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GENERALIZED DETERMINISTIC TRAFFIC RULES

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We study a family of deterministic models for highway traffic flow which generalize cellular automaton rule 184. This family is parameterized by the speed limit m and another parameter k that represents a "degree of aggressiveness" in driving, strictly related to the distance between two consecutive cars. We compare two driving strategies with identical maximum throughput: "conservative" driving with high speed limit and "aggressive" driving with low speed limit. Those two strategies are evaluated in terms of accident probability. We also discuss fundamental diagrams of generalized traffic rules and examine limitations of maximum achievable throughput. Possible modifications of the model are considered.

Keywords: Cellular Automata; Traffic Models; Phase Transitions.

1. Introduction

Transport phenomena in complex systems, in particular models of highway traffic flow, attracted much attention in recent years. Much of the effort was concentrated on fluid-dynamics Navier-Stokes-like models,[1] as well as discrete stochastic models, first proposed by Nagel and Schreckenberg,[2] and subsequently studied by many other authors using a variety of techniques.[3–6] In what follows, we shall study a family of purely deterministic discrete traffic models. The only randomness comes from the fact that the initial configuration of cars is chosen at random. This family of models represents various driving strategies, either chosen by drivers (distance between cars) or externally imposed (such as the speed limit). Our "artificial highway" consists of an array of L cells. Each cell is either occupied by a single car or empty. Cars can move only to the right, and we assume periodic boundary conditions. Time is discrete. At each time step, each driver moves his car according to some specified rule. The evolution is synchronous, that is, all cars move at the

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same time. In the simplest model of the family, car at site i can either move to site $i+1$ if this site is empty, or not move if site $i+1$ is occupied. Thus, the state of a given cell i depends only on cells $i-1$, i and $i+1$. This model is equivalent to cellular automaton[7] rule 184 if the state of an occupied site is 1, whereas the state of an empty site is 0. Under this rule, which rule table is

$$\begin{gathered} 000 \to 0,\ 001 \to 0,\ 010 \to 0,\ 011 \to 1, \\ 100 \to 1,\ 101 \to 1,\ 110 \to 0,\ 111 \to 1, \end{gathered}$$

the density of 1's is conserved, meaning that the number of "cars" does not change with time. Moreover, each car can move at most one site to the right during one time step, so the "speed limit" for this model is $m = 1$. For a lattice of length L, the average speed (sum of all speeds divided by the number of all cars) is, therefore, always less or equal to $m = 1$.

Let us now assume that we start from a random configuration of density ρ, and that cars move according to rule 184. It has been recently proved[8] that for large L and t, the average speed $\overline{v}_t$ at time t equals

$$\overline{v}_t = \begin{cases} \Theta(\rho, t) & \text{if } \rho < \frac{1}{2}, \\ \dfrac{1-\rho}{\rho}\Theta(\rho, t) & \text{otherwise,} \end{cases} \tag{1}$$

where

$$\Theta(\rho, t) = 1 - \frac{[4\rho(1-\rho)]^t}{\sqrt{\pi t}}. \tag{2}$$

Since $\lim_{t\to\infty} \Theta(\rho, t) = 1$, the average speed in the long time limit $\overline{v}_\infty = 1$ when the car density ρ is less than $1/2$ and $\overline{v}_\infty = (1-\rho)/\rho$ otherwise. In statistical physics terminology, the system exhibits a second order kinetic phase transition, where ρ is the control parameter, and $\overline{v}$ the order parameter. The critical point is at exactly $\rho = 1/2$, and at the critical point $\overline{v}_t$ approaches its stationary value as $t^{-1/2}$. Away from the critical point, the approach is exponential, and it slows down as ρ comes closer to $1/2$. In fact, when the lattice is finite (as in real life), the behavior of this model is not significantly different from the $L = \infty$ case. It is sufficient to perform $L/2$ iterations in order to reach the stationary state.[8] These considerations are illustrated in Fig. 1, representing the average car speed versus car density after 500 iterations of a 1000-site lattice. Numerical simulations are compared to the theoretical expression of $\overline{v}_\infty$ given by

$$\overline{v}_\infty = \begin{cases} 1 & \text{if } \rho < \frac{1}{2}, \\ \dfrac{1-\rho}{\rho} & \text{otherwise.} \end{cases} \tag{3}$$

It is straightforward to generalize rule 184 to higher velocities. Let us consider two cars A and B such that A follows B, and denote by g the gap between them.

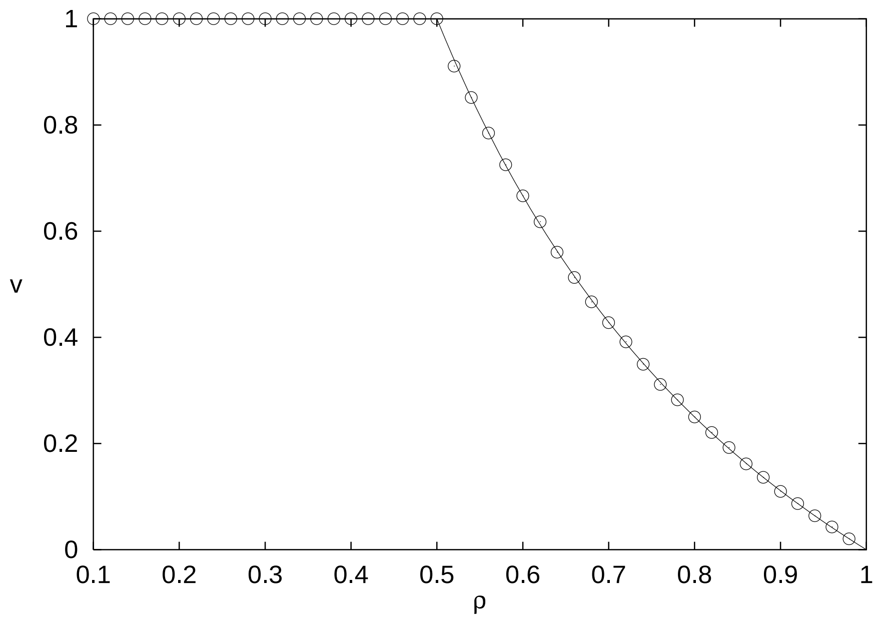


Fig. 1. Average speed of cars as a function of density for rule 184. Solid line represents theoretical $\overline{v}(\rho)$ curve.

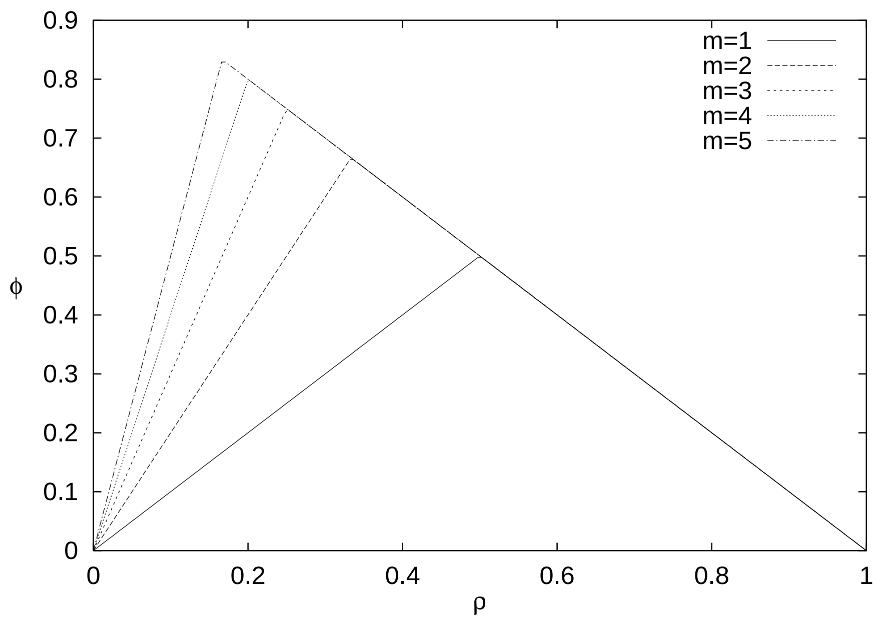


Fig. 2. Fundamental diagram for $\mathcal{R}_{m,1}$ for several different values of m. Vertical axis represents the flow $\phi = \rho\overline{v}$.

Then, at the next time step, car A moves g sites to the right if $g \leq m$, and m sites otherwise. Such a rule will be referred to as $\mathcal{R}_{m,1}$, where m denotes the speed limit. With this notation, $\mathcal{R}_{1,1}$ represents rule 184. The reason for the additional subscript "1" will become clear later. In this case, the phase transition occurs at $\rho_c = 1/(1+m)$. In the free moving phase, i.e., below the critical point, all cars are moving with maximum velocity. In the "jammed phase" (above ρ_c), the average speed is equal to $(1-\rho)/\rho$, just as for rule 184. The plot of the flow $\phi = \rho\overline{v}$ versus ρ for several different values of m, called the fundamental diagram, is shown in Fig. 2. It is clear that when the speed limit m increases, the total throughput of the highway increases.

2. Anticipatory Driving

How can we increase the flux without increasing the speed limit? It is obvious that the source of inefficiency in rule 184 is the "cautious attitude" of the drivers. Consider, for example, the following configuration:

$\cdots$	0	A	B	0	C	0	0	0	$\cdots$
$\cdots$	0	A	0	B	0	C	0	0	$\cdots$

The first line represents locations of cars A, B, C at time t, and the second line their location at time $t+1$. Zeros represent empty sites. Since driver A doesn't know whether car B is going to move or not, it is safer for him not to move. If he could see further than just one site forward, he could predict that car B will move forward, and that the site in front of him will be vacant at the next time step.

A simple rule which incorporates such a "prediction" mechanism can be constructed in a following way. Let us say that a driver at site i first checks whether in a block of k sites directly in front of him — i.e., from site $i+1$ to site $i+k$ — at least one site is empty. If it is, he moves his car by one site to the right, even if site $i+1$ is occupied since he knows that the car at site $i+1$ will move (as all drivers follow the same rule). If all sites from $i+1$ to $i+k$ are occupied, the car at site i does not move. Such a rule will be denoted by $R_{1,k}$. For $k=2$, the configuration discussed before will evolve as

$\cdots$	0	A	B	0	C	0	0	0	$\cdots$
$\cdots$	0	0	A	B	0	C	0	0	$\cdots$

As we see, cars A and B now move as one block, as long as the site in front of car B is empty. Such "blocking" significantly improves fluidity, as the fundamental diagram in Fig. 3 demonstrates. We immediately notice that fundamental diagrams corresponding to rules $\mathcal{R}_{1,n}$ and $\mathcal{R}_{n,1}$ are mirror images of each other with respect to the $\rho = 0.5$ line. Indeed, $\mathcal{R}_{1,n}$ and $\mathcal{R}_{n,1}$ are closely related: If cars are moving to the right according to rule $\mathcal{R}_{1,n}$, then empty sites are moving to the left according to rule $\mathcal{R}_{n,1}$. The critical density, which corresponds to a periodic configuration with a period consisting of k cars followed by one empty site equals $\rho_c = k/(k+1)$.

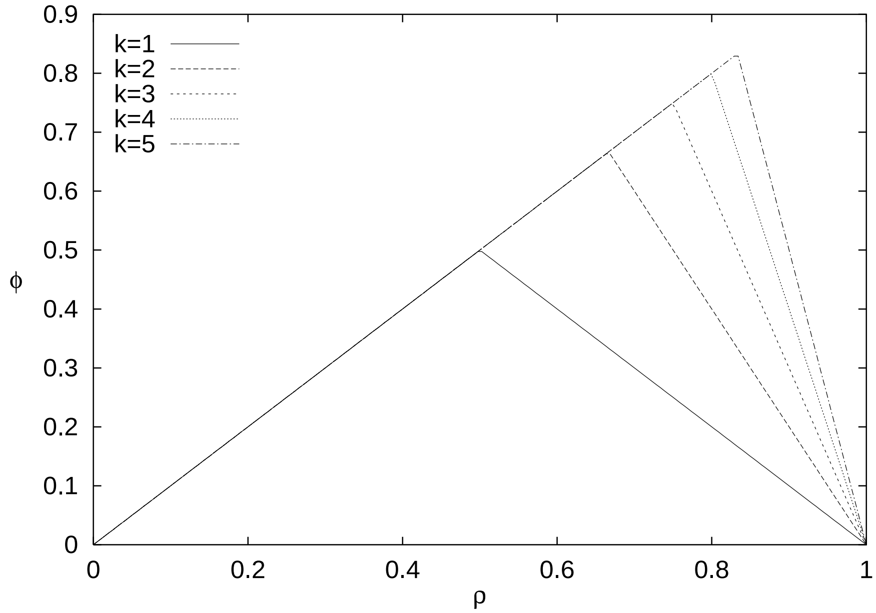


Fig. 3. Fundamental diagram for $\mathcal{R}_{1,k}$ for several different values of k.

The stationary ($t \to \infty$ limit) average velocity can be obtained using the following simple argument: Since now a block of k cars moves at the same time, from (1) it follows that

$$\overline{v}_{\infty} = \begin{cases} 1 & \text{if } \rho < \rho_c, \\ k(1-\rho)/\rho & \text{otherwise,} \end{cases} \tag{4}$$

where $\rho_c = k/(k+\overline{v})$.

3. Accidents

Rules $\mathcal{R}_{n,1}$ and $\mathcal{R}_{1,n}$ may be considered as two complementary driving strategies. Drivers obeying rule $\mathcal{R}_{n,1}$ drive with a higher speed, but they keep the distance to the preceding car larger. Those who comply with rule $\mathcal{R}_{1,n}$ drive more slowly, however they keep a smaller distance between consecutive cars. Since the maximum possible flow for both rules is the same (it equals $n/(n+1)$), one could ask which rule is better in terms of driving safety. Of course, if all drivers follow the same rule, either $\mathcal{R}_{n,1}$ or $\mathcal{R}_{1,n}$, there is no problem, since no accident can occur. But, in the real world, some drivers are careless, have slower reaction time, defective brakes etc., and accidents do occur. On the single-lane highway, only one type of accident may occur: a car bumps into the preceding car which abruptly decreased its velocity. Thus, we can say that cars which decrease their velocity are potential causes of accidents. In our model, such potentially dangerous cars can be identified as cars which, at time t, have a smaller velocity that at time $t-1$ (by definition the velocity at time t is $x(t) - x(t-1)$, where $x(t)$ is a position of a given car at time t). It should be stressed that none of the rules considered here allow us to simulate accidents, and we will only evaluate the risk of accidents, represented by the number of cars which are slowing down. A simple mean-field estimate of the number of such cars can be carried out if we neglect time correlations between velocities, i.e., if we assume that v_{t-1} and v_t are not correlated. For rule $\mathcal{R}_{m,1}$, the probability that a given car is slowing down will then be given by

$$P(v_t < v_{t-1}) = \sum_{i=1}^{m} \sum_{j=0}^{i-1} P(v_t = j) P(v_{t-1} = i), \tag{5}$$

where $P(v_t = j)$ denotes the probability that a car has velocity j at time t. Let us further assume that the length of the lattice is L, the number of cars is N, and the number of cars which have velocity i is N_i. In the stationary state N_i is time-invariant, thus we can write

$$P(v_t < v_{t-1}) = \sum_{i=1}^{m} \sum_{j=0}^{i-1} n_j n_i, \tag{6}$$

where $n_i = N_i/N$. The values of n_i's for $\mathcal{R}_{m,1}$ are given by[9]

$$\begin{aligned} n_k &= n_0(1-n_0)^k \text{ if } k < m \\ n_m &= (1-n_0)^m. \end{aligned} \tag{7}$$

Using these expressions, we obtain

$$\begin{aligned} P(v_t < v_{t-1}) &= \sum_{j=0}^{i-1} n_j n_m + \sum_{i=1}^{m-1} \sum_{j=0}^{i-1} n_j n_i \\ &= n_0(1-n_0)^m \sum_{j=0}^{m-1} (1-n_0)^j \\ &\quad + n_0^2 \sum_{i=1}^{m} \sum_{j=0}^{i-1} (1-n_0)^j (1-n_0)^i, \end{aligned}$$

and after computing all sums,

$$\begin{aligned} P(v_t < v_{t-1}) &= 1 - n_0 - (1-n_0)^{2m} \\ &\quad + \frac{(1-n_0)^{2m} + 2n_0 - 1 - n_0^2}{2 - n_0}. \end{aligned}$$

As shown by Fukui and Ishibashi,[9] when $\rho > \rho_c$, n_0 satisfies the following relation:

$$\frac{(1-n_0)[1-(1-n_0)^m]}{n_0} = \frac{1}{\rho} - 1. \tag{8}$$

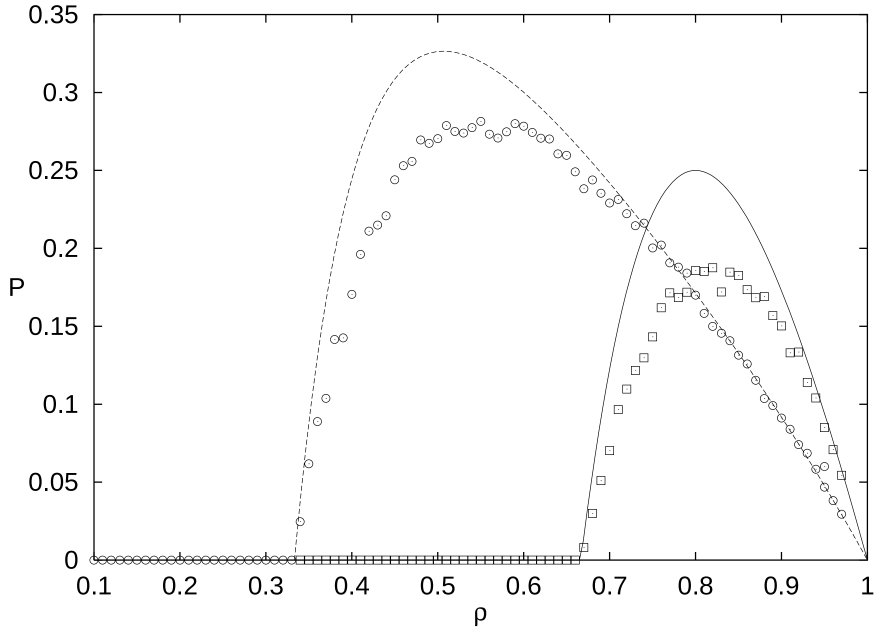


Fig. 4. Fraction of cars which are slowing down for $\mathcal{R}_{1,2}$ (□) and $\mathcal{R}_{2,1}$ (○). Solid and dashed lines represent the mean-field approximation for, respectively, rules $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,1}$.

If, for example, $m = 2$, (8) can be solved for n_0 which is given by

$$n_0 = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{4-3\rho}{\rho}}. \tag{9}$$

The fraction of cars which are slowing down is

$$P(v_t < v_{t-1}) = \frac{3\rho^2 - 5\rho - 2}{2\rho^2} + \frac{3+\rho}{2\rho}\sqrt{\frac{4-3\rho}{\rho}}. \tag{10}$$

Below ρ_c, $P(v_t < v_{t-1})$ is of course zero, since all cars are moving with constant speed m.

For $\mathcal{R}_{1,k}$, the mean-field approximation is much simpler, because all cars are either stopped or moving with speed 1 and $\overline{v} = 0 \times n_0 + 1 \times n_1 = n_1$, thus

$$P(v_t < v_{t-1}) = n_0 n_1 = (1 - n_1)n_1 = (1 - \overline{v})\overline{v}. \tag{11}$$

For $k = 2$, Eq. (4) yields $\overline{v} = 2(1-\rho)/\rho$ (above the critical density), therefore

$$P(v_t < v_{t-1}) = \frac{1(1-\rho)(3\rho-2)}{\rho}. \tag{12}$$

As before, below ρ_c the fraction of cars which are slowing down is zero.

The fraction of slowing cars obtained from computer simulations for both $\mathcal{R}_{2,1}$ and $\mathcal{R}_{1,2}$ is shown in Fig. 4, together with mean-field approximation curves given by Eqs. (10) and (12). Although mean-field predictions overestimate $P(v_t < v_{t-1})$, they are not very far from numerical results (further improvement could be achieved by using an n-cluster approximation,[6] also known as the local structure theory[10] in cellular automata theory). One feature, however, is apparent: Accident probability, which should be proportional to $P(v_t < v_{t-1})$, is much higher for $\mathcal{R}_{2,1}$ than for $\mathcal{R}_{1,2}$ if the density of cars is below 0.8. Above 0.8, rule $\mathcal{R}_{1,2}$ becomes more "dangerous", although the difference between rules diminishes as ρ approaches 1. It is also remarkable that in the case of rule $\mathcal{R}_{1,2}$ we have no risk of accidents up to $\rho = 2/3$, while rule $\mathcal{R}_{2,1}$ becomes very dangerous quite fast, being worst at approximately $\rho = 0.5$.

4. Generalization

Let us now construct a general rule which allows both higher speed limit m and blocking of kth order, with m and k both larger than 1. Consider a car located at site i. Its driver first locates the nearest gap (cluster of empty sites) in front of him whose length is g. If the first empty site (i.e., the first site belonging to the mentioned gap) is farther than $i + k$, the car does not move, otherwise, it moves to site $i + v$ site, where $v = \min(g, m)$. We will refer to this rule as $\mathcal{R}_{m,k}$. We could now expect that, as before, the "perfect configuration" is a periodic sequence whose period consists of k cars followed by m empty sites. The critical density is then $\rho_c = k/(k + m)$, and this should hold for any ρ larger than ρ_c, i.e., $\rho = k/(k + \overline{v})$. Below ρ_c, we expect $\overline{v} = m$. For the average velocity, this leads to the following expression:

$$v = \begin{cases} m & \text{if } \rho < \rho_c, \\ k(1-\rho)/\rho & \text{otherwise.} \end{cases} \tag{13}$$

Similarly, the flow is given by

$$\phi = \begin{cases} \rho m & \text{if } \rho < \rho_c, \\ (1-\rho)k & \text{otherwise.} \end{cases} \tag{14}$$

Unfortunately, this simple reasoning has some flaws. Although in the vicinity of $\rho = 0$ and $\rho = 1$, Eqs. (13) and (14) describe the behavior of $\mathcal{R}_{m,k}$ correctly, they fail to give a correct description in the intermediate region. This is clearly illustrated by the fundamental diagram, which, according to (14), should be always tent-shaped, with a peak at $\rho_c = k/(k + m)$. In practice, however, the shape is quite different, as shown in Fig. 5. There is no single peak at ρ_c, but instead, the flow stays at its maximum value over an extended interval of ρ values, and the fundamental diagram looks like "a tent with a flat roof." This means that, unlike rules $\mathcal{R}_{1,k}$ and $\mathcal{R}_{m,1}$, rule $\mathcal{R}_{m,k}$, for $m, k > 1$, does not exhibit a phase transition at ρ_c. This is due to the fact that the "perfect configuration" with period consisting of k cars followed by m empty sites is no longer stable, like this was the case for rules $\mathcal{R}_{1,k}$ and $\mathcal{R}_{m,1}$. In fact, the phase transition does occur,

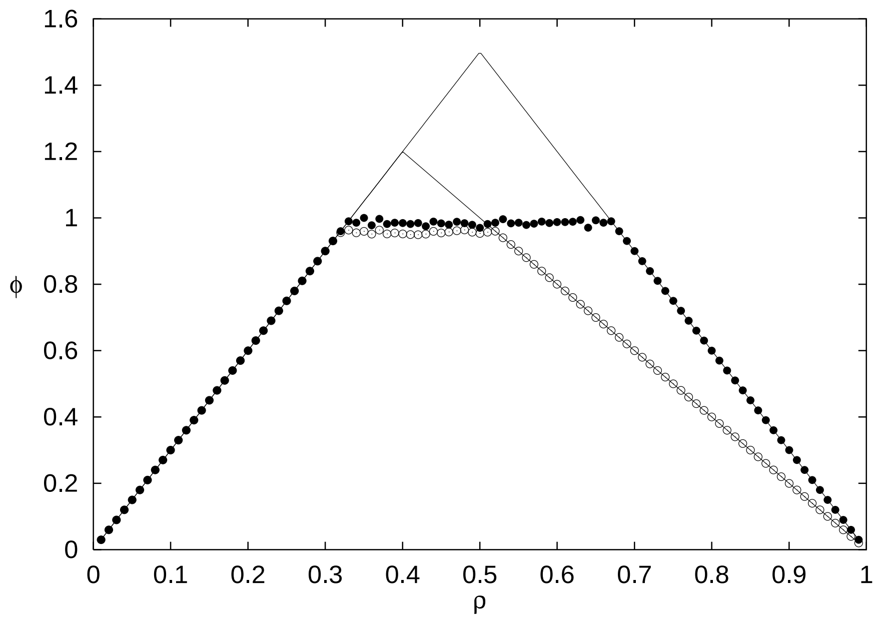


Fig. 5. Fundamental diagram for $\mathcal{R}_{3,3}$ (•) and $\mathcal{R}_{3,2}$ (∘). Solid lines represent theoretical flow obtained from Eq. (14).

but it is, in a sense, spread over an interval. To be more precise, let us first note that $\overline{v}$ is a linear combination of n_i's. Consequently, if, as functions of ρ, any of the n_i has a discontinuous derivative, this will show up in $\overline{v}$. Figure 6 shows an example of such a "velocity spectrum", i.e., plots of $n_i(\rho)$ for $i = 0, 1, 2, 3$. We can see that derivatives of all n_i's have discontinuities at some point, but they occur at different values of ρ. The first transition occurs at $\rho = 0.33$ ($n_3 = 1 \to n_3 \neq 1$, $n_1 = 0 \to n_1 \neq 0$ and $n_2 = 0 \to n_2 \neq 0$), whereas at $\rho \approx 0.51$ we have another transition $n_0 = 0 \to n_0 \neq 0$. For rule $\mathcal{R}_{m,1}$, all these transitions occurred at the same point, and we consequently observed a single transition in $\overline{v}$.

Another interesting feature of the fundamental diagram of $\mathcal{R}_{m,k}$ is the maximum flow value. From (14) one would expect $\phi_{\max} = km/(k+m)$, which could be as large as we want if we choose the right m and k values. Fig. 5, however, clearly shows that this is not the case: there seems to be a cutoff at ϕ slightly below 1. In fact, simulations performed with a wide array of k and m values suggest that the flow can never be larger that 1, regardless of k and m. For large k and m, the cutoff occurs almost exactly at 1, so the flow is well approximated by the piecewise linear function

$$\phi = \begin{cases} \rho m & \text{if } \rho \leq 1/m, \\ 1 & \text{if } 1/m < \rho < (k-1)/k, \\ (1-\rho)k & \text{if } \rho \geq (k-1)/k. \end{cases} \tag{15}$$

"Large" means equal or larger than 3, as even for $m = k = 3$ the above formula is fairly accurate (the cutoff occurs at 0.98, instead of 1). In any case, $\phi = 1$ is the

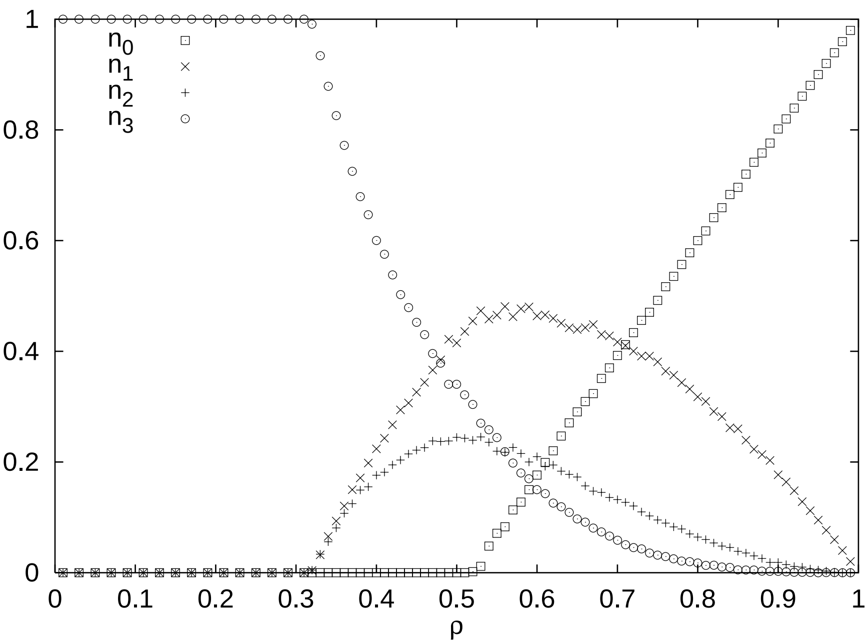
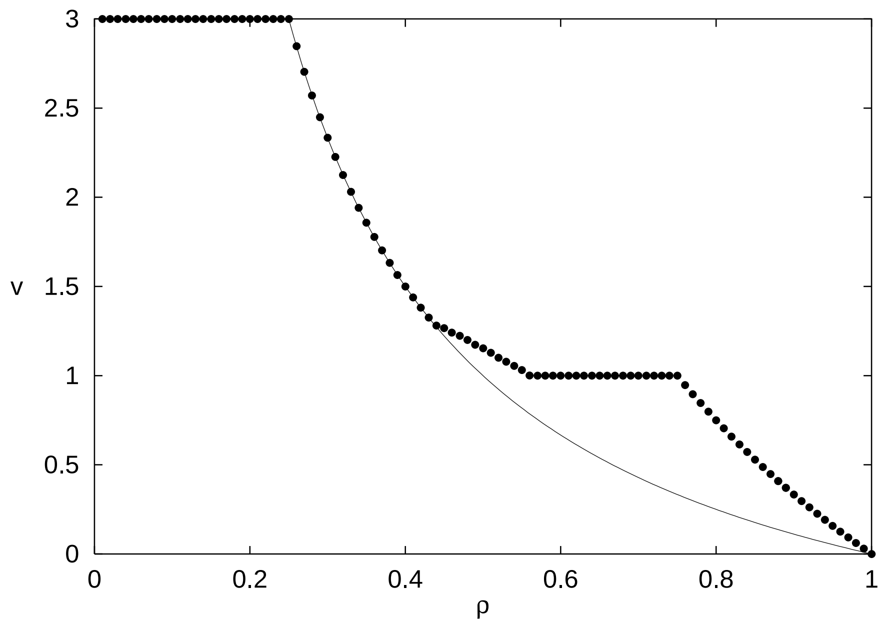


Fig. 6. Velocity spectrum for $\mathcal{R}_{3,2}$. Spectrum computed after 5000 iterations, lattice size $L = 2000$.

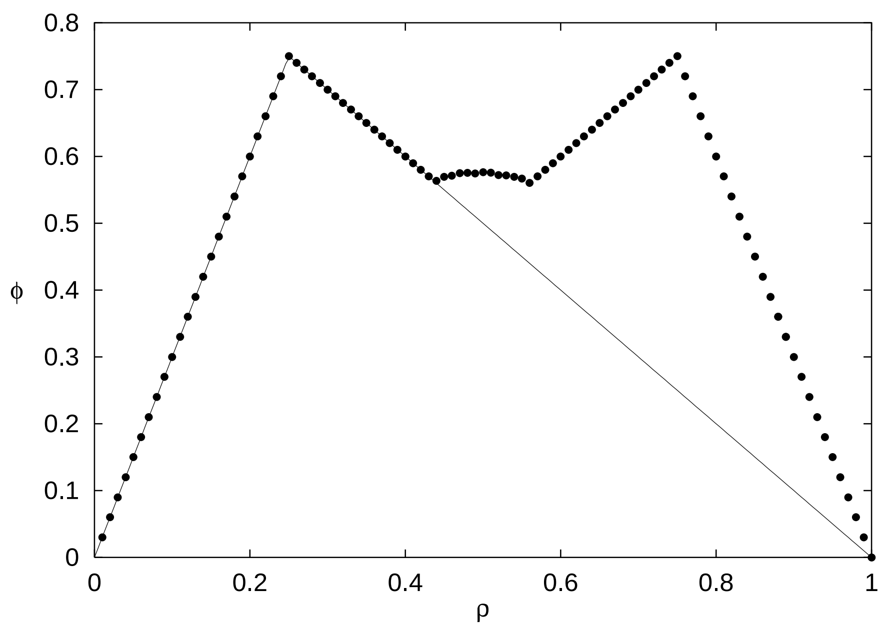
absolute maximum flow in the stationary state. It is not possible to make it larger by increasing either k or m.

5. Possible Modifications

Driving strategies based on $\mathcal{R}_{m,k}$ are, of course, not the only possible ways of increasing traffic fluidity. In the case of rule $\mathcal{R}_{m,k}$, the decision of a driver where to move the car at the next time step is based only on the size of the nearest gap g_{near}, located within the $k+m-1$ sites in front of the car. Instead of considering the nearest gap, drivers could base their decisions on the largest gap. More precisely, let us say that $g_{\max}$ is the largest continuous block of empty sites, entirely located between sites i and $i+k+m$. If, for example, $m = 3$, $k = 7$, consider the configuration $\ldots\underline{1}0100110000\ldots$, the largest continuous block of zeros within $m+k-1 = 9$ sites in front of the underlined car has a length $g_{\max} = 3$. We now define rule $\hat{\mathcal{R}}_{m,k}$ such that drivers move their cars by $\min(g_{\max}, m)$ sites to the right. Since $g_{\max} \geq g_{\text{near}}$, we can expect that the average speed (or flow) for $\hat{\mathcal{R}}_{m,k}$ should be larger than or equal to the average speed (or flow) for $\mathcal{R}_{m,k}$. To demonstrate that this is indeed the case, let us compare rules $\hat{\mathcal{R}}_{3,1}$ and $\mathcal{R}_{3,1}$. Figure 7(a) shows that up to $\rho \approx 0.43$, $\hat{\mathcal{R}}_{3,1}$ behaves exactly like $\mathcal{R}_{3,1}$. When the density increases beyond this value, additional phase transitions occur. While rule $\mathcal{R}_{3,1}$ exhibits just a single phase transition at $\rho = 0.25$, $\hat{\mathcal{R}}_{3,1}$ exhibits four transitions, at $\rho = 0.25$, 0.43, 0.57, and 0.75. Interestingly, looking at the fundamental diagram



(a)



(b)

Fig. 7. (a) Plot of average velocity $\overline{v}$ as a function of density ρ for $\hat{\mathcal{R}}_{3,1}$ (•) and $\mathcal{R}_{3,1}$ (solid line). (b) Fundamental diagram of $\hat{\mathcal{R}}_{3,1}$ (•) and $\mathcal{R}_{3,1}$ (solid line).

(Fig. 7(b)) we can see that it is fully symmetric with respect to $\rho = 0.5$. This means that the second peak has the same shape as the peak in fundamental diagram of $R_{3,1}$ (as a consequence of the duality between $\mathcal{R}_{3,1}$ and $\mathcal{R}_{1,3}$). Therefore, $\hat{\mathcal{R}}_{3,1}$ can be considered as a sort of the "mix" of the aforementioned rules,

$$\hat{\mathcal{R}}_{3,1} \approx \begin{cases} \mathcal{R}_{3,1} & \text{if } \rho < 0.5, \\ \mathcal{R}_{1,3} & \text{if } \rho > 0.5. \end{cases} \tag{16}$$

The $\approx$ sign reflects the fact that this relation is not true in the vicinity of 0.5.

Another interesting detail which should be mentioned pertains to the maximum flow value. As in the $\mathcal{R}_{m,k}$ case, we investigated the fundamental diagram of $\hat{\mathcal{R}}_{m,k}$ for many different m and k values, and it seems that also in this case ϕ cannot exceed 1. Nevertheless, we found no simple explanation of this seemingly general feature.

6. Conclusion and Remarks

We have studied a family of deterministic traffic flow models which generalize cellular automaton rule 184. In addition to the speed limit, this family is parameterized by a quantity which represents "aggressiveness" in driving, strictly related to the distance between consecutive cars. We compared two complementary driving strategies resulting in identical maximum throughput: slow driving with small distance between cars (represented by $\mathcal{R}_{1,2}$) and faster driving with larger distance between cars (represented by $\mathcal{R}_{2,1}$). We found that at small car densities $\mathcal{R}_{2,1}$ is more dangerous in terms of an accident probability, while for larger densities (above $\rho = 0.8$), $\mathcal{R}_{1,2}$ becomes more dangerous. We also demonstrated that deterministic traffic rules can be further generalized, and the resulting fundamental diagrams exhibit multiple phase transitions as well as many interesting symmetry features. This indicates that those rules certainly deserve further study.

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