NEW SETS WITH LARGE BORSUK NUMBERS

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ABSTRACT. We construct finite sets in $\mathbb{R}^n$, $n \geq 298$, which cannot be partitioned into $n+11$ parts of smaller diameter thus decreasing the smallest dimension in which Borsuk's conjecture is known to be false.

1. Introduction and notation

Borsuk's famous conjecture stated in [1] asks whether every bounded set in $\mathbb{R}^n$ can be partitioned into at most $n+1$ sets of smaller diameter. Believed by many to be true for some decades, but proved only for $d \leq 3$, see [8, 4], it came as a surprise when Kahn and Kalai [6] constructed finite sets showing the contrary.

The Borsuk number $b(M)$ of a bounded set M in $\mathbb{R}^n$ containing at least two points is the smallest positive integer m such that M can be partitioned into m sets of smaller diameter. Let also $b(n)$ be the maximal $b(M)$ where M ranges over all finite subsets of $\mathbb{R}^n$ containing at least two points. The result of Kahn and Kalai states that $b(n) \geq 1.1^{\sqrt{n}}$ for large n, and that Borsuk's conjecture $b(n) \leq n+1$ fails already for $n = 1325$. Improvements on the least dimension n with $b(n) > n+1$ were obtained by Nilli ($n = 946$, [7]), Raigorodski ($n = 561$, [10]), Weißbach ($n = 560$, [13]), the first author ($n = 323$, [5]), and Pikhurko ($n = 321$, [9]). A nice recent survey on Borsuk's problem and related questions is [12].

In fact, it is known that $b(n) > n+1$ for all $n \geq 321$, see [11, 5, 9]. Here we show that this is even true for $n \geq 298$.

Theorem 1. *For $n \geq 298$, there exists a finite set in the unit sphere in $\mathbb{R}^n$ which cannot be partitioned into $n+11$ sets of smaller diameter.*

As usual, given $x, y \in \mathbb{R}^d$, the euclidian norm of x and the inner product of x and y are denoted by $\|x\|$ and $\langle x, y\rangle$, respectively. We write $M^\perp$ for the linear space of all points orthogonal to a set $M \subset \mathbb{R}^d$. The standard unit vectors in $\mathbb{R}^d$ are denoted by $e_1, \ldots, e_d$.

We now recall and introduce some definitions from the theory of spherical codes. We mainly use notations as can be found in [2]. Ω_d is the unit sphere in $\mathbb{R}^d$. Given

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$C_1, C_2 \subset \Omega_d$, we let $\langle C_1, C_2\rangle := \{\langle x_1, x_2\rangle \ : \ x_1 \in C_1 \text{ and } x_2 \in C_2\}$. If $S \subset [-1, 1)$, a set $C \subset \Omega_d$ is called S-code if $\langle C, C\rangle \subset S \cup \{1\}$. The largest cardinality of an S-code in Ω_d is denoted by $A(d, S)$.

We also need the following definition. If $T \subset [-1, 1]$, we set

$$A(d, S, T) = \max\{|C_1| + |C_2| \ : \ C_1, C_2 \text{ are } S\text{-codes in } \Omega_d \text{ with } \langle C_1, C_2\rangle \subset T\}.$$

Here $|C|$ is the cardinality of the set C. Given a set S of real numbers and another real number c, we let

$$cS = \{cs \ : \ s \in S\} \quad \text{and} \quad c + S = \{c + s \ : \ s \in S\}.$$

Naturally, $S + c = c + S$ and $\frac{S}{c} = \frac{1}{c}S$.

The proof of Theorem 1 will be based on the following result, which we recall from [5].

Theorem 2. *Let S be a finite subset of $[-1, 1)$, $d \in \mathbb{N}$, $n = d(d + 3)/2$, and define $\alpha = \max S \cap [-1, 0)$ and $\beta = \min S \cap [0, 1)$. If $\alpha + \beta < 0$, then*

$$b(n - 1)\ A(d, S \setminus \{\alpha, \beta\}) \geq A(d, S).$$

Later on we shall exploit the following detail.

Remark 1. *The proof of Theorem 2 gives a finite subset M of the sphere $\Omega_n \cap \{(\xi_i)_{i=1}^n : \sum_{i=1}^d \xi_i = (1-\alpha-\beta)^{-\frac{1}{2}}\}$ in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ with $b(M)\ A(d, S \setminus \{\alpha, \beta\}) \geq A(d, S)$. Furthermore, two points $x, y \in M$ represent the diameter of M if and only if $\langle x, y\rangle = \frac{-\alpha\beta}{1-\alpha-\beta}$, provided that $A(d, S \setminus \{\alpha, \beta\}) < A(d, S)$.*

The remainder of the paper is organized as follows. In the next section, we prove some results which allow the reduction of cardinality estimates of certain spherical codes to lower dimensions by carefully studying the geometry of the involved codes. In Section 3, we estimate some concrete cardinalities of codes relevant for our purposes via the nowadays well established linear algebra methods, which appear in almost every estimate on Borsuk numbers obtained by now. Finally, in Section 4 we put the things together to show that an appropriate embedding of a finite set in Ω_{23} is a counterexample to Borsuk's conjecture in $\mathbb{R}^{298}$. As in [5], we use vectors of minimal length in a lattice, here it is the laminated lattice Λ_{23}, see [2]. This set may be alternatively obtained as the subset of the vectors of minimal length in the Leech lattice used in [5] which have equal first and second coordinates. The only relevant parameters for our purposes are its size (93150) and that, after normalization, it is a $\{-1, 0, \pm\frac{1}{2}, \pm\frac{1}{4}\}$-code in Ω_{23}.

2. Reductions for cardinality estimates of codes

The next three propositions can be used to reduce cardinality estimates of spherical codes to lower dimensions or to smaller sets of admissible scalar products. These reductions become possible by studying the geometry of the involved codes. To avoid trivial cases, we always assume throughout the rest of the paper that $d \geq 2$.

Proposition 1. *Let $S \subset [-1,1)$ be such that $-1 \in S$ and $S \cap (-S) = \{a, -a\}$ with $0 < a < 1$. Define*

$$\overline{S} = \frac{S - a^2}{1 - a^2} \cap [-1, 1) \quad \text{and} \quad \overline{T} = \frac{S + a^2}{1 - a^2} \cap [-1, 1].$$

Then

$$A(d, S) = \max\{A(d, S \setminus \{-1\}), 2 + A(d-1, \overline{S}, \overline{T})\}.$$

Proof. $A(d, S \setminus \{-1\}) \leq A(d, S)$ is trivial.

If C_1 and C_2 are $\overline{S}$-codes in Ω_{d-1} with $\langle C_1, C_2 \rangle \subset \overline{T}$, we define $D_1, D_2 \subset \Omega_d$ by

$$D_1 = \sqrt{1 - a^2}\ C_1 \times \{a\} \quad \text{and} \quad D_2 = \sqrt{1 - a^2}\ C_2 \times \{-a\}.$$

Then

$$\langle D_i, D_i \rangle \setminus \{1\} = (1 - a^2)(\langle C_i, C_i \rangle \setminus \{1\}) + a^2 \subset (1 - a^2)\overline{S} + a^2 \subset S$$

for $i = 1, 2$. Moreover,

$$\langle D_1, D_2 \rangle = (1 - a^2)\langle C_1, C_2 \rangle - a^2 \subset (1 - a^2)\overline{T} - a^2 \subset S.$$

Hence altogether $D_1 \cup D_2 \cup \{e_d, -e_d\}$ is an S-code in Ω_d, which implies that

$$2 + |C_1| + |C_2| = 2 + |D_1| + |D_2| \leq A(d, S).$$

We are left to show that

$$A(d, S) \leq \max\{A(d, S \setminus \{-1\}), 2 + A(d-1, \overline{S}, \overline{T})\}.$$

To this end, choose a maximal S-code C in Ω_d, i.e. $|C| = A(d, S)$. If C does not contain an antipodal pair $\{x, -x\}$ then C is actually an $(S \setminus \{-1\})$-code and $|C| \leq A(d, S \setminus \{-1\})$. So we may finally assume that there is $x \in C$ such that also $-x \in C$. This implies that $\langle x, y \rangle \in \{-a, a\}$ for all $y \in C \setminus \{x, -x\}$.

Let us now define

$$D_1 = \left\{ \frac{y - ax}{\sqrt{1 - a^2}} \ : \ y \in C \text{ and } \langle x, y \rangle = a \right\}$$

and

$$D_2 = \left\{ \frac{y + ax}{\sqrt{1 - a^2}} \ : \ y \in C \text{ and } \langle x, y \rangle = -a \right\}.$$

Then $D_1, D_2 \subset \Omega_d \cap \{x\}^\perp$ which we may identify with Ω_{d-1}. Moreover,

$$\langle D_i, D_i \rangle = \frac{\langle C, C \rangle - a^2}{1 - a^2} \quad \text{for } i = 1, 2 \quad \text{and} \quad \langle D_1, D_2 \rangle = \frac{\langle C, C \rangle + a^2}{1 - a^2}.$$

So we find that $\langle D_i, D_i \rangle \subset \overline{S} \cup \{1\}$ for $i = 1, 2$ and $\langle D_1, D_2 \rangle \subset \overline{T}$, which implies that $|D_1| + |D_2| \leq A(d-1, \overline{S}, \overline{T})$. Thus we finally arrive at

$$A(d, S) = |C| = 2 + |D_1| + |D_2| \leq 2 + A(d-1, \overline{S}, \overline{T}),$$

which finishes the proof. $\square$

Proposition 2. *Let $S \subset [-1,1)$ and $T \subset [-1,1]$ be such that $1 \in T$ and $S \cap T = \{a\}$ with $|a| < 1$. Define*

$$\overline{S} = \frac{S - a^2}{1 - a^2} \cap [-1,1) \quad \text{and} \quad \overline{T} = \frac{T - a^2}{1 - a^2} \cap [-1,1].$$

Then

$$A(d,S,T) = \max\{A(d,S,T \setminus \{1\}), 2 + A(d-1,\overline{S},\overline{T})\}.$$

Proof. $A(d,S,T \setminus \{1\}) \leq A(d,S,T)$ is trivial.

If C_1 and C_2 are $\overline{S}$-codes in Ω_{d-1} with $\langle C_1, C_2 \rangle \subset \overline{T}$, we define $D_1, D_2 \subset \Omega_d$ by

$$D_1 = \sqrt{1-a^2}\; C_1 \times \{a\} \quad \text{and} \quad D_2 = \sqrt{1-a^2}\; C_2 \times \{a\}.$$

Then

$$\langle D_i, D_i \rangle \setminus \{1\} = (1-a^2)(\langle C_i, C_i \rangle \setminus \{1\}) + a^2 \subset (1-a^2)\overline{S} + a^2 \subset S$$

for $i = 1,2$. So $D_1 \cup \{e_d\}$ and $D_2 \cup \{e_d\}$ are S-codes in Ω_d. Moreover,

$$\langle D_1, D_2 \rangle = (1-a^2)\langle C_1, C_2 \rangle + a^2 \subset (1-a^2)\overline{T} + a^2 \subset T.$$

Also, $\langle x, e_d \rangle = \langle e_d, y \rangle = a$ for all $x \in D_1$ and $y \in D_2$. Hence altogether $\langle D_1 \cup \{e_d\}, D_2 \cup \{e_d\} \rangle \subset T$, which implies that

$$2 + |C_1| + |C_2| = 2 + |D_1| + |D_2| \leq A(d,S,T).$$

We are left to show that

$$A(d,S,T) \leq \max\{A(d,S,T \setminus \{1\}), 2 + A(d-1,\overline{S},\overline{T})\}.$$

To this end, let C_1, C_2 be S-codes in Ω_d such that $\langle C_1, C_2 \rangle \subset T$ and $A(d,S,T) = |C_1| + |C_2|$. If $C_1 \cap C_2 = \emptyset$, then $\langle C_1, C_2 \rangle \subset T \setminus \{1\}$, so $|C_1| + |C_2| \leq A(d,S,T \setminus \{1\})$. Hence we may assume that there is $x \in C_1 \cap C_2$. It follows that, for any $y \in (C_1 \cup C_2) \setminus \{x\}$,

$$\langle x, y \rangle \in S \cap T = \{a\}.$$

So $\langle x, y \rangle = a$ for all $y \in (C_1 \cup C_2) \setminus \{x\}$.

Let us now define

$$D_1 = \left\{ \frac{y - ax}{\sqrt{1-a^2}} \; : \; y \in C_1 \setminus \{x\} \right\} \quad \text{and} \quad D_2 = \left\{ \frac{z - ax}{\sqrt{1-a^2}} \; : \; z \in C_2 \setminus \{x\} \right\}.$$

Then $D_1, D_2 \subset \Omega_d \cap \{x\}^\perp$ which we may identify with Ω_{d-1}. Moreover,

$$\langle D_i, D_i \rangle = \frac{\langle C_i, C_i \rangle - a^2}{1 - a^2} \quad \text{for } i = 1,2 \quad \text{and} \quad \langle D_1, D_2 \rangle = \frac{\langle C_1, C_2 \rangle - a^2}{1 - a^2}.$$

So we find that D_1 and D_2 are $\overline{S}$-codes in Ω_{d-1} and $\langle D_1, D_2 \rangle \subset \overline{T}$, which implies that $|D_1| + |D_2| \leq A(d-1,\overline{S},\overline{T})$. Thus we finally arrive at

$$A(d,S,T) = |C_1| + |C_2| = 2 + |D_1| + |D_2| \leq 2 + A(d-1,\overline{S},\overline{T}),$$

which finishes the proof. $\square$

Proposition 3. *Let $S \subset [-1,1)$ and $T \subset [-1,1]$ be such that $-1 \in T$ and $S \cap (-T) = \{a\}$ with $|a| < 1$. Define*

$$\overline{S} = \frac{S - a^2}{1 - a^2} \cap [-1,1) \quad \text{and} \quad \overline{T} = \frac{T + a^2}{1 - a^2} \cap [-1,1].$$

Then

$$A(d,S,T) = \max\{A(d,S,T \setminus \{-1\}), 2 + A(d-1,\overline{S},\overline{T})\}.$$

Proof. $A(d,S,T \setminus \{-1\}) \leq A(d,S,T)$ is trivial.

If C_1 and C_2 are $\overline{S}$-codes in Ω_{d-1} with $\langle C_1, C_2 \rangle \subset \overline{T}$, we define $D_1, D_2 \subset \Omega_d$ by

$$D_1 = \sqrt{1-a^2}\; C_1 \times \{a\} \quad \text{and} \quad D_2 = \sqrt{1-a^2}\; C_2 \times \{-a\}.$$

Then

$$\langle D_i, D_i \rangle \setminus \{1\} = (1-a^2)(\langle C_i, C_i \rangle \setminus \{1\}) + a^2 \subset (1-a^2)\overline{S} + a^2 \subset S$$

for $i = 1, 2$. So $D_1 \cup \{e_d\}$ and $D_2 \cup \{-e_d\}$ are S-codes in Ω_d. Moreover,

$$\langle D_1, D_2 \rangle = (1-a^2)\langle C_1, C_2 \rangle - a^2 \subset (1-a^2)\overline{T} - a^2 \subset T.$$

Also, $\langle x, -e_d \rangle = \langle e_d, y \rangle = -a$ for all $x \in D_1$ and $y \in D_2$. Hence altogether $\langle D_1 \cup \{e_d\}, D_2 \cup \{-e_d\} \rangle \subset T$, which implies that

$$2 + |C_1| + |C_2| = 2 + |D_1| + |D_2| \leq A(d,S,T).$$

We are left to show that

$$A(d,S,T) \leq \max\{A(d,S,T \setminus \{-1\}), 2 + A(d-1,\overline{S},\overline{T})\}.$$

To this end, let C_1, C_2 be S-codes in Ω_d, such that $\langle C_1, C_2 \rangle \subset T$ and $A(d,S,T) = |C_1| + |C_2|$. If $C_1 \cap (-C_2) = \emptyset$, then $\langle C_1, C_2 \rangle \subset T \setminus \{-1\}$, so $|C_1| + |C_2| \leq A(d,S,T \setminus \{-1\})$. Hence we may assume that there is $x \in C_1$ with $-x \in C_2$. It follows that, for any $y \in C_1 \setminus \{x\}$,

$$\langle x, y \rangle \in S \cap (-T) = \{a\}.$$

So $\langle x, y \rangle = a$ for all $y \in C_1 \setminus \{x\}$. Similarly, $\langle x, z \rangle = -a$ for all $z \in C_2 \setminus \{-x\}$.

Let us now define

$$D_1 = \left\{ \frac{y - ax}{\sqrt{1-a^2}} \;:\; y \in C_1 \setminus \{x\} \right\} \quad \text{and} \quad D_2 = \left\{ \frac{z + ax}{\sqrt{1-a^2}} \;:\; z \in C_2 \setminus \{-x\} \right\}.$$

Then $D_1, D_2 \subset \Omega_d \cap \{x\}^{\perp}$ which we may identify with Ω_{d-1}. Moreover,

$$\langle D_i, D_i \rangle = \frac{\langle C_i, C_i \rangle - a^2}{1 - a^2} \quad \text{for } i = 1,2 \quad \text{and} \quad \langle D_1, D_2 \rangle = \frac{\langle C_1, C_2 \rangle + a^2}{1 - a^2}.$$

So we find that D_1 and D_2 are $\overline{S}$-codes in Ω_{d-1} and $\langle D_1, D_2 \rangle \subset \overline{T}$, which implies that $|D_1| + |D_2| \leq A(d-1, \overline{S}, \overline{T})$. Thus we finally arrive at

$$A(d,S,T) = |C_1| + |C_2| = 2 + |D_1| + |D_2| \leq 2 + A(d-1,\overline{S},\overline{T}),$$

which finishes the proof. □

3. Application of the linear algebra method

Proposition 4. $A(d, \{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}) \leq \frac{d(d+3)}{2}$.

Proof. Let C be a $\{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$-code in Ω_d. For every $c \in C$, we consider the polynomial $P_c : \mathbb{R}^d \to \mathbb{R}$, $P_c(x) = (2\langle x, c\rangle - 1)(4\langle x, c\rangle - 1)$. The proposition will be proved once it is shown that the set $\{P_c : c \in C\} \cup \{1\}$ consists of linearly independent functions. Indeed, all these functions belong to the $\frac{(d+1)(d+2)}{2}$-dimensional space of polynomials of total degree at most 2 in d indeterminates. Then $|C|+1 \leq \frac{(d+1)(d+2)}{2} = \frac{d(d+3)}{2}+1$, which shows that $A(d, \{-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}) \leq \frac{d(d+3)}{2}$.

Assume that

$$\sum_{c\in C} \lambda_c P_c + \lambda_1 = 0. \tag{1}$$

The quadratic part of this expression is $\sum_{c\in C} 8\lambda_c \langle \cdot, c\rangle^2 = 0$. Summation over the unit vectors e_i and using that $\sum_{i=1}^d \langle e_i, c\rangle^2 = \|c\|^2 = 1$ yields $\sum_{c\in C} \lambda_c = 0$. Now evaluation of the constant part of (1) gives $\lambda_1 = -\sum_{c\in C} \lambda_c = 0$. Substituting $f \in C$ in (1) then leads to

$$\sum_{c\in C} \lambda_c P_c(f) = 0 \quad \text{for all} \quad f \in C. \tag{2}$$

Let $A = (P_c(f))_{c,f\in C}$ be the matrix of this homogenous system of linear equations for $\lambda_c, c \in C$. Since $P_c(f) \equiv \delta_{c,f} \mod 2$, we find for the determinant of that system that $\det(A) \equiv 1 \mod 2$. So the determinant cannot vanish, and the only solution of (2) is the trivial solution, showing the independence of the functions in question. □

4. Conclusion

To simplify our still complex presentation of the example, we use the following two easy lemmas.

Lemma 5. *Let $S \subset [-1, 1)$ and $T \subset [-1, 1]$. Then*

(i) *if $S \cap T = \{-1\}$ and $1 \in T$ then $A(d, S, T) = \max\{4, A(d, S, T \setminus \{1\})\}$.*
(ii) *if $T \cap (-T) = \emptyset$ then $A(d, S, T) = \max\{A(d, S), A(d, S \setminus \{-1\}, T)\}$.*

Proof. In both cases, let C_1 and C_2 be S-codes in Ω_d, such that $\langle C_1, C_2\rangle \subset T$ and $A(d, S, T) = |C_1| + |C_2|$.

To prove (i) note that if $C_1 \cap C_2 = \emptyset$ then $\langle C_1, C_2\rangle \subset T \setminus \{1\}$. If there exists $x \in C_1 \cap C_2$ then any $y \in (C_1 \cup C_2) \setminus \{x\}$ satisfies $\langle x, y\rangle \in S \cap T = \{-1\}$. So $C_1 \cup C_2 \subset \{x, -x\}$ and $|C_1| + |C_2| \leq 4$.

To verify (ii) observe that if neither C_1 nor C_2 contains an antipodal pair $\{x, -x\}$ then they are actually $(S \setminus \{-1\})$-codes, hence $|C_1| + |C_2| \leq A(d, S \setminus \{-1\}, T)$. If

$x \in C_1 \cap (-C_1)$, say, then $C_2 = \emptyset$ by $T \cap (-T) = \emptyset$. Thus C_2 is empty and $|C_1| + |C_2| = |C_1| \le A(d, S)$. $\square$

Lemma 6. *For $S \subset [-1, 1)$, $T \subset [-1, 1]$, and $a \in (0, 1)$, we have*

$$A(d, S) \le A(d + 1, (1 - a)S + a)$$

and

$$A(d, S, T) \le A(d + 1, ((1 - a)S + a) \cup ((1 - a)T - a)).$$

Proof. If C is an S-code in Ω_d, then $\sqrt{1-a}\,C \times \{\sqrt{a}\}$ is a $((1 - a)S + a)$-code in Ω_{d+1}. This proves the first inequality. For the second inequality, given S-codes C_1, C_2 in Ω_d with $\langle C_1, C_2\rangle \subset T$, let

$$C = (\sqrt{1-a}\,C_1 \times \{\sqrt{a}\}) \cup (\sqrt{1-a}\,C_2 \times \{-\sqrt{a}\}).$$

Then C is indeed a $(((1 - a)S + a) \cup ((1 - a)T - a))$-code in Ω_{d+1}. $\square$

We are also going to use the next estimate.

Proposition 7. *For all $a, b \in [-1, 1)$ and $c \in [-1, 1]$,*

$$A(d, \{a, b\}, \{c\}) \le \frac{d(d + 3)}{2}.$$

Proof. First, we recall the general estimate on cardinalities of 2-distance sets in spheres from [3] which states that

$$A(d, \{a, b\}) \le \frac{d(d + 3)}{2} \quad \text{for all} \quad a, b \in [-1, 1) \quad \text{and} \quad d \ge 1. \tag{3}$$

Let now C, D be $\{a, b\}$-codes in Ω_d such that $\langle x, y\rangle = c$ for all $x \in C$ and $y \in D$ and $|C| + |D| = A(d, \{a, b\}, \{c\})$. If C or D is empty, (3) immediately implies the claimed inequality. If C is a singleton, then D is contained in the intersection of Ω_d with a sphere centered at the point in C. Hence D is either a singleton itself or lies in a sphere in a proper affine subspace. In the latter case, (3) gives that

$$|C| + |D| \le 1 + \frac{(d - 1)(d + 2)}{2} \le \frac{d(d + 3)}{2}.$$

If $|D| = 1$, we trivially have that $|C| + |D| = 2 \le \frac{d(d+3)}{2}$. The same argument applies if D is a singleton.

Finally, we assume that both C and D contain at least 2 points. The affine hull of a set in $\mathbb{R}^d$ is the intersection of all affine subspaces containing it. Let E, F be the affine hulls of C, D, respectively. Since all points in D have the same distance to all points in C, the affine subspaces E and F are orthogonal to each other. If the dimension of E is k, the dimension of F is at most $d - k$. The cardinality assumption on C and D implies that $k \ge 1$ and $d - k \ge 1$. Since C and D are 2-distance sets in spheres in E and F, the inequality (3) now yields

$$|C| \le \frac{k(k + 3)}{2} \quad \text{and} \quad |D| \le \frac{(d - k)(d - k + 3)}{2}.$$

It is an elementary exercise to check that this gives

$$|C| + |D| \le \frac{d(d + 3)}{2},$$

thus proving the proposition. □

Let now C be the set of normalized vectors of minimal length in the Leech lattice which are orthogonal to a fixed vector of minimal length in that lattice. Then C is a $\{-1, 0, \pm\frac{1}{2}, \pm\frac{1}{4}\}$-code of cardinality 93150 in a unit sphere in dimension 23, see [2, ch. 14.4].

We are going to apply Theorem 2 with $d = 23$, $n = 299$, and $S = \{-1, 0, \pm\frac{1}{2}, \pm\frac{1}{4}\}$. The code C shows that $A(23, S) \geq 93150$. To estimate $A(23, \{-1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\})$, we prove the following result, which is the main technical part of the present paper using all the previously established methods.

Proposition 8. $A(d, \{-1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}) \leq \frac{d^2+3d+4}{2}$.

Proof. The proof for $d \geq 8$ is outlined in Figure 1. Here a dashed arrow means that the expression in the box at the arrowhead is not smaller than the expression in the box at the root of the arrow. Continuous arrows mean that the expression at the root is equal to the maximum of the expressions at the arrowheads. Finally, close to the arrow is the name of the theorem which has to be applied to prove the corresponding inequality or equality. If $2 \leq d \leq 7$ some of the reduction steps are obviously to be dropped. The details for the verification of the inequality in this case are left to the attentive reader. □

Now the crucial estimate

$$b(298) \geq 310$$

is a consequence of Theorem 2, since $A(23, S) \geq 93150$ as above and

$$A(23, S \setminus \{-\tfrac{1}{4}, 0\}) = A(23, \{-1, -\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}\}) \leq 301$$

by Proposition 8.

According to Remark 1, the estimate $b(M) \geq 310$ is realized by a finite set M which is contained in the intersection of Ω_{299} with the 298-dimensional affine subspace $\{(\xi_i)_{i=1}^{299} : \sum_{i=1}^{23} \xi_i = (\frac{5}{4})^{-\frac{1}{2}}\}$. Moreover, $\|x - y\| = \mathrm{diam}(M)$ if and only if $\langle x, y \rangle = 0$. This yields $\mathrm{diam}(M) = \sqrt{2}$. Clearly, after rescaling we find a finite set $K \subset \Omega_{298}$ with $b(K) \geq 310$ and $\mathrm{diam}(K) > \sqrt{2}$. Now inductive application of the following lemma shows that

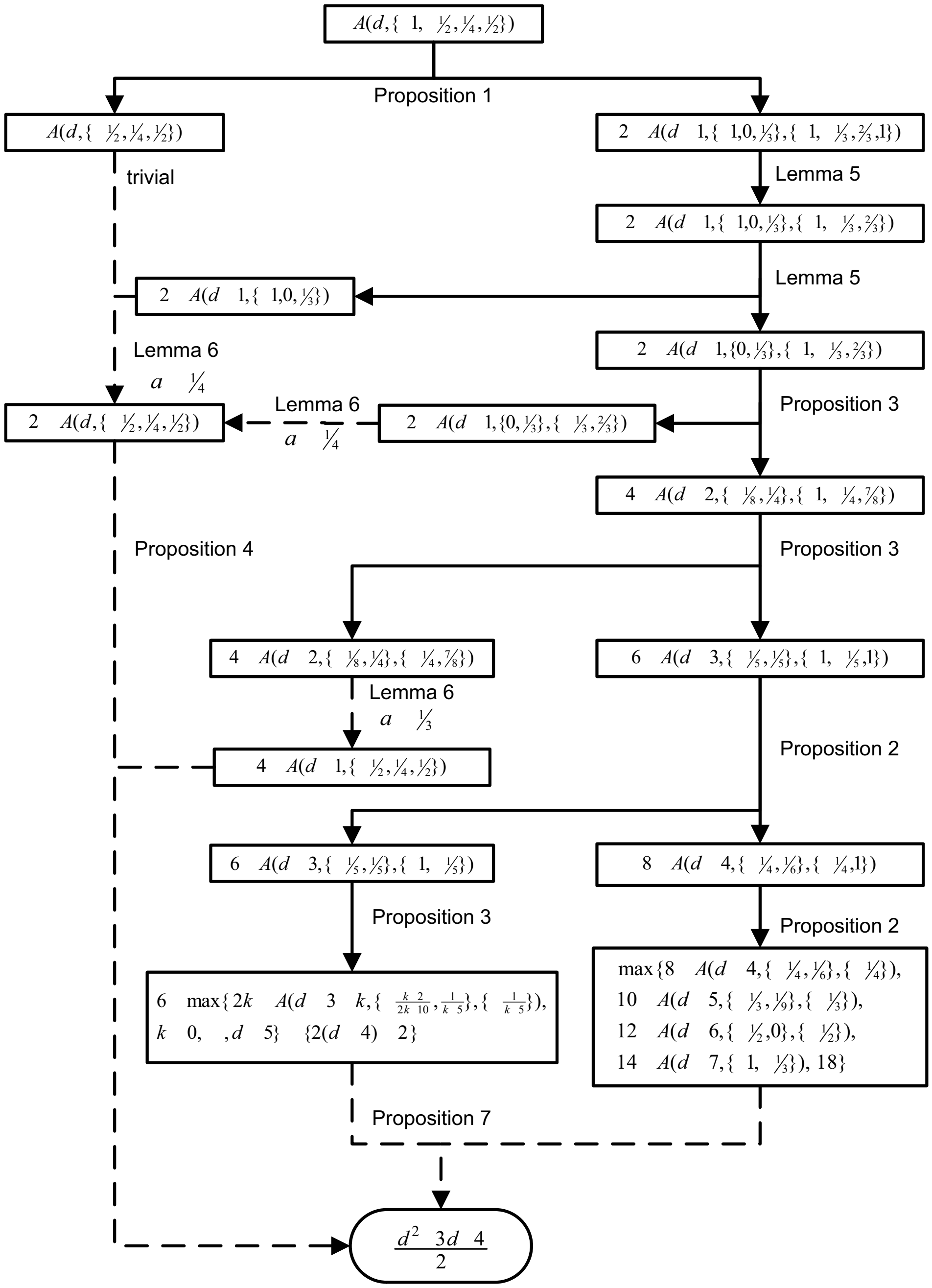
$$b(n) \geq n + 12 \quad \text{for all} \quad n \geq 298,$$

thus completing the proof of Theorem 1. A related method of the transfer of codes with large Borsuk number into higher dimensions is used in [11].

Lemma 9. *Let $K \subset \Omega_{n-1}$ be a set with $\mathrm{diam}(K) \geq \sqrt{2}$. Then there exists $L \subset \Omega_n$ with $\mathrm{diam}(L) \geq \sqrt{2}$ and $b(L) \geq b(K) + 1$. If K is finite then L can be assumed to be finite, too.*

Proof. Let $\delta = \mathrm{diam}(K)$. We put $K' = \frac{2\sqrt{\delta^2-1}}{\delta^2} K \times \{\frac{2-\delta^2}{\delta^2}\}$ and $L = K' \cup \{e_n\}$. One easily checks that $L \subset \Omega_n$ and that $\|e_n - x\| = \frac{2\sqrt{\delta^2-1}}{\delta} = \mathrm{diam}(K')$ for all $x \in K'$. Thus $\mathrm{diam}(L) = \mathrm{diam}(K') = 2\sqrt{1 - \frac{1}{\delta^2}} \geq \sqrt{2}$, since $\delta \geq \sqrt{2}$. Moreover,

FIGURE 1. Structure of the proof



every partition of L into sets of smaller diameter splits into the singleton $\{e_n\}$ and a corresponding partition of K'. Hence $b(L) = b(K') + 1 = b(K) + 1$. □

Remark 2. *Using the same method for the set C of all vectors of minimal norm in the Leech lattice, we obtain $A(24, \{-1, -\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}) \leq 326$ and consequently $b(323 + k) \geq 603 + k$ for all $k \geq 0$ improving also Theorem 1 in* [5].

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