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Appendix A

Unit Quaternion Representation of Rotation

Quaternions were first defined by W.R. Hamilton in the 1800's. Quaternions can be used in describing rotations in 3D real vector space and Lorentz space [Hermann 73]. We will show how a unit quaternion is used to represent a rotation of a vector in 3D space in this chapter. First we will define properties and operations of quaternions, and then show the relationships with the rotation matrix.

Using complex number notation, a quaternion $\mathring{q}$ can be represented by the following:

$$\mathring{q} = q_0 + iq_x + jq_y + kq_z$$

where, q_0, q_x, q_y and q_z are real numbers. The basis of the imaginary components i, j, and k have the following properties:

$$i^2 = -1, j^2 = -1, k^2 = -1$$
$$ij = k, \ jk = i, \ ki = j$$
$$ji = -k, kj = -i, ik = -j.$$

If $\mathring{r} = r_0 + ir_x + jr_y + kr_z$, quaternion multiplication becomes

$$\begin{aligned}\mathring{r}\mathring{q} = \; & (r_0q_0 - r_xq_x - r_yq_y - r_zq_z) \; + i(r_0q_x + r_xq_0 + r_yq_z - r_zq_y) \\ & + j(r_0q_y - r_xq_z + r_yq_0 + r_zq_x) + k(r_0q_z + r_xq_y - r_yq_x + r_zq_0). \qquad \text{(A.1)}\end{aligned}$$

It can be easily verified that the multiplication operation is not commutative.

The conjugate of $\mathring{q}$ has the same real part with negated imaginary part, $\mathring{q}^* = q_0 - iq_x - jq_y - kq_z$. The conjugate of a product of two quaternions $(\mathring{q}\mathring{r})^*$ is $\mathring{r}^*\mathring{q}^*$. The dot product of two quaternions is the sum of products of corresponding components.

$$\mathring{p}\cdot\mathring{q} = p_0q_0 + p_xq_x + p_yq_y + p_zq_z = (\mathring{p}\mathring{q}^* + \mathring{q}\mathring{p}^*)/2$$

It is obvious that the dot product is a real number. The square of the magnitude of a quaternion is the dot product of the quaternion with itself.

$$|\mathring{q}|^2 = \mathring{q}\cdot\mathring{q} = \mathring{q}\mathring{q}^* = \mathring{q}^*\mathring{q}$$

A multiplicative inverse of a nonzero quaternion is

$$\mathring{q}^{-1} = (1/\mathring{q}\cdot\mathring{q})\mathring{q}^*.$$

A vector in 3D space $\mathbf{t} = (t_x \; t_y \; t_z)^T$ can be represented by a purely imaginary quaternion

$$\mathring{t} = 0 + it_x + jt_y + kt_z.$$

We will denote $\mathring{t} = \mathbf{t}$, if the corresponding components are the same between a 3D vector and a pure imaginary quaternion. Let $\mathbf{s}$ be a vector in 3D space, and $\mathring{s}$ be its pure imaginary representation; then we can verify the following properties, which will be used in the following proofs.

$$\mathbf{s}\cdot\mathbf{t} = \mathring{s}\cdot\mathring{t} = (\mathring{s}\mathring{t}^* + \mathring{t}\mathring{s}^*)/2$$

$$\mathbf{s}\times\mathbf{t} = (\mathring{s}\mathring{t} - \mathring{t}\mathring{s})/2$$

We will validate the claim that $\mathring{t}' = \mathring{r}\,\mathring{t}\,\mathring{r}^*$ is a rotation operation when $\mathring{t}$ is a pure imaginary quaternion equivalent to a 3D vector $\mathbf{t}$, and $\mathring{r}$ is a unit quaternion. The inverse mapping is $\mathring{r}^*\,\mathring{t}'\,\mathring{r}$ and the mapping is bijective. First it will be shown that $\mathring{t}'$ is pure imaginary by checking if $\mathring{t}' + \mathring{t}'^* = 0$.

$$\mathring{t}' + \mathring{t}'^* = \mathring{r}\,\mathring{t}\,\mathring{r}^* + (\mathring{r}\,\mathring{t}\,\mathring{r}^*)^* = \mathring{r}\,\mathring{t}\,\mathring{r}^* + \mathring{r}\,\mathring{t}^*\,\mathring{r}^* = \mathring{r}(\mathring{t} + \mathring{t}^*)\mathring{r}^* = 0$$

from the fact that $\mathring{t}$ is pure imaginary.

The rotation operation must preserve the dot product.

$$\begin{aligned}
\mathbf{s}'\cdot\mathbf{t}' &= (\mathring{s}'\,\mathring{t}'^* + \mathring{t}'\,\mathring{s}'^*)/2 \\
&= \{\mathring{r}\,\mathring{s}\,\mathring{r}^*(\mathring{r}\,\mathring{t}\,\mathring{r}^*)^* + \mathring{r}\,\mathring{t}\,\mathring{r}^*(\mathring{r}\,\mathring{s}\,\mathring{r}^*)^*\}/2 \\
&= \{\mathring{r}\,\mathring{s}\,(\mathring{r}^*\mathring{r})\,\mathring{t}^*\,\mathring{r}^* + \mathring{r}\,\mathring{t}\,(\mathring{r}^*\mathring{r})\,\mathring{s}^*\,\mathring{r}^*\}/2 \\
&= (\mathring{r}\,\mathring{s}\mathring{t}^*\,\mathring{r}^* + \mathring{r}\,\mathring{t}\mathring{s}^*\,\mathring{r}^*)/2 \\
&= \mathring{r}\left(\frac{\mathring{s}\mathring{t}^* + \mathring{t}\mathring{s}^*}{2}\right)\mathring{r}^* \\
&= \mathring{r}(\mathring{s}\cdot\mathring{t})\mathring{r}^* = (\mathring{s}\cdot\mathring{t})(\mathring{r}\mathring{r}^*) = (\mathring{s}\cdot\mathring{t}) = \mathbf{s}\cdot\mathbf{t}
\end{aligned}$$

Here we use the fact that the dot product $\mathring{s}\cdot\mathring{t}$ is a real number, and the product of a real number with a quaternion is commutative. It is a special case of the above that $\mathbf{t}'\cdot\mathbf{t}'$ is the same as $\mathbf{t}\cdot\mathbf{t}$, which means the length of a vector is invariant after the rotation.

The members of the set of linear coordinate transformations which leave the Euclidean norm invariant are either rotations or reflections [Cartan 66]. The reflection operation switches right-handed coordinates to left-handed coordinates or vice versa, which results in sign change of the cross product [Hermann 70]. The cross product of two rotated vectors is shown to be the same as the rotation of the cross product

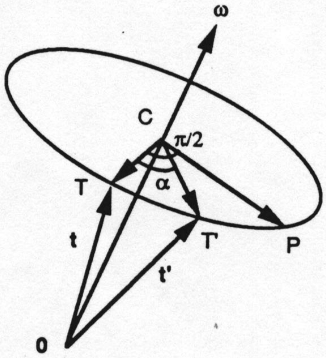

Figure A.1: Rotation of a vector $\mathbf{t}$ by an angle α around the unit vector $\boldsymbol{\omega}$. $\vec{OC} = (\mathbf{t}\cdot\boldsymbol{\omega})\boldsymbol{\omega}$ is the projection of $\mathbf{t}$ to the $\boldsymbol{\omega}$ direction. Then $\vec{CT} = \mathbf{t} - (\mathbf{t}\cdot\boldsymbol{\omega})\boldsymbol{\omega}$, and $\vec{CP} = \boldsymbol{\omega} \times \mathbf{t}$. Now $\vec{CT'}$ becomes $\cos\alpha\vec{CT} + \sin\alpha\vec{CP}$, and the rotated vector $\mathbf{t}' = \vec{OC} + \vec{CT'}$.

of the two vectors, i.e.

$$\begin{aligned}
\mathbf{s}' \times \mathbf{t}' &= (\mathring{s}'\mathring{t}' - \mathring{t}'\mathring{s}')/2 \\
&= (\mathring{r}\,\mathring{s}\,\mathring{r}^*\mathring{r}\,\mathring{t}\,\mathring{r}^* - \mathring{r}\,\mathring{t}\,\mathring{r}^*\mathring{r}\,\mathring{s}\,\mathring{r}^*)/2 \\
&= \mathring{r}\left(\frac{\mathring{s}\mathring{t} - \mathring{t}\mathring{s}}{2}\right)\mathring{r}^*
\end{aligned}$$

which is the rotation of the vector $\mathbf{s} \times \mathbf{t}$.

If $\mathbf{t}'$ is taken from $\mathbf{t}$ rotated by an angle α around the $\boldsymbol{\omega}$ axis which is a unit vector, then it has the following relationship as a result of analysis on Fig. A.1.

$$\mathbf{t}' = \cos\alpha\mathbf{t} + \sin\alpha(\boldsymbol{\omega} \times \mathbf{t}) + (1 - \cos\alpha)(\boldsymbol{\omega}\cdot\mathbf{t})\boldsymbol{\omega}$$

Its quaternion equivalent is

$$\cos\alpha\,\mathring{t} + \sin\alpha\,(\mathring{\omega}\mathring{t} - \mathring{t}\mathring{\omega})/2 + (1 - \cos\alpha)(\mathring{\omega}\mathring{t}^* + \mathring{t}\mathring{\omega}^*)/2 \qquad (A.2)$$

If we represent a unit quaternion $\mathring{r}$ in the following form

$$\cos(\alpha/2) + \sin(\alpha/2)\,\mathring{\omega}$$

then $\mathring{r}\,\mathring{t}\,\mathring{r}^*$ reduces to Eq. A.2 after simple calculations.

A component by component equation for $\mathring{t}'=\mathring{r}\,\mathring{t}\,\mathring{r}^*$ with matrix notation becomes

$$\begin{bmatrix} t'_x \\ t'_y \\ t'_z \end{bmatrix} = \begin{bmatrix} (r_0^2+r_x^2-r_y^2-r_z^2) & (-2r_0r_z+2r_xr_y) & (2r_0r_y+2r_xr_z) \\ (2r_0r_z+2r_xr_y) & (r_0^2-r_x^2+r_y^2-r_z^2) & (-2r_0r_x+2r_yr_z) \\ (-2r_0r_y+2r_xr_z) & (2r_0r_x+2r_yr_z) & (r_0^2-r_x^2-r_y^2+r_z^2) \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}.$$

This equation shows the relationship to the familiar rotation matrix from the quaternion. Verification of the orthonormality of the matrix is trivial.

The following equations for the derivation of a unit quaternion from a rotation matrix comes from [Salamin 74]. From the diagonal elements of the rotation matrix we obtain

$$\begin{aligned} r_0^2 &= (1+R_{11}+R_{22}+R_{33})/4, \\ r_x^2 &= (1+R_{11}-R_{22}-R_{33})/4, \\ r_y^2 &= (1-R_{11}+R_{22}-R_{33})/4, \\ r_z^2 &= (1-R_{11}-R_{22}+R_{33})/4, \end{aligned} \tag{A.3}$$

where R_{ij} is the i-th row and j-th column element of the rotation matrix $\mathbf{R}$. From the off-diagonal elements we obtain

$$\begin{aligned} r_0r_x &= (R_{32}-R_{23})/4, \\ r_0r_y &= (R_{13}-R_{31})/4, \\ r_0r_z &= (R_{21}-R_{12})/4, \\ r_xr_y &= (R_{12}+R_{21})/4, \\ r_xr_z &= (R_{13}+R_{31})/4, \\ r_yr_z &= (R_{23}+R_{32})/4. \end{aligned} \tag{A.4}$$

Choose the equation from Eq. A.3 which yields the largest magnitude. Then the rest of the three components are calculated from Eq. A.4.

References

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