STAGE 2

Specialist Mathematics

CURRICULUM STATEMENT 2007

SSABSA
SENIOR SECONDARY ASSESSMENT BOARD OF SOUTH AUSTRALIA

SACE
SOUTH AUSTRALIAN CERTIFICATE OF EDUCATION

EQUITY STATEMENT

This curriculum statement has been accredited by the Senior Secondary Assessment Board of South Australia and is consistent with equal opportunity and human rights legislation.

Each curriculum statement is constructed using the principles of the SSABSA Curriculum and Assessment Policy which identify the student as the centre of the teaching, learning, and assessment processes within the SACE. Inclusivity and flexibility are guiding standards that SSABSA uses in determining curriculum and assessment practices that support students in achieving the requirements of the SACE.

SACE STUDENT QUALITIES

It is intended that a student who completes the SACE will:

1. be an active, confident participant in the learning process (*confidence*).
2. take responsibility for his or her own learning and training (*responsibility*, *self-direction*).
3. respond to challenging learning opportunities, pursue excellence, and achieve in a diverse range of learning and training situations (*challenge*, *excellence*, *achievement*).
4. work and learn individually and with others in and beyond school to achieve personal or team goals (*independence*, *collaboration*, *identity*).
5. apply logical, critical, and innovative thinking to a range of problems and ideas (*thinking*, *enterprise*, *problem-solving*, *future*).
6. use language effectively to engage with the cultural and intellectual ideas of others (*communication*, *literacy*).
7. select, integrate, and apply numerical and spatial concepts and techniques (*numeracy*).
8. be a competent, creative, and critical user of information and communication technologies (*information technology*).
9. have the skills and capabilities required for effective local and global citizenship, including a concern for others (*citizenship*, *interdependence*, *responsibility towards the environment*, *responsibility towards others*).
10. have positive attitudes towards further education and training, employment, and lifelong learning (*lifelong learning*).

ESSENTIAL LEARNINGS AND KEY COMPETENCIES

This curriculum statement offers a number of opportunities to develop essential learnings and key competencies as students engage with their learning. Curriculum support materials, with examples of how these can be developed while students are undertaking programs of study in this subject, will be available on the SSABSA website (www.ssabsa.sa.edu.au).

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CONTENTS

1-unit and 2-unit Subjects

A 1-unit subject consists of 50 to 60 hours of programmed time. It is normally considered to be a one-semester or half-year subject. A 2-unit subject consists of 100 to 120 hours of programmed time. It is considered to be a full-year subject.

Accreditation

This curriculum statement was accredited by the Board from 2003. This accreditation is effective until further notice.

SSABSA informs schools of changes and amendments approved during the period of accreditation. Refer to the curriculum statement on the SSABSA website (www.ssabsa.sa.edu.au) for future changes.

RATIONALE

Mathematics is an integral part of the education and training of all students. It is an important component of the senior secondary studies of many students as they complete their schooling and prepare for further study, training, or work, and for personal and community life.

Mathematics is a diverse and growing field of human endeavour. Mathematics can make a unique contribution to the understanding and functioning of our complex society. By facilitating the current and new technologies and institutional structures, mathematics plays a critical role in shaping that society.

In a time of major change, nations, states, and their citizens are having to operate successfully in an emerging global, knowledge-based economy. Major social, cultural, and environmental changes are occurring simultaneously with changing commercial relationships, new computing and communications technologies, and new sciences such as biotechnology and nanotechnology. Mathematics plays an important part in all of these.

By virtue of mathematics, computers can offer a wide and ever-changing variety of services to individuals and enterprises. It is therefore important that citizens have confidence in their mathematical abilities to understand the services offered and make informed judgments about them.

It is equally important that students have the opportunity to gain the grasp of mathematics that will allow them to be designers of the future, and leaders in various fields. They may be involved in product design, industrial design, production design, engineering design, or the design of new financial and commercial instruments.

The same considerations apply to the new sciences and the new technologies that they support. As systems for information-searching, data-handling, security, genetic design, molecular design, and smart systems in the home and at work, become more and more sophisticated, users will need to have a basic facility with mathematics and the designers of such technologies will need to have increasing understanding of mathematics.

The elements of traditional mathematics — arithmetic, geometry, algebra, calculus, and probability — are involved in the creation and understanding of technologies. It is important that students understand these elements in the context of current and emerging technologies because they transform our understanding of mathematics. Statistics and statistical arguments abound in the mass media and other sources of information, and many students will use statistics in their chosen careers. It is vital that citizens, as they make decisions that affect their lives and future, are able to interpret and question claims of advertisers and other advocates.

Computers and graphics calculators allow a whole new range of mathematical experience to be brought into the classroom through, for example, interactive geometry and drawing tools, techniques of numerical computation, and the analysis of graphs. Some concepts and calculations that were once too difficult to describe or implement can now be demonstrated relatively easily. Technology opens up new applications, allows new explanations, and provides new learning environments for the development of mathematical understanding and confidence.

All students, regardless of gender or background, should have access to mathematical opportunities that accommodate and extend their experiences, broaden their perspective on mathematics, and allow them to appreciate the variety of its past and present roles in society.

ADVICE FOR STUDENTS

Specialist Mathematics will enable students to experience and understand mathematics as a growing body of knowledge for creative use in application to an external environment — a view of mathematics that students are likely to find relevant to their world. This subject deals with phenomena from the students' common experiences, as well as from scientific, professional, and social contexts.

Students can gain from Specialist Mathematics the insight, understanding, knowledge, and skills to follow pathways that will lead them to become designers and makers of technology. The subject will provide pathways into university courses in mathematical sciences, engineering, computer science, physical sciences, and surveying. Students envisaging careers in other related fields, including economics and commerce, may also benefit from studying this subject.

For accurate information about courses, prerequisites, and assumed knowledge, students should consult current publications from the institutions or providers and the South Australian Tertiary Admissions Centre.

In recognition that much of mathematics is conducted through the medium of electronic technology, graphics calculators and/or computers will be used routinely in this subject, and learning activities will be constructed in such a way that students will acquire technological skills concurrently with the relevant traditional skills.

Assessment in Specialist Mathematics consists of two school-based components, together weighted at 50% (skills and applications tasks, and directed investigation(s)), and an external examination component weighted at 50%.

Students who undertake this subject should have successfully completed the following topics in Stage 1 Mathematics:

- Topic 8: Geometry and Mensuration, Subtopics 8.3 and 8.4
- Topic 9: Models of Growth, Subtopics 9.1 to 9.6
- Topic 10: Quadratic and Other Polynomials, all subtopics
- Topic 11: Coordinate Geometry, Subtopics 11.1 to 11.6
- Topic 12: Functions and Graphs, Subtopics 12.4 and 12.5
- Topic 13: Planar Geometry, all subtopics
- Topic 14: Periodic Phenomena, all subtopics.

This subject is designed to be taken in conjunction with Stage 2 Mathematical Studies.

Student Research

When conducting research, in both the school and the wider community, students must be aware that their actions have the potential to affect other people positively or negatively. SSABSA's policy on students as researchers sets out SSABSA's commitment to supporting students in ethical research. Students who are conducting research should follow SSABSA's Guidelines on Conducting Ethical Research for the SACE. See the SSABSA website (www.ssabsa.sa.edu.au).

SACE Classification

For the purposes of SACE completion, Specialist Mathematics is classified as a Group 2 subject.

GOALS

This subject is designed to develop students':

- confidence with mathematical concepts and relationships, and use of mathematical skills and techniques in a range of contexts;
- appreciation of the power, applicability, and elegance of mathematics in analysing, investigating, modelling, and describing aspects of the world;
- facility with mathematical language in communicating ideas and reasoning;
- skills of problem-solving and abstract thinking;
- appreciation of the importance of electronic technology in mathematics;
- mathematical knowledge and skills so that they may become informed citizens capable of making sound decisions in the world of work and in their personal environments.

STRANDS

The study of Specialist Mathematics is described in the following six strands:

- Exploring, Analysing, and Modelling Data
- Measurement
- Number
- Patterns and Algebraic Reasoning
- Spatial Sense and Geometric Reasoning
- Analysing and Modelling Change.

Exploring, Analysing, and Modelling Data

This strand deals with ways in which dynamic situations generate data, which are investigated with new mathematical tools such as vectors, alternative coordinate forms, and differential equations. Students reach and present appropriate conclusions to mathematical questions based on data that change with time, and model a variety of consequent phenomena. Students' means of enquiry and predictions from time-dependent data are enhanced as they seek appropriate resolutions to dynamic problems.

Measurement

This strand deals with additional insights into the theme of 'measures', including the modulus and argument of a complex number, angles between lines and between planes, and volumes represented as scalar triple products. Students address matters of practical importance in the everyday world.

Number

This strand deals with new understandings of number and the structure of number in different contexts. The arithmetic of surds is extended to reintroduce complex numbers, and the geometry of the real number line is extended to the plane geometry of the complex numbers. The complex numbers are studied as the ultimate system for solving polynomial equations, and as the number system that unites models of growth with periodic phenomena. As an alternative to simple coordinates, the ordered pair of numbers is used to represent vectors or complex numbers. Students extend their understanding of the nature and structure of number systems, in relation to the solutions of equations, and use numbers in new ways to represent and operate with entities such as vectors.

Patterns and Algebraic Reasoning

This strand deals with functions that describe change, interconnecting data-fitting with graphical and algebraic modelling and leading to a study of calculus. The function and calculus work is extended to include a description and analysis of ‘motion’. Students fit real data and information with an apt mathematical description in order to conjecture and predict how various changes might affect a dynamic situation.

Spatial Sense and Geometric Reasoning

This strand deals with visual learning and conceptual understanding, and the development of skills appropriate to the study of dynamic situations, using vectors and differential equations. Geometric reasoning has a major role in this strand. Students extend their capacity to understand and analyse events that change with time by exploring, conjecturing, examining, and validating geometric, algebraic, and dynamic relationships in several different ways and in different contexts.

Analysing and Modelling Change

This strand deals with applications of elementary calculus, alternative representations of functions, vector geometry, and differential equations. Students use graphics calculators and/or computers to examine mathematical relationships visually, dynamically, and algebraically.

LEARNING OUTCOMES

At the end of the program in Specialist Mathematics, students should be able to:

1. practise mathematics by analysing data and any other relevant information elicited from the study of situations taken from social, scientific, economic, or historical contexts;
2. understand fundamental concepts, demonstrate mathematical skills, and apply mathematical procedures in routine and non-routine contexts;
3. think mathematically through enquiry, evaluation, and proof;
4. make informed and critical use of electronic technology to provide numerical results and graphical representations, and to refine and extend mathematical knowledge;
5. communicate mathematically, and present mathematical information in a variety of ways;
6. work both individually and cooperatively in planning, organising, and carrying out mathematical activities.

STRUCTURE AND ORGANISATION

This 2-unit subject consists of the following five topics, with the approximate allocation of programmed time shown:

• Topic 1: Trigonometric Preliminaries	6 hours
• Topic 2: Polynomials and Complex Numbers	24 hours
• Topic 3: Vectors and Geometry	24 hours
• Topic 4: Calculus	26 hours
• Topic 5: Differential Equations	20 hours.

Approximately 4 hours of programmed time should be spent on the directed investigation(s), which is one of the assessment components. Additional programmed time should be allocated to consolidation and 6 to 8 hours to skills and applications tasks.

Each topic consists of a number of subtopics. The subtopics are presented in the 'Scope' section in two columns, as a series of key questions and key ideas side-by-side with considerations for developing teaching and learning strategies.

A problems-based approach is integral to the development of the mathematical models and associated key ideas in each topic. Through key questions teachers can develop the key concepts and processes that relate to the mathematical models required to address the problems posed. The considerations for developing teaching and learning strategies present suitable problems for consideration, and guidelines for sequencing the development of the ideas. They also give an indication of the depth of treatment and emphases required. It is hoped that this form of presentation will help teachers to convey the concepts and processes to their students in relevant social contexts.

The key questions and key ideas cover the prescribed areas for teaching, learning, and assessment in this subject. It is important to note that the considerations for developing teaching and learning strategies are provided as a guide only. Although the material for the external examination will be based on the key questions and key ideas outlined in the five topics, the applications described in the considerations for developing teaching and learning strategies may provide useful contexts for examination questions.

SCOPE

Mathematics is a key enabling science for the technologies that are driving the new global economy. Much of the power of computers derives from their ability, in the hands of mathematically knowledgeable people, to harness the subject in new and creative ways. Specialist Mathematics presents three traditional topics, complex numbers, vectors and geometry, and the calculus of trigonometric functions, in a way that promotes their fundamental concepts as a paradigm for models of interacting quantities. The aim is to provide students with an appreciation of certain mathematical ideas that are both elegant and profound, and at the same time to allow them to understand how this kind of mathematics enables computers to model, for example, chemical, biological, economic, and climatic systems.

Specialist Mathematics will develop ideas that are new to the student, and give a new emphasis to familiar ones, by featuring the modelling capabilities of the topics presented.

First among these is the idea that functions describe time-varying quantities. The trace on a heart-monitoring machine and a seismograph record the history of the values of a time-varying quantity, look like the graph of a function, and can be interpreted in such a way. By this interpretation the derivative of the function describes the instantaneous rate of change of the values of the quantity. Students therefore acquire an interpretation of derivatives in addition to the usual ‘slope of the tangent’.

The trigonometric functions exemplify this interpretation. Their treatment in Stage 1 is reviewed, as they are reintroduced as time-varying quantities associated with a point moving round a circle of unit radius at unit speed. Trigonometric functions are heavily emphasised because of their overwhelming application and importance as basic models of cyclical phenomena.

A second idea in Specialist Mathematics is that when several quantities need to be considered simultaneously they should be regarded as properties of a single entity of some kind. For example, pairs of quantities can be regarded as the Cartesian coordinates of a single point, or the components of its position vector, or the real and imaginary parts of a single complex number. A pair of time-varying quantities can therefore be regarded as a moving point, moving vector, or moving complex number, and each interpretation brings particular insights. The rates of change of a pair of quantities can equally well be regarded as the components of a vector or the real and imaginary parts of a complex number. The vector in this case describes the velocity of the moving point.

The third idea in Specialist Mathematics builds on the first two by treating cosine and sine as a single entity, rather than separately. When this approach is adapted to the calculus of cosine and sine, it is the pattern $x' = -y,\ y' = x$ satisfied by the pair of functions $x(t) = \cos t,\ y(t) = \sin t$ that is important, treated as a relationship between the velocity $[x', y']$ of a moving point and its position vector $[x, y]$.

It expresses the fact that the velocity of the circular motion defining cosine and sine is the rotation of the radius vector anticlockwise through a right angle. For complex numbers this rotation is implemented by multiplication by i, and the identity $i(x+iy) = -y+ix$ corresponds to the appearance of $\left[-y, x\right]$ in the pattern for the derivatives of cosine and sine. This indicates some of the interrelationships that Specialist Mathematics seeks to bring out by its particular approach to the three topics.

The immense modelling capability of mathematics derives from the ability to determine the complete time behaviour of a collection of quantities from laws by which they interact or influence each other. When treated as functions, and their rates of change by derivatives, these laws are called ‘differential equations’.

The calculus of cosine and sine provides a fundamental example of this modelling capability. Moreover, their law of interaction $x' = -y,\ y' = x$ is not merely derivable from circular motion, it is characteristic of it. Any *two* quantities that interact by this law — whether two enzyme concentrations, or voltage and current, or velocity and displacement — can be represented by a point undergoing circular motion, and their time behaviour described using cosine and sine. The treatment of the calculus of trigonometric functions in Specialist Mathematics is designed to culminate in, and provide access to, this ubiquitous and useful model of basic cyclical behaviour and its simple variants. This is the purpose of the last topic, ‘Differential Equations’. This topic also highlights a review of the fundamental theorem of calculus from the point of view of differential equations, intended to deepen students’ perspective on indefinite integrals, and an explanation of the logistic functions that were introduced in Mathematical Studies, now derived from a simple law of evolution for self-interacting species.

The focus on new ideas and perspectives in Specialist Mathematics does not detract from the importance of the basic material. The arithmetic of complex numbers is developed and their geometric interpretation as an expansion of the number line into a number plane is emphasised. Their fundamental feature, that every polynomial equation has a solution over the complex numbers, is promoted and De Moivre’s theorem is exploited to find *n*th roots. Topic 3: Vectors and Geometry extends the two-dimensional ideas developed in Stage 1 to the study of lines and planes in three dimensions, their intersections, and the angles formed by them. It also extends the study of circular geometry and tangents, and develops vector methods of proof. The calculus of trigonometric functions necessitates a review of the rules of calculus and their operation in this new context, providing the opportunity for increased skill and a deeper appreciation of how they work.

Specialist Mathematics puts students in a position to understand what lies behind computer models of two interacting quantities, represented as a point moving on a computer screen. Students are encouraged to explore the laws of circular motion with suitable computer packages, and to extend their experiments to more complex laws of interaction that pertain to competing biological species, simple enzymatic systems, or electrical feedback.

By these explorations, Specialist Mathematics opens up pathways to a variety of ‘mathematics futures’ including: geometry and vector methods beyond two and three dimensions to provide models of systems with many degrees of freedom; the general theory of differential equations, chaos theory, and dynamic systems to analyse the consequences of more complex laws of interaction. Students who are inspired to consider ideas such as these and become familiar with this kind of mathematics will be well placed to contribute to the new sciences and technologies, and to win a place for themselves in the new global economy.

TOPIC 1: TRIGONOMETRIC PRELIMINARIES

Subtopic 1.1: Graphs of Trigonometric Functions

Key Questions and Key Ideas

What trigonometric knowledge is needed?

Sine and cosine function values are the *x* and *y* coordinates of a moving point on the unit circle,

$$\sin^2(t) + \cos^2(t) = 1$$

Algebraic identities satisfied by trigonometric functions and derived from the symmetry of the unit circle to establish special cases:

- $\cos(2\pi + t) = \cos t$
- $\sin(2\pi + t) = \sin t$
- $\cos(\pi + t) = -\cos t$, $\sin(\pi + t) = -\sin t$
- $\cos(-t) = \cos t$
- $\sin(-t) = -\sin t$

and consequent identities such as

- $\sin(\pi - t) = \sin t$
- $\cos(\pi - t) = -\cos t$
- $\sin(\frac{\pi}{2} + t) = \cos t$
- $\cos(\frac{\pi}{2} + t) = -\sin t$
- $\sin(\frac{\pi}{2} - t) = \cos t$
- $\cos(\frac{\pi}{2} - t) = \sin t$.

Definition of odd function and even function

A brief treatment of the tangent function, including its period

Considerations for Developing Teaching and Learning Strategies

Students would review the definition of trigonometric functions as developed in Stage 1 Mathematics, Subtopics 14.4 and 14.5 (sine function), 14.6 and 14.7 (cosine function), and 14.8 (tangent function), and the concepts of radian measure and arc length. The *x* and *y* coordinates of the moving point are considered as functions of the arc length *t*. (Note that this is a radian measure of the angle subtended by the arc at the centre of the circle.)

Students can derive these symmetry properties from symmetry operations on the model of a point moving round the unit circle, and interpret them as properties of the graphs of cosine and sine.

It can be seen from the identities that cosine is an even function and sine is an odd function. This is compared with instances of these properties for polynomial functions and possibly the absolute value function.

Subtopic 1.2: Properties of Trigonometric Graphs

Key Questions and Key Ideas

How can transformations be represented trigonometrically?

Horizontal and vertical dilations, translations, and/or reflections, and their effect on trigonometric functions and their graphs, including a brief treatment of all three reciprocal functions: cosec x, sec x, cot x

Considerations for Developing Teaching and Learning Strategies

Exploration, through the use of graphing technology and interactive geometry, of the effect of:

- vertical dilations on the equation and amplitude of trigonometric functions;
- horizontal dilations on the equation and period of trigonometric functions;
- vertical translations on the equation and range of trigonometric functions;
- horizontal translations on the equation and range of trigonometric functions;
- reflection in the x-axis on the equation and graph of trigonometric functions.

The graphs and periodicity of the reciprocal trigonometric functions can also be considered using electronic technology.

Subtopic 1.3: Trigonometric Identities

Key Questions and Key Ideas

What trigonometric identities for sine and cosine functions are used?

- Addition formulae
- Double angle formulae.

Considerations for Developing Teaching and Learning Strategies

Students derive the formula for $\cos(a - b)$ by the dot product (see Subtopic 13.9 of Stage 1 Mathematics). Students compare this derivation with the approach using the cosine rule (see Subtopic 14.7 of Stage 1 Mathematics). The other addition formulae are derived using the properties established in Subtopic 1.1.

The identity

$$\sin A - \sin B = 2\cos\frac{(A+B)}{2}\ \sin\frac{(A-B)}{2}$$

could be considered at this point for later use in Topic 4: Calculus.

TOPIC 2: POLYNOMIALS AND COMPLEX NUMBERS

Subtopic 2.1: Complex Numbers

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Why were complex numbers 'invented'?	Revision of the introduction of complex numbers from Subtopic 10.1 of Stage 1 Mathematics, in order to be able to describe solutions to all quadratic equations with real coefficients. (Students note that the sum and product of the roots of a real polynomial are real.) Analogy can be drawn with the extension of the natural numbers to the integers, integers to rationals, rationals to reals — in each case with a view to ensuring that certain kinds of equations have solutions. A parallel is drawn with the arithmetic of surds and how surds arise — for example, solving the quadratic equation $x^2 + 2x - 1 = 0$ leads to numbers of the form $\{a + b\sqrt{2}: a, b \text{ rational}\}$.

Subtopic 2.2: Complex Conjugation

Key Questions and Key Ideas

Calculating with complex numbers and their conjugates

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{and} \quad \overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$$

Considerations for Developing Teaching and Learning Strategies

Revision of addition, multiplication, and division of complex numbers from Subtopic 10.1 of Stage 1 Mathematics.

Since the emphasis in this topic as a whole is intended to be on the geometric representation of complex numbers, this revision of basic operations is as brief as possible. Teachers may wish to introduce the Cartesian plane representation as detailed in Subtopic 2.4 simultaneously with the revision.

Subtopic 2.3: Inductive Argument

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
How are the conjugates of sum and product of any number of complex numbers found?	Students must interpret, with emphasis on inductive argument, statements such as $\overline{z_1 + z_2 + z_3 + \ldots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \ldots + \overline{z_n}$ and $\overline{z_1.z_2.z_3.z_n} = \overline{z_1}\,.\,\overline{z_2}\,.\,\overline{z_3}\ \ldots\ \overline{z_n}$ as infinite lists of statements labelled by $n = 1, 2, 3, \ldots$ From this perspective they should consider strategies for deriving any statement in the list from the previous one.
Examples, in this context, using induction as a method of argument to establish generalisations	Induction is used to establish generalisations of the triangle inequality, that the modulus of a product is the product of the moduli, and to prove De Moivre's theorem (this subtopic and Subtopics 2.4, 2.5, 2.6, and 2.7). There are some examples of this type of argument later in this curriculum statement. There are also examples in connection with the derivative of a sum of n functions being the sum of the derivatives, and the general formula for the derivative of x^n (see Subtopic 2.7 of Mathematical Studies). Note the special case of the product formula $\overline{z^n} = \overline{z}^n$ which arises when all the zs are equal. The following example provides two possible approaches to an inductive argument: Let $f(x) = \ln x$. Use an inductive argument to prove that, if the nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$, then $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ for $n = 1, 2, \ldots$ Solution: First check the truth of the statement for the first value of n: $f'(x) = \frac{1}{x} = x^{-1} = (-1)^0(0!)x^{-1}$, as required.

Key Questions and Key Ideas

Considerations for Developing Teaching and Learning Strategies

For the next step,

either

- assume $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$, and then differentiate again to get
  $f^{(n+1)}(x) = (-1)^{n-1}(n-1)!(-n)x^{-n-1} = (-1)^n n!\,x^{-(n+1)}$
  as required,

or

- differentiate $f'(x)$ to get
  $f''(x) = -x^{-2} = (-1)^1 1!\,x^{-2}$,
  then again,
  $f^{(3)}(x) = (-1)^1 1!(-2)x^{-3} = (-1)^2 2!\,x^{-3}$,
  $f^{(4)}(x) = (-1)^2 2!(-3)x^{-4} = (-1)^3 3!\,x^{-4}$
  and so on, with the power of (–1) and the size of the factorial increasing, and the power of *x* decreasing, at each step. The pattern is described by the formula $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$, as required.

Subtopic 2.4: The Complex Number Plane

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
How can complex numbers be represented geometrically? • Cartesian form • Conjugate • Modulus.	The Cartesian plane as extension of the real number line to two dimensions. Correspondence between the complex number $a + bi$, the coordinates (a, b), and the vector $[a, b]$. It is noted that complex number addition corresponds to vector addition via the parallelogram rule (Subtopic 13.7 of Stage 1 Mathematics). To this extent complex numbers can be regarded as two-dimensional vectors with rules for multiplication and division to supplement addition and subtraction.
Geometric notion of $\|z - w\|$ as the distance between points in the plane representing complex numbers	Relative positions of z and its conjugate; their sum is real and difference purely imaginary.
Triangle inequality	The triangle inequality is a content thread that also appears in other parts of this curriculum statement, and the related ideas should be noted. Extension to the sum of several complex numbers can be argued by induction as noted above.

Subtopic 2.5: Polar Form

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Is the Cartesian form always the most convenient representation for: • describing sets of points in the plane? • multiplying and dividing complex numbers?	
The properties $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ and $\text{cis}\,(\theta_1 + \theta_2) = \text{cis}\ \theta_1 \,.\, \text{cis}\ \theta_2$. They are the basis on which multiplication by $r\ \text{cis}\ \theta$ is interpreted as dilation by r and rotation by θ.	These properties make polar form the most powerful representation for dealing with multiplication. Students observe that the real and imaginary parts of the identity $\text{cis}\,(\theta_1 + \theta_2) = \text{cis}\ \theta_1 \,.\, \text{cis}\ \theta_2$ are the addition of angles formulae for cosine and sine. Thus complex multiplication encodes these trigonometric identities in a remarkable and simple way. The geometric significance of multiplication and division as dilation and rotation should be emphasised, as should the geometric interpretation of modulus as distance from the origin. Geometry from Stage 1 can be used — for example, using properties of a rhombus to determine the polar form of $z + 1$ from that of z.
Multiplication by i as anticlockwise rotation through a right angle	Although this follows from the addition formula for $\text{cis}\,\theta$, it should also be demonstrated directly: $i(x+iy) = -y + ix$ shows that multiplication by i sends (x, y) to $(-y, x)$, which can be shown to be the effect on the coordinates of rotating a point about the origin anticlockwise through a right angle.
Conversion between Cartesian form and polar form	The conversions $x = r\cos\theta,\ y = r\sin\theta$, where $r = \sqrt{x^2+y^2}$ and $\tan\theta = \left(\dfrac{y}{x}\right)$, along with $\cos\theta = \left(\dfrac{x}{\sqrt{x^2+y^2}}\right)$ and $\sin\theta = \left(\dfrac{y}{\sqrt{x^2+y^2}}\right)$, and their use in converting between Cartesian form and polar form. Calculators can be used, both to check calculations and to enable students to consider examples that are not feasible by hand.

Key Questions and Key Ideas	**Considerations for Developing Teaching and Learning Strategies**
Graphical interpretation and solution of equations describing circles, lines, rays, and simple spirals, and inequalities describing associated regions. Obtaining equivalent Cartesian equations and inequalities where appropriate.	Graphical solution of equations and inequalities such as $\|z\| < 2$, $\|z + i\| = 3$, $\|z + i\| = \|z - 1\|$, $\theta = \pi/4$, $r = \theta$, $0 \le \mathrm{Im}(z) < 1$, strengthening geometric interpretation. An important aim is the conversion of such equations and inequalities into Cartesian form in the case of circles and lines (as a link with work in other subtopics), through geometric understanding of the descriptions used above directly in terms of modulus and argument. Dynamic geometry software can be used as an aid. Students could investigate various polar graphs using graphing technology.
The utility of the polar form in calculating powers of complex numbers	The properties $\| z_1 z_2 \ldots z_n \| = \|z_1\|\|z_2\| \ldots \|z_n\|$ and $\mathrm{cis}\,(\theta_1 + \theta_2 + \theta_n) = \mathrm{cis}\ \theta_1 \,.\, \mathrm{cis}\ \theta_2 \ldots \mathrm{cis}\ \theta_n$ should be argued by induction, as noted above.
• De Moivre's theorem for positive integral n (argument by induction)	Note also the special case $\| z^n \| = \| z \|^n$ when all the zs are equal, and $\mathrm{cis}\,(n\,\theta) = (\mathrm{cis}\ \theta)^n$ when all the θs are equal, which is De Moivre's theorem for positive integral n.
• Extension to negative integral powers and fractional powers • Solution of $z^n = c$ (c real or purely imaginary), in particular the case $c = 1$.	Extension to negative powers via $$\mathrm{cis}\,(-n\theta) = \frac{1}{\mathrm{cis}\,(n\theta)} = \frac{1}{(\mathrm{cis}\,\theta)^n} = (\mathrm{cis}\,\theta)^{-n}$$ and to fractional powers via $$\left(\mathrm{cis}\left(\frac{\theta}{q}\right)\right)^q = \mathrm{cis}\,\theta = \mathrm{cis}\,(\theta + 2n\pi)\,.$$ As a particular example of the use of De Moivre's theorem, the symmetric disposition of the nth roots of unity in the complex plane should be appreciated. The fact that their sum is zero can be linked in the vector section of the curriculum statement to the construction of a regular n-gon.

Subtopic 2.6: An Application — Quadratic Iteration

Key Questions and Key Ideas

How do simple quadratic iterations on complex numbers affect position in the plane?

Position relative to the unit circle

- Iterating the transformation $z \longrightarrow z^2$, and the effect on the final result for values of z with:

  $|z| < 1$

  $|z| = 1$

  $|z| > 1$.

Iterating the transformation

$$z \longrightarrow z^2 + c$$

- Introduction of the complex constant c into the transformation above has a marked effect on the iteration process. Behaviours to note are:

  invariant points (one-cycles)
  cycles
  convergence
  divergence
  chaotic behaviour.

  The criterion for divergence (escape criterion) of the iteration $z_{n+1} = z_n{}^2 + c$ with $z_0 = 0 : |z| > 2$ for some n

Considerations for Developing Teaching and Learning Strategies

This key question involves considering the geometric and algebraic properties of complex numbers, and examining the effect of repeated operations (iteration) on the position of the final result in the Cartesian plane. The use of interactive geometry, spreadsheets, and/or a graphics calculator are possible strategies for supporting concept development here.

Most students at this level would appreciate the effect of repeated squarings on a real number. The key ideas involved allow exploration of such concepts on the field of complex numbers. Students could explore the effect of repeated squarings algebraically, using Cartesian form for the first few results and De Moivre’s theorem for the higher powers. The use of electronic technology would visually and numerically support these explorations, allowing students to see a sequence of results for a range of complex numbers. The effect of the process on the modulus and argument of the results and the difference in behaviour for points on, in, and outside the unit circle are pertinent observations here.

Pen-and-paper manipulation for simple examples of invariant points, as well as two-cycles and three-cycles, could be followed by further investigations into these and other properties with the aid of electronic technology.

Use of one of the commonly available Mandelbrot software programs would be a good starting-point for this whole subtopic. After the Mandelbrot set has been explored visually, a more analytical exploration could be made, using electronic technology to plot orbit diagrams. In this way the Mandelbrot set could be explored, with students noting the different behaviour inside, near, and outside the chaotic boundary of the set. The relevance of the circle $|z| = 2$ and the role of the lobes in the Mandelbrot set could also be explored in this way.

Students who want to investigate further examples of complex iteration could look at Julia sets and/or the fractal produced by iterating the so-called ‘Newton’s method transformation’,

$$z \longrightarrow \frac{1}{2}\left(z + \frac{c}{z}\right).$$

Subtopic 2.7: Fundamental Theorem of Algebra

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Operations on real polynomials: polynomials can be added and multiplied What can be said about division of polynomials? (It is sufficient to consider polynomials of degree ≤ 4.)	Review of the multiplication process. The division algorithm for polynomials. Polynomial long division technique. The use of undetermined coefficients and equating coefficients in factoring when one factor is given.
• Roots, zeros, factors • Remainder theorem, with proof; its use in verifying zeros	The correspondence between the roots of a polynomial equation, the zero of a polynomial, and the linear factor of a polynomial should be understood.
• Factorisation of cubics and quartics with real coefficients (given a zero), using long division or the multiplication process/inspection. With the introduction of complex numbers, all real polynomials have a factorisation into linear factors; this is the statement of the fundamental theorem of algebra	The existence of a formula for the roots of a cubic and a quartic, and some of the associated history, could be mentioned as background. Real polynomials can be factored into real linear and quadratic factors, and into linear factors with the use of complex numbers. The connection between the zeros and the shape and position of the graph of a polynomial should be explored. There should be many opportunities to make use of graphing technology. Special examples: $x^n = 1$ or -1 as solved above by De Moivre's theorem; the factorisation of $x^3 \pm a^3$, $x^4 \pm a^4$, and so on, as an illustration of the use of the remainder theorem. The statement of the fundamental theorem can be considered to answer the question 'Why were complex numbers "invented"?'. Though not every real polynomial of degree n has n real zeros, in the field of complex numbers every real or complex polynomial of degree n has exactly n zeros (counting multiplicity).

TOPIC 3: VECTORS AND GEOMETRY

Subtopic 3.1: Algebraic and Geometric Treatments of Three-dimensional Vectors

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
What is a vector?	Revision of vectors as directed line segments (arrows) in space (generalising from the two-dimensional treatment in Stage 1).
What is a parallelogram in three dimensions? Equality of vectors Coordinate systems and position vectors; components	Recall that vectors are equal if they form opposite sides of a parallelogram. The notion of a parallelogram in three dimensions requires a discussion of when the opposite sides of a quadrilateral are coplanar so that it is not twisted; the simplest criterion is that its diagonals intersect. Applications (e.g. navigation and force) as encountered in Stage 1 can readily be extended to three dimensions.
Operations with vectors Addition of vectors, multiplication by scalars, length of vectors: both geometrically and algebraically	Contexts and applications as mentioned above.
Parallel vectors, collinearity, ratio of division of a line segment (internal and external)	Ratio of division is needed in the treatment of Bézier curves (see Topic 4: Calculus).
Can vectors be multiplied together? What is the meaning of the product? Scalar (dot) product and vector (cross) product: their properties and their interpretation in context	These two operations on pairs of vectors provide important geometric information. The scalar and vector products should be treated in terms of coordinates and of length and angle. Conditions for perpendicularity and parallelism, and construction of perpendiculars.
Cross-product calculation using the determinant	Though determinants may not have been encountered previously, this is a suitable place for their introduction, since they arise naturally in the formulae for area and volume. (Note that the area of a parallelogram in two dimensions can be expressed as a 2×2 determinant in terms of the coordinates of the defining vectors.) (Determinants are treated in Subtopic 3.4 of Mathematical Studies.) The cross-product of two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ in three dimensions is a vector, mutually perpendicular to $\boldsymbol{b}$ and to $\boldsymbol{c}$, whose length is the area of the parallelogram determined by $\boldsymbol{b}$ and $\boldsymbol{c}$. The right-hand rule determines its sense. The components and the vector itself may be expressed using determinants. (Applications of scalar and vector products include work and moments.)
When and how is the scalar triple product used? • Volume of a parallelepiped • Test for coplanarity.	Scalar triple product as volume of a parallelepiped, again using determinant notation; test for coplanarity.

Subtopic 3.2: Lines and Planes

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
How is the equation of a line in three dimensions written? • Vector, parametric, and Cartesian forms.	Geometric considerations lead to the vector equation of a line; from this can be derived the parametric form and (less importantly) the Cartesian form. Exercises should highlight the construction of parallel lines, perpendicular lines, and the phenomenon of skewness. Computation of the point of a given line that is closest to a given point; distance between skew lines.
The next most simple subset of three dimensions is a plane. How can a plane be described by an equation? • Cartesian form $ax + by + cz = d$.	The equation of a plane, also developed using geometric ideas as a vector equation from which the Cartesian form $ax + by + cz = d$ is derived.
Relationships between lines and planes	Intersection of a line and a plane, lines parallel to or coincident with planes. Computation of the point of a given plane that is closest to a given point.
Intersections of planes: algebraic and geometric descriptions	Finding the intersection of a set of two or more planes amounts to solving a system of linear equations in three unknowns; this is covered in Mathematical Studies. The geometric interpretation in the case of unique solution, infinite solution, or no solution should be discussed (but may need to be delayed until the end of the course to fit in with the timing of Mathematical Studies Topic 3: Working with Linear Equations and Matrices).
Angles formed by lines and planes	Calculation of the angle between lines, even if they are skew, and of the angle between a line and a plane, and the angle between two planes.

Subtopic 3.3: Geometry of Circles and Tangents

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Circles and their tangents arise often in applications involving motion	Revision of theorems on angles in circles and tangents.
What basic properties should be familiar? • The radius is perpendicular to the tangent at the point of contact • The perpendicular from the centre of a circle to a chord bisects the chord • Tangents from an external point are equal in length • The angle in a semicircle is a right angle • The angle at the centre is twice the angle at the circumference • Angles in the same segment are equal • Alternate segment theorem.	Formal proofs of these theorems are not required, although the informal process of logical justification should be rigorous, with the emphasis on clearly communicating the sound justification of a general rule. (*Note*: Theorems on intersecting chords are not required.)
Cyclic quadrilaterals and concyclic points	
Simple deductive problems involving circles, tangents, and cyclic quadrilaterals	Examples will normally be expected to involve only one circle.

Subtopic 3.4: Vector Methods of Proof

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
The use of vector methods of proof, particularly in establishing parallelism, perpendicularity, and properties of intersections	Students should see some examples of proof by vector methods, and appreciate their power. Suitable examples include: • the angle in a semicircle is a right angle; • medians of a triangle intersect at the centroid. The result: If $k_1\boldsymbol{a} + k_2\boldsymbol{b} = l_1\boldsymbol{a} + l_2\boldsymbol{b}$, then $k_1 = l_1$ and $k_2 = l_2$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

TOPIC 4: CALCULUS

Subtopic 4.1: Functions and Quantities Varying with Time

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
How exactly do functions describe time-varying quantities?	The trace on a heart-monitoring machine and a seismograph record the history of the values of a time-varying quantity, and look like the graph of a function. Students can explore how a seismograph or the trace on a heart-monitoring machine is produced in a way that connects directly with the graph of a function. For example, consider a vertical line moving from left to right at unit speed on a Cartesian plane, and carrying a point whose directed height at any instant gives the value of the quantity of interest. Such a model may well have been introduced to produce the graphs of cosine and sine from the circular motion model (see Subtopic 1.1, or Subtopics 14.4 to 14.7 of Stage 1 Mathematics). Their interpretation as functions of time and the production of their graphs in this fashion are well worth reviewing at this point.
The derivative of a function as the instantaneous rate of change of the time-varying quantity that the function describes	Given a chord on the graph of the function, its 'rise' gives the change in the value of the quantity, and its 'run' gives the interval of time over which the change took place. The gradient of the chord is therefore the average rate of change over this interval of time. The gradient of a tangent, as a limiting value of the gradient of chords, naturally measures the rate of change of the quantity at the instant determined by the point of contact (connecting with Subtopic 2.6 of Mathematical Studies).

Subtopic 4.2: Pairs of Uniformly Varying Quantities and Their Representation as a Moving Point

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
How can a pair of time-varying quantities be modelled as a point moving in a plane?	Given two time-varying quantities, their values at any instant can be interpreted as the Cartesian coordinates $x(t)$ and $y(t)$ of a point moving in the plane.
When the quantities vary uniformly with time, what kind of curve do they trace out? • Coordinate representation (parametric) • Vector interpretation.	Being uniform, the quantities are functions of the form $x(t) = x_0 + at, y(t) = y_0 + bt.$ These describe the parametric equation of a line in two dimensions. This is understood with reference to the vector and parametric equations of lines in three dimensions studied in Subtopic 3.2. The vector form of the two-dimensional equation is $$\left[x_0 + ta, y_0 + tb\right] = \left[x_0\ y_0\right] + t\left[a, b\right].$$ Emphasis is placed on the interpretation of the parameter *t* as time. It is noted that vector $\left[a,b\right]$ gives the change in position vector occurring in each unit of time, and that this is the fundamental notion of velocity. Interpretation of complex numbers (optional). The pair $x(t)$ and $y(t)$ can also be considered the real and imaginary parts of a moving complex number $z(t) = x(t) + iy(t)$. Then $z(t) = z_0 + \alpha t$, where $z_0 = x_0 + iy_0$ and $\alpha = a + ib$. Note that $z(t)$ is increased by the complex number α in each unit of time.
Examples involving pairs of uniformly varying quantities	Students may like to consider examples such as production costs with two components, capital and labour, and a simplistic model of predator–prey relationships, defined by two uniformly varying populations.

Subtopic 4.3: Pairs of Non-uniformly Varying Quantities — Polynomials of Degree 2 and 3

Key Questions and Key Ideas

What curves are traced out by a moving point $(x(t), y(t))$ in which the functions $x(t)$ and $y(t)$ are polynomials of degree 2 and 3?

Examples of applications to:

- objects in free flight;
- Bézier curves;

and equivalent examples should be considered.

Considerations for Developing Teaching and Learning Strategies

The position of an object in free flight is given by the equations

$$x(t) = x_0 + at, y(t) = y_0 + bt - \tfrac{1}{2}gt^2$$

where (x_0, y_0) is the initial position, [*a*, *b*] is the initial velocity, and *g* is the acceleration due to gravity. Students observe the parabolic shape of the curve.

This equation can be used to answer questions such as: ‘At what angle should a basketball be thrown to score a goal?’ or ‘At what angle should a cricket ball be struck to clear the fence?’

Degree 3 polynomials feature in computer-aided design. The use of Bézier curves attempts to mimic freehand drawing using cubic polynomials, and motion along Bézier curves is used to simulate motion in computer animations. They are constructed using four control points, two marking the beginning and end of the curve and two others controlling the shape. They can be constructed with interactive geometry software.
(www.moshplant.com/direct-or/bezier /)

Although the curves can be drawn interactively and intuitively by the designer, an accurate mathematical description is needed for: editing the curve; zooming in by the program; passing the design on to be printed or processed by specialised machining equipment.

Subtopic 4.4: Related Rates, Velocity, and Tangents

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Where two functions of time $x(t)$ and $y(t)$ are related by $y(t) = f(x(t))$, their rates of change are related by the chain rule $y' = \frac{\mathrm{d}f}{\mathrm{d}x} x'$.	Examples of calculating y' from x' and the derivative of *f*. For instance, $x(t)$ might be the length of the side of a square that is changing over time, and $y(t)$ its area. Or they might be the length of the side of a cube and its volume.
The notation for derivatives concerning time is: $$x' = \frac{\mathrm{d}x}{\mathrm{d}t},\ y' = \frac{\mathrm{d}y}{\mathrm{d}t}.$$ For a moving point $(x(t), y(t))$, the vector of derivatives $[x'(t),\ y'(t)]$ is naturally interpreted as its instantaneous velocity	For uniform motion the velocity [*a*, *b*] has for its components the rates of change of the components of the position vector. The idea that $[x'(t),\ y'(t)]$ is interpreted as the instantaneous velocity vector of the moving point $(x(t),\ y(t))$ is a natural extension of this idea, in which average rates of change are replaced by instantaneous rates of change, or derivatives.
	If, given a function $f(x)$ and a function of time $x(t)$, you set $y(t) = f(x(t))$ then the moving point $(x(t), y(t))$ travels along the graph of $f(x)$.
The velocity vector is always tangent to the curve traced out by a moving point	The chain rule $y' = \frac{\mathrm{d}f}{\mathrm{d}x} x'$ shows that the ratio of y' to x' is $\frac{\mathrm{d}f}{\mathrm{d}x}$ and that the velocity vector is tangent to the graph. Since effectively every curve is the graph of a function (by rotation if necessary), the velocity vector is always tangent to the curve traced out by a moving point.
Parametric equations of tangents to parametric curves	At time $t = a$ the moving point passes through $(x(a),\ y(a))$ and the velocity vector $[x'(a),\ y'(a)]$ is tangent at this point to the curve traced out. The parametric line $x(a) + tx'(a),\ y(a) + ty'(a)$ passes through $(x(a),\ y(a))$ at time t = 0 and has velocity $[x'(a),\ y'(a)]$, so it is tangent to the curve at this point. Students can use this formula to calculate the parametric equations of tangents to parametric curves, for example, Bézier curves.
Speed of the moving point as magnitude of velocity vector, that is $$\sqrt{x'^2(t) + y'^2(t)}$$	Calculation of the speed of projectiles or of points moving along Bézier curves.

Subtopic 4.5: Derivatives of the Circular Functions

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Cosine and sine considered as functions of time	Students review from Subtopic 1.1 their understanding of cosine and sine in terms of the circular motion model, this time with an emphasis on the coordinates of the moving point as functions of time.
The derivatives of cosine and sine explained in terms of the relationship $x' = -y,\ y' = x$ derived from circular motion	By inspecting the graphs of cosine and sine, and the gradients of their tangents, students can conjecture that $(\cos t)' = -\sin t$ and $(\sin t)' = \cos t$. Emphasis is placed on the pattern $x' = -y,\ y' = x$ displayed by these formulae and its explanation in terms that the velocity vector $[x', y']$ is obtained from the position vector $[x, y]$ by an anticlockwise rotation through a right angle. This follows because both velocity vector and position vector have length *l*, the motion is anticlockwise, and the velocity vector is tangent to the unit circle and therefore perpendicular to the position vector $[x, y]$ (see Subtopic 3.3). Rotating the vector $[x, y]$ anticlockwise through a right angle produces $[-y, x]$. The geometric explanation of this fact will have been given when interpreting multiplication by the complex number i as anticlockwise rotation through a right angle (see Subtopic 2.5).
Motion round larger and smaller circles	Looking ahead to Subtopic 5.3, students are encouraged at this point to consider the motions $(R\cos t, R\sin t)$, which also satisfy the pattern $x' = -y,\ y' = x$. They describe a point moving round another circle shadowing the point on the unit circle.
Angular velocity, and moving with different speed round the unit circle	The speed of the point is equal to the radius per unit time (i.e. 1 radian). Students might also interpret $(R\cos(t+\varphi), R\sin(t+\varphi))$ as shadowing a different point on the unit circle that is always φ units in advance of the moving point. Students argue geometrically why changing the speed of the point on the unit circle to ω changes the motion to $(\cos\omega t, \sin\omega t)$. On the other hand its velocity should be magnified by a factor ω, so that $x' = -\omega y,\ y' = \omega x$. Students are introduced to the use of the chain rule with cosine and sine by verifying the consistency of these two observations. They explore and interpret the variants $(R\cos(\omega t+\varphi), R\sin(\omega t+\varphi))$.

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
The derivative of sin *t* from first principles	The calculation of the derivatives of cosine and sine through circular motion presumes that these derivatives exist. A first-principles calculation from the limit definition provides a proof that the derivatives exist, and allows students to see the results in the context of the 'gradient of tangent' interpretation of derivatives from which the limit definition is derived. This traditional calculation proceeds by first computing the derivative of sine at zero, which amounts to the limit $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$ which is derived graphically, numerically, and geometrically. The derivatives of the sine and cosine functions in general can be derived from this special case using addition of angles formulae (see Subtopic 1.3).

Subtopic 4.6: Calculus of Trigonometric Functions

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Computing the derivatives of trigonometric functions and functions involving them	Using the rules for differentiation, the derivatives of all the trigonometric functions can be determined as well as composite functions involving the trigonometric functions. Higher order derivatives involving trigonometric functions such as the *n*th derivative of $\sin x$ can be considered to give further examples of argument by mathematical induction.
Applications of the derivative involving: • simple rates of change; • maximum and minimum problems; • further related rates.	The determination of rates of change of quantities such as the: • height of a piston; • voltage at an electrical outlet. The maximum and minimum values in simple problems involving trigonometric functions (e.g. the maximum length of a ladder that can be carried round a corner or in a corridor, or the maximum or minimum height of a tide).
Indefinite integrals and definite integrals of functions of the type $\sin(ax+b)$, $\cos^2 x$, $\sec^2 x$ Extension of the substitution method (covered in Mathematical Studies) to simple examples involving trigonometric functions Applications to the calculations of areas, distance travelled, and so on	The calculation of areas under the graph of trigonometric functions: • using the fundamental theorem of calculus; • with $\sin x^2$ as an example of where numerical methods are required to determine the area.

TOPIC 5: DIFFERENTIAL EQUATIONS

Subtopic 5.1: Introduction to Differential Equations

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
What is a differential equation? A differential equation is an equation that connects the rate of change of an unknown quantity with the values of known quantities, particularly its own	Students review the law of exponential growth (Subtopic 2.12 of Mathematical Studies), interpreting it as stating that the rate of change of a quantity is some fixed multiple of (i.e. proportional to) its own value. Describing the quantity as the function $y(x)$ and the constant of proportion as k, this states that $y'(x) = ky(x)$. Possibilities for $y(x)$ can be described in terms of the exponential function. Students explore the question of what all the possibilities are.
In terms of functions, it is an equation between the derivative of the function, the independent variable, and the function itself	The examples from Subtopic 2.12 of Mathematical Studies, such as population growth, as well as Newton's law of cooling, radioactive decay, and simple capacitor/resistor circuits, are important for emphasising the mathematical modelling role of differential equations and the ability to determine from them the time behaviour of quantities.
Use of the notation y' to denote the derivative of the function y	Note that y' is often used to denote the derivative of *y*, that is, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}s}$, regardless of the label, name, or interpretation of its independent variable as time or whatever else.
Review of integration as an example of solving simple differential equations	Students will gain a much deeper understanding of the fundamental theorem from a review of it in terms of differential equations. The quantity in this case is the area *A* of the strip under the graph of a function f swept out by a line moving from left to right at unit speed, starting from some point $x = a$. The fundamental theorem of calculus states that the rate of change of this quantity at any instant is equal to the height of the leading edge of the strip (i.e. to the function value at that instant). This example may well have been used in Subtopic 2.13 of Mathematical Studies, to motivate study of the fundamental theorem. Integration is the process of finding solutions to the differential equation $A'(x) = f(x)$; these solutions are called 'indefinite integrals'.
Initial conditions and their use in specifying solutions	Students are aware that differential equations rarely have one solution, and usually have many; the fundamental theorem is an appropriate place

Key Questions and Key Ideas	**Considerations for Developing Teaching and Learning Strategies**
	to emphasise the fact. However, because the strip begins at $x = a$, it is known that $A(a) = 0$. This is called an 'initial condition'. Solutions are often determined by initial conditions. In this case, if $F(x)$ satisfies $F'(x) = f(x)$, then the difference between $F(x)$ and $A(x)$ has derivative zero and must be constant.
	Comparing their values at $x = a$, that difference must be $F(a)$. This produces the formula for area $$A(x) = F(x) - F(a)$$ in terms of an indefinite integral $F(x)$.
How can the information about the derivative of a function be described?	An equation $y'(x) = f(x)$ indicates the slope of the graph at each point x but not the value of y. A line of gradient $f(x_o)$ can be drawn at each point on each vertical line $x = x_o$, and one of these is the tangent line to the graph. This family of lines, one through each point in the plane, is called the 'slope field'.
Reconstruct a graph from a slope field both manually and using graphics software	Given an initial value, say, y_o, for y at $x = x_o$, the slope field indicates the direction in which to draw the graph. These ideas should be displayed using graphics calculators or software, and students invited to consider how the computer or graphics calculator might have traced the curves — for example, by following each slope line for a small distance, then following the slope line at the new point.
	These graphical results are compared with known solutions from integration, such as the solutions $y = x^2 + c$ for the equation $y'(x) = 2x$. What role does c play in the geometric picture? What is the key feature of a family of curves of the form $F(x) + c$?
Construct tables of approximate values for the solution numerically using Euler's method	Note that Euler's method is most useful in the many cases when exact solutions cannot be found by integration.

Subtopic 5.2: Separable Differential Equations

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
Separable differential equations • Differential equations expressible in the form $g(y)y' = f(x)$. Examples $y' = ky$ and $y' = k(A - y)$.	The exponential equation $y'(x) = ky(x)$ is the simplest example of a separable differential equation. Other examples such as $y' = k(A - y)$ arise from Newton's law of cooling or as models of the spread of rumours. They are solved in terms of exponential functions. The family of solutions, and the use of initial conditions to determine which one describes a problem, is emphasised.
The logistic differential equation $y' = k(P - y)y$	The logistic function features in Subtopic 2.12 of Mathematical Studies, where it is inferred from data. The differential equation provides an explanation as to why it arises in terms of a law of evolution. *P* represents a total population (of molecules, of organisms, etc.) of which *y* are active or infected and $P - y$ are not. To create more of the 'actives', an active and an inactive must meet. The rate of meeting is proportional to the product of the actives and inactives, and this will determine the rate of increase y' of the actives.
Method of solution of separable differential equations	If *G* is an indefinite integral of *g* and *F* of *f*, then differentiating $G(y) = F(x)$ (using the chain rule on the left-hand side of the equation) reproduces the differential equation. Solving the equation $G(y) = F(x)$ for *y* gives a function that therefore satisfies the equation above, and hence the differential equation. To carry out this method for the logistic differential equation, students will need to check the identity $$\frac{1}{y} + \frac{1}{P - y} = \frac{P}{y(P - y)}.$$

Subtopic 5.3: Interacting Quantities, Systems of Differential Equations, and Cyclical Behaviour

Key Questions and Key Ideas

A system of differential equations: specifying an interaction between two time-varying quantities means specifying the rate of change of each in terms of their two values. Describing the quantities by functions, the derivative of each one is given by a formula in the two functions

Examples should be restricted to the case of two equations for two functions

It is not intended that students have knowledge of the method of solving second order differential equations

Considerations for Developing Teaching and Learning Strategies

This key idea is explained and reinforced by a variety of simple examples from a variety of areas of knowledge such as biology, chemistry, electronics, mechanics, and climate. This is one of the most important points of contact between mathematics and other areas of knowledge, and a critical place where the power of its modelling capabilities is demonstrated by a universal model that applies across a wide range of knowledge. In order to exhibit this, both students and teachers must acquaint themselves with some of the key ideas and elementary quantitative relationships of these other areas of knowledge.

For example, voltage and current are familiar concepts. When a time-varying current I flows through a coil it induces a voltage V across the coil, against the flow and in proportion to its rate of change. This quantitative relationship can be expressed as $-LI' = V$. If the current is produced by discharge from a capacitor then, via another simple quantitative relationship, $-CV' = I$, where C is another constant. These two relationships constitute a law of interaction between, or a system of differential equations for, the voltage and current in a capacitor–coil circuit, the basic device for producing radio signals.

When a spring is stretched or compressed, it experiences a force $-ks$ in proportion to its extension s and in the opposite direction. Its velocity v is the rate of change of this extension so that $s' = v$. If a weight of mass m is attached, then its momentum is mv and, by Newton's second law, $mv' = -ks$. The definition of velocity thus combines with Newton's second law to provide a law of interaction between, or a system of differential equations for, the extension and velocity of the spring, the simplest of oscillatory devices whose variants are the basis of the mechanical watch.

Examples from a variety of areas of knowledge, whether these or others chosen to appeal to the interests of students and teachers, are introduced and discussed.

Key Questions and Key Ideas

The differential equations in the simplest examples have the form

$$x' = -\alpha y,\ y' = \beta x.$$

In the special case where $\alpha = \beta = 1$, it is known that $x(t) = \cos t,\ y(t) = \sin t$ satisfy $x' = -y,\ y' = x$.

What other functions satisfy this case?

The functions

$$x(t) = R\cos(t+\varphi),\quad y(t) = R\sin(t+\varphi)$$

provide a solution that starts at

$$(R\cos\varphi,\ R\sin\varphi).$$

Suitable choice of R and φ to design a solution with prescribed initial values of x and y.

What system of differential equations results from motion round a unit circle at some speed ω, not necessarily equal to 1? What functions might satisfy the system?

On a circle of radius R a point moving according to $x' = -\omega y,\ y' = \omega x$ moves at a speed of ωR or ω radii per unit of time, or ω radians per unit of time

Conversion between 'angular velocity' ω, frequency, and the period of the circular motion

How might solutions of systems $x' = -\alpha y,\ y' = \beta x$ be constructed when α and β are not equal?

Considerations for Developing Teaching and Learning Strategies

It is easy to check that the given functions satisfy the system of differential equations. Students explain how they might have been led to predict this.

The system $x' = -y,\ y' = x$ describes a moving point whose velocity vector is an anticlockwise rotation of its position vector through a right angle. Since its velocity is perpendicular to its position vector, its motion would be expected to be circular. Dilating $(\cos t,\ \sin t)$ by a factor R gives $(R\cos t,\ R\sin t)$, which describes a circular motion on a circle of radius *R*, with speed *R*, and starting on the *x*-axis.

$(R\cos(t+\varphi),\ R\sin(t+\varphi))$ describes the same circular motion but with its clock delayed by φ units of time, so that it starts at $(R\cos\varphi,\ R\sin\varphi)$. In order to start at some prescribed point (a, b), $a = R\cos\varphi,\ b = R\sin\varphi$ is set so that $\tan\varphi = \dfrac{b}{a}$ and $R = \sqrt{a^2 + b^2}$ (this is conversion from Cartesian into polar form, as in Subtopic 2.5).

Another variant on the circular motion that defines cosine and sine is a point travelling round a unit circle at some speed ω, not necessarily equal to 1. The velocity vector $[x',\ y']$ of such a motion is perpendicular to the position vector, and must be some multiple of $[-y,\ x]$. This vector has unit length on the unit circle, but the velocity, which is ω there, must be $\omega[-y,\ x]$ and the system of differential equations must be $x' = -\omega y,\ y' = \omega x$. Speeding up by a factor of ω can be achieved by replacing t with ωt in functions of time. This suggests that $x(t) = R\cos(\omega t+\varphi),\quad y(t) = R\sin(\omega t+\varphi)$ satisfies the system. This is easily confirmed, and such a solution can be designed to have any prescribed starting-point.

To satisfy this system, students design solutions of the form $(A\cos(\omega t+\varphi),\ B\sin(\omega t+\varphi))$ and with a prescribed starting-point. They use graphics software to explore the motion of the point, and recognise that the paths traced out are ellipses.

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
The 'interaction constants' α and β determine the angular velocity ω and the ratio of A to B, which measures the distortion of the ellipse from a circle Derivation of the formulae $\omega=\sqrt{\alpha\beta}$ and $\frac{A}{B}=\sqrt{\frac{\alpha}{\beta}}$	By substituting functions of this form into differential equations, the equations $\omega A=\alpha B,\ \omega B=\beta A$ are obtained. Multiplying them together gives the formula for ω, and substituting this formula into either of them gives the formula for $\frac{A}{B}$. To achieve a prescribed starting-point (a, b) for the motion, the equations $a = A\cos\varphi,\ b = B\sin\varphi$ are solved, using the formula for $\frac{A}{B}$ to eliminate either A or B. Students apply this theory to determine the time behaviour of the systems used to motivate the study of these differential equations, or to determine some of their characteristics, such as the frequency of their oscillation. Thus, in relation to the examples given above: the frequency of the radio signal generated by a given capacitor–coil circuit could be determined, given the values of L (inductance) and C (capacitance); or the entire time behaviour of the voltage and current could be determined, given their starting values. In a Cartesian coordinate system, with axes chosen for voltage and current, graphics software could be used to plot the elliptical motion of the point that represents the pair.
Are the solutions constructed the only ones possible? • There are no other solutions • Any two motions satisfying a given system of differential equations that start at the same point coincide for all time.	This idea is a fitting topic for further work in a directed investigation. Students may be encouraged to see the strength of this idea by noticing that moving points of the form $x(t) = A\cos t - B\sin t,\ y(t) = A\sin t + B\cos t$ satisfy $x' = -y,\ y' = x$. This would mean, however, that they must have the form $x(t) = R\cos\,(t+\varphi),\ y(t) = R\sin\,(t+\varphi)$, which provides another proof of the addition of angles formulae.
Electronic technology is used to explore the motion of points under a given law of interaction by moving the point according to the velocity that the law prescribes for its current position	For two-dimensional systems, computer packages can trace the motion of points under a given law of interaction by moving the point according to the velocity that the law prescribes for its current position. Students experiment with the circular and elliptical motion systems in this subtopic, and extend to more complex systems, noting the new phenomena that can occur (spirals, hyperbolae, limit cycles, etc.).

Key Questions and Key Ideas

Considerations for Developing Teaching and Learning Strategies

In a directed investigation students could develop a sense of the phenomena that current mathematical investigations are seeking to explain, and the kinds of real problems in which they arise by generalising the systems studied in this subtopic to $x' = \alpha x + \beta y,\ \ y' = \gamma x + \delta y$. This would allow for a wide range of behaviour, depending on the choice of coefficients. Allowing cubic expressions will produce limit cycles. Applications of these types of systems can be found in population biology and electronics (Lotka-Volterra and Van der Pol equations).

ASSESSMENT

Assessment is subject to the requirements, policies, and procedures of the Board.

One of the purposes of assessment is to measure the extent to which students have achieved the learning outcomes of a program based on this curriculum statement. The assessment tasks used to determine the SSABSA Subject Achievement Score are summative. Formative tasks are important in the learning process, but do not contribute to final grades.

Assessment in Specialist Mathematics consists of the following components, weighted as shown:

Assessment Component 1: Skills and Applications Tasks (40%)
Assessment Component 2: Directed Investigation Task(s) (10%)
Assessment Component 3: Examination (50%).

Assessment Component 1: Skills and Applications Tasks

This assessment component is designed to assess primarily Learning Outcomes 1 to 5. It is weighted at 40%.

Skills and applications tasks require students to solve mathematical problems that may:

- be routine, analytical, and/or interpretative;
- be posed in familiar and unfamiliar contexts;
- require a discerning use of electronic technology.

In setting skills and applications tasks, teachers may provide the students with information in written format or in the form of numerical data, diagrams, tables, or graphs. The tasks should require the student to demonstrate an understanding of relevant mathematical ideas, facts, and relationships. Students should be able to select appropriate algorithms or techniques and relevant mathematical information to successfully solve routine, analytical, and/or interpretative problems. Some of these problems should be set in a personal, global, or historical context.

Students should be required to provide explanations and arguments, and use notation, terminology, and representation correctly throughout the task. They may be required to use electronic technology appropriately to aid and enhance the solution of some problems.

Skills and applications tasks are to be undertaken under the direct supervision of an invigilator. The total time spent by students on assessment tasks in this assessment component should be between 6 and 8 hours of programmed time.

Criteria for Judging Performance

The student's performance in the skills and applications tasks will be judged by the extent to which he or she demonstrates:

- mathematical skills and understandings (without electronic technology);
- mathematical skills and understandings (with electronic technology);
- analysis and interpretation of results and information;
- the communication of mathematical information;
- the ability to prove conjectures.

Assessment Component 2: Directed Investigation Task(s)

This assessment component is designed to assess all the learning outcomes. It is weighted at 10%.

Students are required to undertake one or more directed investigations totalling approximately 4 hours of programmed time.

A directed investigation is an assessment task that requires students to investigate a mathematical relationship, concept, or problem, which may be set in an applied context. The subject of the directed investigation is usually derived from one or more subtopic(s), although it can also relate to a whole topic or across topic(s).

Typically the teacher provides students with a clear, detailed, and sequential set of instructions for part of the task or to initiate the task. Students are encouraged to demonstrate their knowledge, skills, and understanding in the investigation. The generation of data and the exploration of patterns and structures, or changing parameters, may often provide an important focus. From these, students may recognise patterns or structures and make a conjecture. Notation, terminology, and forms of representation of information gathered or produced, calculations, and results will be important considerations. Students should interpret and justify results, summarise, and draw conclusions. Students should be required to provide appropriate explanations and arguments in a report. A directed investigation may require the use of graphics calculators and/or computer technology. Students can undertake a directed investigation either individually or as part of a group. When the directed investigation is undertaken by a group, each student must make an identifiable contribution to the final product. During a directed investigation, the student's progress is guided and supported by the teacher.

A completed directed investigation should include:

- an introduction that demonstrates an understanding of the features of the problem or situation investigated;
- evidence that the student has followed instructions;
- appropriate representation of information gathered or produced, calculations, and results;
- a summary of results or findings and conclusions drawn.

The mode of presentation of a directed investigation may include:

- a written report or an oral report;
- other multimedia formats.

Criteria for Judging Performance

The student's performance in the directed investigation task(s) will be judged by the extent to which he or she demonstrates:

- mathematical skills and understandings (without electronic technology);
- mathematical skills and understandings (with electronic technology);
- analysis and interpretation of results and information;
- the communication of mathematical information;
- the organisation and presentation of material;
- the ability to prove conjectures;
- the ability to work independently;
- the ability to work cooperatively.

Assessment Component 3: Examination

This assessment component is designed to assess primarily Learning Outcomes 1 to 5. It is weighted at 50%.

The 3-hour external examination will be based on the subtopics and key questions and key ideas outlined in the five topics. The considerations for developing teaching and learning strategies are provided as a guide only, although applications described under this heading may provide useful contexts for examination questions.

The examination will consist of three sections, the first focusing on knowledge and routine skills and applications, the second focusing on more complex questions, and the third focusing on investigative questions. Some questions may require students to interrelate the knowledge, skill, and understanding of some topics. It is also expected that the skills and understanding developed through the directed investigation task(s) will be assessed in the examination.

SSABSA will assume that students will have access to graphics calculators and/or computers during the external examination. It is expected that questions in the examination will be a mixture of the following types:

- Graphics calculator and/or computer *inactive* questions. There is no advantage in using a graphics calculator or computer to answer these questions.
- Graphics calculator and/or computer *neutral* questions. These questions can be solved without a graphics calculator or computer, although the electronic technology may be used.
- Graphics calculator and/or computer *active* questions. These questions require the use of a graphics calculator or computer for their solution.

Students will need to be discerning in their use of electronic technology to solve questions in examinations.

Criteria for Judging Performance

The student’s performance in the examination will be judged by the extent to which he or she demonstrates:

- mathematical skills and understandings (without electronic technology);
- mathematical skills and understandings (with electronic technology);
- analysis and interpretation of results and information;
- the communication of mathematical information;
- the ability to prove conjectures.

MODERATION

Moderation is subject to the requirements, policies, and procedures of the Board. SSABSA publishes the specific moderation requirements annually.

Moderation is a process designed to place different teachers' assessments of their students' performance in the same subject on the same scale so that valid comparisons between performances can be made. The purpose of moderation is to help to ensure fairness to students and to provide the wider community with reliable information about student performance. Moderation is undertaken to ensure that the school-assessed scores or SACE designations given to students who take the subject are comparable from school to school.

Assessment Component 1: Skills and Applications Tasks and Assessment Component 2: Directed Investigation Task(s) will be statistically moderated against Assessment Component 3: Examination.

Assessment Component 3: Examination will be externally marked.

SUPPORT MATERIALS

Useful support materials are available on the SSABSA website (www.ssabsa.sa.edu.au), for example:

- annotated work samples
- assessment exemplars
- assessment plans
- illustrative programs
- performance standards
- resources
- teaching and learning strategies.