix} |\tilde{\tau}|^2, & -\text{Re } \tilde{\tau} \\ -\text{Re } \tilde{\tau}, & 1 \end{pmatrix}. \tag{4.3}$$

One can easily check that the Hamiltonian $H_{\tilde{\tau}}$ is invariant under the following two transformations (Fig. 4)

$$\begin{cases} i'_x = i_x - i_y \\ i'_y = i_y \end{cases} \tag{4.4a}$$

$$\tilde{\tau}' = \tilde{\tau} + 1 \tag{4.4b}$$

and

$$\begin{cases} i'_x = i_y \\ i'_y = -i_x \end{cases} \tag{4.5a}$$

$$\tilde{\tau}' = -\frac{1}{\tilde{\tau}}. \tag{4.5b}$$

Thus $H_{\tilde{\tau}}$, $H_{\tilde{\tau}+1}$ and $H_{-\frac{1}{\tilde{\tau}}}$ describe the same system. The distinct spin models in the family are labeled by $\tilde{\tau}$'s in the fundamental region (moduli space) (Fig. 3). The degenerate ground states of $H_{\tilde{\tau}}$ for each $\tilde{\tau}$ induce a non-Abelian gauge structure over the moduli space. This gauge structure is determined by parallel transportations

$$\begin{aligned} W(\tilde{\tau}+1, \tilde{\tau}) &= \tilde{U} \text{P} e^{-i \int_{\tilde{\tau}}^{\tilde{\tau}+1} (A_{\tilde{\tau}} d\tilde{\tau} + A_{\tilde{\tau}^*} d\tilde{\tau}^*)} \\ W(-\frac{1}{\tilde{\tau}}, \tilde{\tau}) &= \tilde{S} \text{P} e^{-i \int_{\tilde{\tau}}^{-\frac{1}{\tilde{\tau}}} (A_{\tilde{\tau}} d\tilde{\tau} + A_{\tilde{\tau}^*} d\tilde{\tau}^*)} \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} (A_{\tilde{\tau}})_{mn} &= \langle \Phi_m(\tilde{\tau}) | \, i \frac{\partial}{\partial \tilde{\tau}} \, | \Phi_n(\tilde{\tau}) \rangle \\ (A_{\tilde{\tau}^*})_{mn} &= \langle \Phi_m(\tilde{\tau}) | \, i \frac{\partial}{\partial \tilde{\tau}} \, | \Phi_m(\tilde{\tau}) \rangle \end{aligned} \tag{4.7}$$

and the unitary matrices $\tilde{U}$ and $\tilde{S}$ are given by

$$\begin{aligned}\tilde{U}_{mn} &= \langle \Phi_m(\tilde{\tau}) | \Phi_n(\tilde{\tau}+1) \rangle \\ \tilde{S}_{mn} &= \langle \Phi_m(\tilde{\tau}) | \Phi_n(-\frac{1}{\tilde{\tau}}) \rangle .\end{aligned} \tag{4.8}$$

In (4.7) and (4.8), $|\Phi_m(\tilde{\tau})\rangle$ is the ground state wave function. The wave functions $\Phi_m(\tilde{\tau})$ and $\Phi_m(\tilde{\tau}+1)$ ($\Phi_m(\tilde{\tau})$ and $\Phi_m(-\frac{1}{\tilde{\tau}})$) are compared under the transformation (4.4a) ((4.5a)), *i.e.,* as an N electron wave function

$$\langle \Phi_m(\tilde{\tau}) | \Phi_n(\tilde{\tau}+1) \rangle \equiv \sum_{\{i^{(a)},\sigma^{(a)}\}} \Phi_m^*(\ldots, i^{(a)}, \sigma^{(a)}, \ldots | \tilde{\tau}) \Phi_m(\ldots, i'(i^{(a)}), \sigma^{(a)}, \ldots | \tilde{\tau}+1) \tag{4.9}$$

where the function $i'(i)$ is given by (4.4a) and $\sigma^{(a)} = \pm 1$ describes the electron spin. Although $\tilde{\tau}$ in (4.1) and τ in (3.1) may not be identical, there is a one-to-one correspondence between $\tilde{\tau}$ and τ which maps the fundamental region of $\tilde{\tau}$ to the fundamental region of τ. Since the parallel transportation W is invariant under reparametrization, the non-Abelian gauge structures are the same in $\tilde{\tau}$ space and τ space. Especially, the $SU(k)/Z_k$ parts of $W(\tilde{\tau}+1, \tilde{\tau})$ and $W\left(-\frac{1}{\tilde{\tau}}, \tilde{\tau}\right)$ are given by u and s (see (3.38), (3.29) and (3.35)) respectively up to a common unitary transformation. We would like to point out that the definition of the parallel transportations W in (4.6) is independent of choices of the basis of the ground states, up to a unitary transformation.

We would like to remark that the degenerate ground states discussed here are expected to carry the same crystal momentum. This is crucial for the above discussion to be valid. The reason is because (4.4a) is not the only transformation to change $H_{\tilde{\tau}}$ to $H_{\tilde{\tau}+1}$. A more general transformation:

$$\begin{aligned}i'_x &= i_x - i_y + i_{0x} \\ i'_y &= i_y + i_{0y}\end{aligned} \tag{4.10}$$

does the same job, where i_{0x} and i_{0y} are arbitrary integers. Furthermore, there is no natural way to choose i_{0x} and i_{0y}. If $|\Phi_m(\tilde{\tau})\rangle$'s carry the same crystal momentum, different choices of i_{0x} and i_{0y} only change $W(\tilde{\tau}+1, \tilde{\tau})$ by a total phase factor. In this case, the $SU(k)/Z_k$ part of $W(\tilde{\tau}+1, \tilde{\tau})$ is independent of choices of i_{0x} and i_{0y}.

We may use mean field theory of chiral spin state to understand why all degenerate ground states carry the same crystal momentum. The mean field wave function $|\theta_i\rangle_{\text{mean}} = \Phi_{\text{mean}}(i^{(a)}, \sigma^{(a)} | \theta_i)$ is given by the ground state of the mean field Hamiltonian

$$H_{\text{mean}}(\theta_i) = \sum \chi_{ij}^{\theta} \; c_i^\dagger c_j \tag{4.11}$$

where c_i satisfies periodic boundary condition

$$c_i = c_{i+L_x} = c_{i+L_y}.$$

χ_{ij}^{θ} generate $\frac{2\pi p}{q}$ flux per plaquette and satisfy

$$\prod_{i_x=1}^{L_x} \chi_{(i_x, i_y^0),(i_x+1, i_y^0)} = |\chi| e^{i\theta_1}$$

$$\prod_{i_y=1}^{L_y} \chi_{(i_x^0,i_y),(i_x^0,i_y+1)} = |\chi| e^{i\theta_2} \tag{4.12}$$

Note since L_x and L_y are multiples of q, θ_1 and θ_2 defined in (4.12) are independent of i_y^0 and i_x^0. The spin state is obtained from the mean field state by doing the Gutzwiller projection P_G:

$$|\theta_i\rangle = P_G|\theta_i\rangle_{\text{mean}}. \tag{4.13}$$

Since the crystal momentum of $|\theta_i\rangle$ is independent of gauge choice χ_{ij}^{θ}, to show that the x component of the crystal momentum to be zero it is convenient to choose a gauge such that

$$\chi_{ij}^{\theta} = \chi_{i+\hat{x},j+\hat{x}}^{\theta}. \tag{4.14}$$

In this gauge H_{mean} is translation invariant in x direction. Since the electrons satisfy a periodic boundary condition, k_x of single electron state is given by

$$k_x = \frac{2\pi}{L_x}\, n. \tag{4.15}$$

The ground state of H_{mean} has a filled valence band. The mean field electron wave function $\Phi(i^{(a)}, \sigma^{(a)}|\theta_i)$ has a crystal momentum whose x component is given by $2\sum k_x = 0 \bmod 2\pi$ (the factor 2 comes from spin). Since the Gutzwiller projection P_G commutes with translation, the projected state $|\theta_i\rangle$ also has zero crystal momentum in x direction. Similarly one can show $|\theta_i\rangle$ has zero crystal momentum in y direction. Therefore $|\theta_i\rangle$ carries zero crystal momentum (for any θ_i). Although the above result is obtained based on the mean field theory, we believe the result is generally true even beyond mean field theory.

V. APPLICATIONS

In Ref. 1 we have discussed a characterization of topological orders using the ground state degeneracy on compactified space. However, the ground state degeneracy does not provide a complete characterization of the topological order. Two states with the different topological orders may have the same ground state degeneracy on arbitrary Riemann surfaces. For example, consider a rigid state described by the following effective Lagrangian

$$S_{\text{eff}} = \int d^3x \left[\frac{k_1}{4\pi} a_{1\mu}\partial_\nu a_{1\lambda}\epsilon^{\mu\nu\lambda} + \frac{k_2}{4\pi} a_{2\mu}\partial_\nu a_{2\lambda}\epsilon^{\mu\nu\lambda} \right] + \dots \tag{5.1}$$

where $a_{1\mu}$ and $a_{2\mu}$ are two independent $U(1)$ gauge fields. The ground state degeneracy of (5.1) on Riemann surface Σ_g with genus g is given by $k_1^g k_2^g$, which is the same with the ground state degeneracy of (2.1), k^g, if we set $k = k_1 k_2$. Thus, the ground state degeneracy alone can not distinguish the two different topological orders described by (2.1) and (5.1).

However, the non-Abelian gauge structure studied in Section 3 contains much more information about topological orders than the ground state degeneracy does, and gives a more complete description about topological orders. We will illustrate this point by showing that the topological orders described by (2.1) and (5.1) give rise to different non-Abelian gauge structures over the moduli space.

The non-Abelian gauge structure on the moduli space is determined by two parallel transportations $W(\tau+1,\tau)$ and $W(-\frac{1}{\tau},\tau)$. For the topological order given by (2.1), the $SU(k)/Z_k$ parts of W's are

$$\begin{aligned} W(\tau+1,\tau) &= e^{i\varphi_1}U(k) \\ W(-\frac{1}{\tau},\tau) &= e^{i\varphi_2}S(k) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} (U(k))_{mn} &\equiv e^{i\frac{\pi}{k}m^2}\delta_{mn}(-)^{mk} \\ (S(k))_{mn} &\equiv e^{-i\frac{2\pi}{k}mn} \end{aligned} \tag{5.3}$$

The phases of the eigenvalues of $W(\tau+1,\tau)$ take the following values

$$\theta = 2\pi\frac{m^2}{2k} + mk\pi + \varphi_1 \quad , \quad m = 1,\ldots,k \tag{5.4}$$

For the topological order described by (5.1) we have

$$\begin{aligned} W(\tau+1,\tau) &= e^{i\varphi_1'}U(k_1)\otimes U(k_2) \\ W(-\frac{1}{\tau},\tau) &= e^{i\varphi_2'}S(k_1)\otimes S(k_2) \end{aligned} \tag{5.5}$$

and the phases of the eigenvalues of $W(\tau+1,\tau)$ are given by

$$\theta = 2\pi\left(\frac{m^2}{2k_1} + mk_1\pi + \frac{n^2}{2k_2} + nk_2\pi\right) - \varphi_1' \quad , \quad m = 1,\ldots,k_1 \ ; \ \ n = 1,\ldots,k_2 \tag{5.6}$$

The phases in (5.4) and (5.6) do not match each other no matter how we choose φ_1 and φ_1'. For example, for the case $k_1 = k_2 = 2$ and $k = k_1k_2 = 4$, the phases in (5.4) are

$$\theta = \left\{\frac{\pi}{4},\pi,\frac{\pi}{4},0\right\} - \varphi_1 \tag{5.7}$$

and the phases in (5.6) are

$$\theta = \left\{\pi,\frac{\pi}{2},\frac{\pi}{2},0\right\} - \varphi_1' \tag{5.8}$$

They are definitely different sets of numbers for any choice of φ_1 and φ_1'. Therefore, by measuring the eigenvalues of $W(\tau+1,\tau)$ we can distinguish the topological orders described by (2.1) and (5.1), even when $k = k_1k_2$.

Having a more complete characterization of topological order in a rigid state, we are able to determine some qualitative properties of the quasi-particle excitations, *e.g.,* the statistics of the quasi-particles. Let us first consider the rigid state described by (2.1). A charge q (q is an integer) quasi-particle in such a state has a fractional statistics

$$\theta = 2\pi\frac{q^2}{2k} + \varphi \tag{5.9}$$

where φ describes the statistics of a possible neutral particle bounded to the quasi-particle. The different statistics are given by $q = 1, \ldots, k$ for k even and $q = 1, \ldots, 2k$ for k odd. In both cases, the possible statistical angles are given by the phases of the eigenvalues of $W(\tau + 1, \tau)$ (see (5.4)), up to a common phase factor. (For odd k, φ_1 in (5.4) has to take two different values differing by π in order to account for $2k$ different statistical angles.) For the topological order described by (5.1) we reach a similar result. For generic rigid states in 1+2 dimensions, we would like to make the following conjecture. The possible statistics of quasi-particles in a rigid state always appear in groups. The statistical angles in each group are given by the phases of the eigenvalues of $W^{-1}(\tau + 1, \tau)$ plus a common constant.

In the above we have studied a family of spin models defined on torus. Similarly, we can using the same method to study the spin models defined on general Riemann surfaces Σ_g.[3] The moduli space parametrized by τ can be generalized to describe a family of the models defined on the general Riemann surfaces Σ_g. Again the non-Abelian gauge structure on the moduli space of the models on Σ_g contain a lot of information about the topological orders in the ground states. The Abelian part of the induced gauge connection (see (3.4)) also contains information about the topological orders. We would like to conjecture that the total gauge structures (the Abelian one plus the non-Abelian one) on the moduli spaces of the models defined on generic Riemann surfaces Σ_g completely characterize (or classify) the topological orders in 1+2 dimensions.

VI. THE SYMMETRY PROPERTIES OF CHIRAL SPIN STATES

In this section we are going to study the symmetry properties of the degenerate ground states of chiral spin states studied in Section 2. We already showed in Section 4 that chiral spin states carry zero crystal momentum. Here we would like to consider the properties of chiral spin states under rotations.

First, $H_{\tilde{\tau}}$ in (4.1) respects 180° rotation symmetry. We would like to study the quantum numbers of the degenerate ground states of chiral spin states under 180° rotation (Note the level of the chiral spin state, k, is even). Certainly, we assume that $H_{\tilde{\tau}}$ is defined on an unfrustrated lattice. Let us first study the property of $\psi_m(\theta_i)$ under 180° rotation. The Hamiltonian, gauge condition and the boundary condition (3.12)–(3.14) are invariant under 180° rotation $\theta_i \to -\theta_i$. Therefore $\psi_m(\theta_i|\tau)$ form a simple representation of 180° rotation

$$R_{180^\circ}\psi_m(\theta_i|\tau) = \psi_m(-\theta_i|\tau). \tag{6.1}$$

From (2.21) and (2.15) we find that

$$R_{180^\circ}\psi_m(\theta_i|\tau) = \psi_{-m}(\theta_i|\tau). \tag{6.2}$$

Thus among k ground states of H in (3.12)–(3.14), $\frac{k}{2} + 1$ states are even under R_{180° and $\frac{k}{2} - 1$ states are odd under R_{180°.

The quantum number of R_{180° for the wave functional of the local gauge excitations can be shown to be even under 180° rotation (see Appendix). therefore among k total ground state wave functionals $\Phi_m[a_i]$, $\frac{k}{2} + 1$ are even and $\frac{k}{2} - 1$ are odd under 180° rotation.

When $\tilde{\tau} = i$, $H_{\tilde{\tau}}$ also respects 90° rotation symmetry. Let us again first study the properties of $|\psi_m\rangle$ under 90° rotation $\theta_1 \to \theta_2, \theta_2 \to -\theta_1$. Although H in (3.12) is invariant under 90° rotation, both the gauge condition (3.13) and the boundary condition (3.14) are not invariant. We need to perform a gauge transformation to make $\psi_m(\theta_1, \theta_2|\tau)$ and $\psi_m(\theta_2, -\theta_1|\tau)$ satisfy the same boundary condition. However, we may use the magnetic translation T_1 and T_2 to simplify the calculation. Under 90° rotation

$$|\psi_m\rangle \to |\psi'_m\rangle = R_{90^\circ}|\psi_m\rangle. \tag{6.3}$$

$|\psi'_m\rangle$ form a standard representation of the magnetic translation T'_1 and T'_2 in (3.32)

$$\begin{aligned} T'_1|\psi'_m\rangle &= -e^{i\frac{2\pi m}{k}}|\psi'_m\rangle \\ T'_2|\psi'_m\rangle &= |\psi'_{m+1}\rangle \end{aligned} \tag{6.4}$$

From (3.32) and (2.19) we find that

$$\langle\psi_m|\psi'_n\rangle = (R_{90^\circ})_{mn} = \eta' e^{-i\frac{2\pi mn}{k}} \tag{6.5}$$

Using the relation

$$(R_{180^\circ})_{mn} = (R^2_{90^\circ})_{mn} = {\eta'}^2 \delta_{m,-n} \tag{6.6}$$

we determine η' to be ± 1. Let us first assume $\eta' = 1$, (6.6) implies that $\frac{k}{2}+1$ eigenvalues of R_{90° are either $+1$ or -1 and $\frac{k}{2}-1$ eigenvalues are either $+i$ or $-i$. When $\frac{k}{2}$ is odd we have

$$\text{Tr } R_{90^\circ} = \sum_{m=1}^{k} e^{-i\frac{2\pi}{k}m^2} = 0. \tag{6.7}$$

Thus R_{90° has $\frac{1}{2}(\frac{k}{2}+1)$ eigenvalues to be $+1$, $\frac{1}{2}(\frac{k}{2}+1)$ eigenvalues to be -1, $\frac{1}{2}(\frac{k}{2}-1)$ eigenvalues to be $+i$ and $\frac{1}{2}(\frac{k}{2}-1)$ eigenvalues to be $-i$. The above result remains the same for $\eta' = -1$. We expect that the quantum number of 90° rotation of the total ground state wave functions, $\Phi_m[a_i]$, are the same as those for ψ_m, up to ± 1.

Let us summerize the above results for a level $k = 2$ chiral spin state. The level 2 chiral spin state has four fold degenerate vacua on torus. Two of them have $E_{123} = \langle \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_3)\rangle > 0$ and the other two have $E_{123} < 0$. All of the four ground states are even under 180° rotation. Among the two states with $E_{123} > 0$, one is even under 90° rotation and another is odd. Similar results hold for the two states with $E_{123} < 0$. The four ground state carry the following quantum numbers of parity and 90° rotation

$$(P, R_{90^\circ}):\ (++),\ (+-),\ (-+),\ (--)$$

We would like to remark that the degeneracy between the ground states carrying different R_{90° quantum numbers, in our case, does not imply the 90° rotation symmetry to be spontaneously broken. This is because that the different R_{90° quantum numbers come from the global excitations $\psi_m(\theta_i)$ with finite degrees of freedom. Thus, all the degenerate ground states with different R_{90° quantum numbers (and the same E_{123}) belong to the same world. In contrast, the ground states with opposite signs of E_{123} belong to different worlds. The appearance of the degenerate states with different R_{90° quantum numbers is due to our compactification of the space, because the degeneracy comes from the global excitations. The coorelation functions between local operators will show no sign of spontaneous breaking of the 90° rotation symmetry.

Recent numerical studies on a 4×4 spin system[11] (with periodic boundary condition) finds two closely degenerate low lying states with $R_{90^\circ} = \pm 1$. To conclude that the 90° rotation symmetry is broken one needs more information, for example the calculations of proper coorelation functions.

VII. CONCLUSIONS

In this paper, we study a new kind of ordering, *i.e.,* the topological order, in rigid states. As an example, we discuss chiral spin states in $2+1$ dimensional spin systems. We demonstrate that a topologically ordered state in general induces a nontrivial non-Abelian gauge structure over the moduli space. The non-Abelian gauge structure gives a detailed characterization of the topological order present in that state. For example, the non-Abelian gauge structure determines possible statistics of the quasi-particles. The results obtained in this paper are useful for using numerical results on finite systems to determine the appearance of the topological orders and to characterize the topological orders.

We also demonstrate that the topological orders are very robust. The ground state degeneracy and the non-Abelian gauge structure in the topologically ordered states are independent of random spatial dependent perturbations in the short distance coupling constants.

In Ref. 1 and in this paper we probe the topological orders by putting the systems on compactified space and by twisting the systems. One may ask whether there are other ways to characterize topological orders, *e.g.,* using correlation functions between operators. However due to the topological nature of the problem the correlation functions between local operators may not be able to characterize topological orders. One probably needs to use the correlation "functionals" between non-local operators to characterize topological orders. One possible choice of the non-local operators is to use operators defined on loops in space-time[2] (Wilson lines)

$$O[C] = e^{i \oint_C dx^\mu a_\mu}$$

On lattice $O[C]$ has a form[4]

$$O[C] = \prod_{\langle ij \rangle} c^\dagger_{\sigma i}(t_i) c_{\sigma j}(t_j)$$

where $\langle ij \rangle$ are links on the loop C. The dynamical correlation functional between the two loop-operators $\langle O[C_1]O[C_2]\rangle$ depends on how the two loops are linked to each other. The correlation functional can be used to characterize the topological orders.

We also discuss the symmetries of the degenerate ground states of the chiral spin states defined on torus. We find that the four degenerate ground states of the level two chiral spin state carry the following quantum numbers of the parity and the 90° rotation:

$$(P, R_{90^\circ}): \ (++),\ (+-),\ (-+),\ (--)$$

We emphasize that the above results do *not* imply that the 90° rotation symmetry is spontaneous broken.

Certainly it is most important to see whether topologically ordered states have experimental accessible predictions, and to experimentally test whether the spin liquid state in high T_c superconductors contain topological order or not. Topologically ordered states might appear in other systems and other dimensions as well. It would be interesting to find the systems which support topologically ordered states.

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APPENDIX

(2.14) is not the most general boundary condition. Two phase factors $e^{i\varphi_1}$ and $e^{i\varphi_2}$ may be included in the right hand side of the two equations in (2.14).[7] The specific boundary condition (2.14) and the gauge condition (2.12) (or equivalently (2.27) and (2.22)) imply that the parallel transportations along the two loops given by $\theta_1 = 0$ and $\theta_2 = 0$ are chosen to be 1. Here we are going to show that for chiral spin states (k is even) if θ_1 and θ_2 are chosen in the way described in Section 4, the parallel transportation along the loops, $\theta_1 = 0$ and $\theta_2 = 0$, are indeed equal to 1.

Our approach is based on the mean field theory of chiral spin state.[4] Repeating some analysis at the end of Section 4, we know that the mean field wave function $|\theta_i;\tilde\tau\rangle_{\text{mean}}$ is given by the ground state of the following mean field Hamiltonian:

$$H_{\text{mean}}(\theta_i,\tilde\tau) = \sum \chi^{\theta}_{ij}(\tilde\tau) c^{\dagger}_i c_j \tag{A.1}$$

where χ^{θ}_{ij} generate $\frac{2\pi p}{q}$ flux per plaquette (q is even) and satisfy (4.12). Furthermore $\chi^{\theta}_{ij}(\tilde\tau)$ have form

$$|\chi^{\theta}_{ij}(\tilde\tau)| = f_\chi\left(R^{\tau}_{ij}\eta(\tilde\tau)R_{ij}\right) \tag{A.2}$$

where $R^T_{ij} = (i_x - j_x, i_y - j_y)$ and $\eta(\tilde\tau)$ is given by (4.3). The spin state labeled by θ_i is given by $|\theta_i;\tilde\tau\rangle = P_G|\theta_i;\tilde\tau\rangle_{\text{mean}}$.

We may choose the phase of $|\theta_i;\tilde\tau\rangle$ such that $|\theta_i;\tilde\tau\rangle$ is periodic in θ_1 when $\theta_2 = 0$ and periodic in θ_2 when $\theta_1 = 0$ (with period 2π). (This corresponds to the boundary condition (2.27).) The parallel transportations along the loops $\theta_1 = 0$ and $\theta_2 = 0$ are just the Berry phases obtained by changing θ_2 and θ_1. They are given by

$$\begin{aligned} W_2(\tilde\tau) &= e^{i\int_{-\pi}^{\pi} d\theta_2 A^{\theta}_2(\theta_1=0,\theta_2)} = e^{i\varphi_2(\tilde\tau)} \\ W_1(\tilde\tau) &= e^{i\int_{-\pi}^{\pi} d\theta_1 A^{\theta}_1(\theta_1,\theta_2=0)} = e^{i\varphi_1(\tilde\tau)} \end{aligned} \tag{A.3}$$

where

$$A^{\theta}_i = \langle\theta_i;\tilde\tau|i\frac{\partial}{\partial\theta_i}|\theta_i;\tilde\tau\rangle. \tag{A.4}$$

A^{θ}_i in (A.4) is the gauge potential experienced by collective modes described by θ_i. The gauge potential in (2.9) is given by (A.4).

Under 180° rotation:

$$\begin{aligned} \theta_i &\to -\theta_i \\ R_{180^\circ}H_{\text{mean}}(\theta_i)R_{180^\circ} &= H_{\text{mean}}(-\theta_i) \\ R_{180^\circ}|\theta_i;\tau\rangle &= e^{i\varphi(\theta_i)}|-\theta_i;\tau\rangle \end{aligned} \tag{A.5}$$

The first equation of (A.5) implies $W_i \to W_i^{-1}$ under 180° rotation. However, the Hamiltonian system parametrized by θ_i is invariant under 180° rotation. Since W_i are physical quantities, they should be invariant under 180° rotation, *i.e.*,

$$W_i = W_i^{-1}$$

which gives us

$$W_i = \pm 1$$

or

$$\varphi_i(\tilde{\tau}) = 0, \pi. \tag{A.6}$$

More mathematically we may rewrite φ_1 as

$$\varphi_1 = \frac{1}{2}\int_{-\pi}^{\pi} d\theta_1 \left(A_1^\theta(\theta_1, 0) + A_1^\theta(-\theta_1, 0) \right). \tag{A.7}$$

Using (A.5) we find that

$$\begin{aligned} A_i^\theta(\theta_i) &= i\langle \theta_i; \tilde{\tau} | R_{180^\circ} \frac{\partial}{\partial \theta_i} R_{180^\circ} | \theta_i; \tilde{\tau}\rangle \\ &= -A_i^\theta(-\theta_i) - \frac{\partial}{\partial \theta_i}\varphi(\theta_i). \end{aligned} \tag{A.8}$$

Therefore

$$\begin{aligned} \varphi_1 &= -\frac{1}{2}\int_{-\pi}^{\pi} d\theta_1 \frac{\partial}{\partial \theta_1}\varphi(\theta_1, 0) \\ &= -\frac{1}{2}(\varphi(\pi, 0) - \varphi(-\pi, 0)) \end{aligned} \tag{A.9}$$

Since $R_{180^\circ}^2 = 1$ we have

$$\varphi(\theta_i) + \varphi(-\theta_i) = 0 \mod 2\pi \tag{A.10}$$

and

$$\begin{aligned} \varphi(\pi, 0) = 0 \mod \pi \\ \varphi(-\pi, 0) = 0 \mod \pi \end{aligned} \tag{A.11}$$

because $|(\pi, 0); \tilde{\tau}\rangle = |(-\pi, 0); \tilde{\tau}\rangle$. (A.10) and (A.11) imply

$$\varphi(\pi, 0) - \varphi(-\pi, 0) = 0 \mod 2\pi. \tag{A.12}$$

Thus, $\varphi_1 = 0, \pi$ or $W_1 = \pm 1$. Similarly one can show that $W_2 = \pm 1$.

(A.6) implies that φ_i are quantized. Since φ_i are expected to be continuous functions of $\tilde{\tau}$, φ_i must be constant equal to either 0 or π. In the following we will show $\varphi_i = 0$ for the chiral spin state.

Notice that (after relabeling the lattice (4.4))

$$H_{\text{mean}}(\theta_i, \tilde{\tau}) = H_{\text{mean}}(\theta'_i, \tilde{\tau} + 1) \tag{A.13}$$

where θ'_i are given by (3.15). Thus the parallel transportation $W_i(\tilde{\tau})$ and $W_i(\tilde{\tau}+1)$ are related. In particular

$$W_{12}(\tilde{\tau}) = e^{i\varphi_{12}} = W_2(\tilde{\tau} + 1) \tag{A.14}$$

where $W_{12}(\tilde{\tau})$ is the parallel transportation along the loop $\theta_1 - \theta_2 = \theta_1' = 0$ (Fig. 5). We also have the relation

$$W_1(\tilde{\tau}) W_2(\tilde{\tau}) W_{12}^{-1}(\tilde{\tau}) = e^{i\frac{1}{2}\Phi} \tag{A.15}$$

where $\Phi = 2\pi q$ is the total flux going through the torus parametrized by θ_i. Since q is even (A.15) implies

$$\varphi_1(\tilde{\tau}) + \varphi_2(\tilde{\tau}) - \varphi_{12}(\tilde{\tau}) = 0 \mod 2\pi. \tag{A.16}$$

(A.14) tells us $\varphi_{12}(\tilde{\tau}) = \varphi_2(\tilde{\tau}+1) = \varphi_2(\tilde{\tau})$. Thus

$$\varphi_1(\tilde{\tau}) = 0 \mod 2\pi. \tag{A.17}$$

At $\tilde{\tau} = i$, $H_{\text{mean}}(\theta_i, \tilde{\tau})$ has 90° rotation symmetry. We can show that

$$\varphi_2(\tilde{\tau}) = \varphi_2(i) = \varphi_1(i) = 0 \mod 2\pi \tag{A.18}$$

(A.17) and (A.18) give us

$$W_i(\tilde{\tau}) = 1. \tag{A.19}$$

In the following we are going to show that the wave functional of the local excitations, $\tilde{\Phi}[\tilde{a}^i]$, is even under 180° rotation. First notice that $\theta_i = 0$ is a fix point of 180° rotation. Thus $\tilde{\Phi}[\tilde{a}^i]$ and $|\theta_i = 0\rangle$ have the same quantum number of 180° rotation. Because the spin up electrons and the spin down electrons are independent in H_{mean}, the mean field ground state can be written as

$$|\theta_i = 0\rangle_{mean} = |\Phi_\uparrow\rangle \otimes |\Phi_\downarrow\rangle \tag{A.20}$$

where $|\Phi_\uparrow\rangle$ ($|\Phi_\downarrow\rangle$) is the wave function of the up (down) spin electrons. Both up spin and down spin electrons have the same orbital wave function, thus $|\Phi_\uparrow\rangle$ and $|\Phi_\downarrow\rangle$ have the same quantum number of 180° rotation. This implies that $|\theta_i = 0\rangle_{mean}$ is even under 180° rotation. Since the Gutzwiller projection P_G commutes with 180° rotation, $|\theta_i = 0\rangle$ has the same 180° rotation quantum number as $|\theta_i = 0\rangle_{mean}$. Therefore $|\theta_i = 0\rangle$, as well as $\tilde{\Phi}[\tilde{a}^i]$, is even under 180° rotation.

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FIGURE CAPTIONS

Figure 1: The torus parametrized by $z = x + iy$.

Figure 2: The torus parametrized by θ_i (and θ_i' in (3.15)).

Figure 3: The moduli space of $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$.

Figure 4: The two Hamiltonian $H_{\tilde{\tau}}$ and $H_{\tilde{\tau}+1}$ are equivalent under the transformation (4.4a). (a) represents the Hamiltonian with $\tilde{\tau} = i$ and (b) with $\tilde{\tau} = i + 1$. $J_{ij} = J_1$ on the links represented by solid lines. $J_{ij} = J_2$ on the links represented by doted lines. The Hamiltonians represented by (a) and (b) can be continuously deformed into each other by changing $\tilde{\tau}$ from i to $i + 1$.

Figure 5: The parallel transportations in θ-space. $W_i = W_i(\tilde{\tau})$ and $W_2' = W_2(\tilde{\tau} + 1)$.