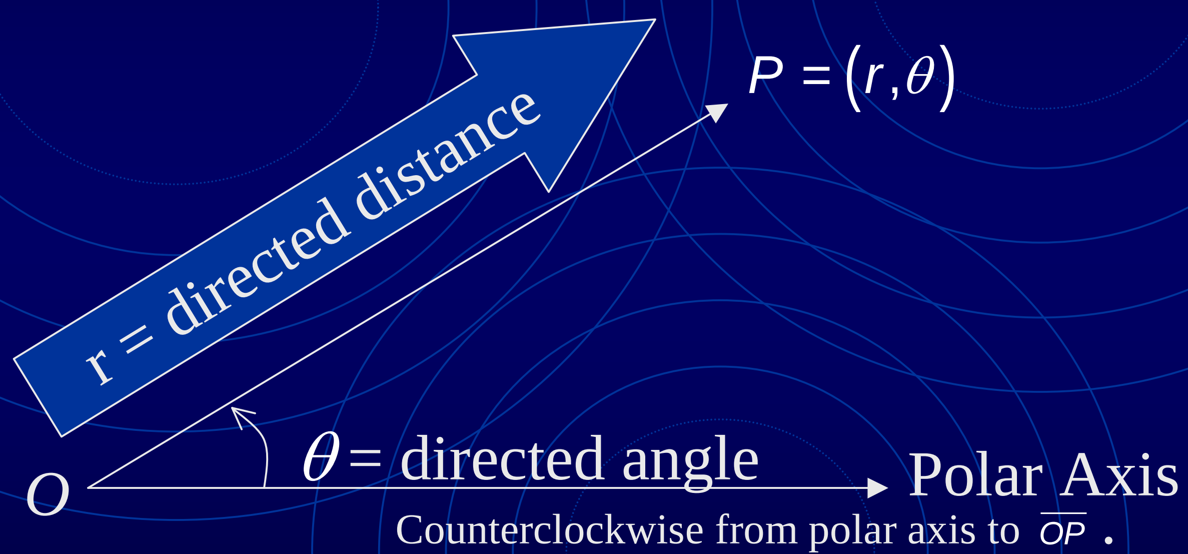
Polar Coordinates and Graphing

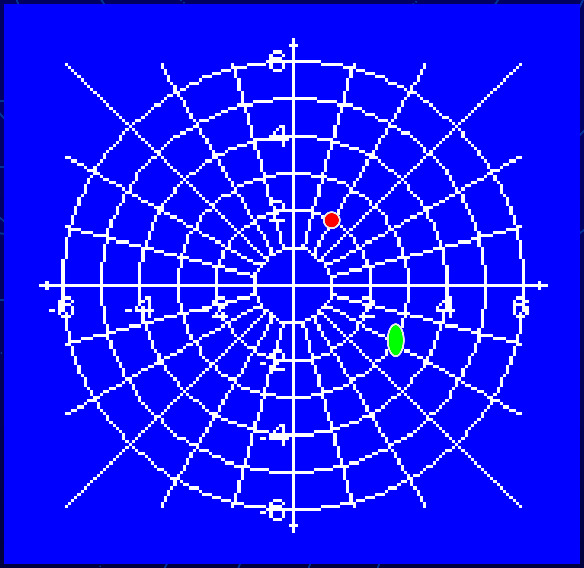


Plotting Points in the Polar Coordinate System

The point $(r,\theta) = \left(2, \frac{\pi}{3}\right)$ lies two units from the pole on the terminal side of the angle $\theta = \frac{\pi}{3}$.

The point $(r,\theta) = \left(3, -\frac{\pi}{6}\right)$ lies three units from the pole on the terminal side of the angle $\theta = -\frac{\pi}{6}$.

The point $(r,\theta) = \left(3, \frac{11\pi}{6}\right)$ coincides with the point $\left(3, -\frac{\pi}{6}\right)$.



Multiple Representations of Points

In the polar coordinate system, each point does not have a unique representation. In addition to $\pm 2\pi$, we can use negative values for r. Because r is a directed distance, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point.

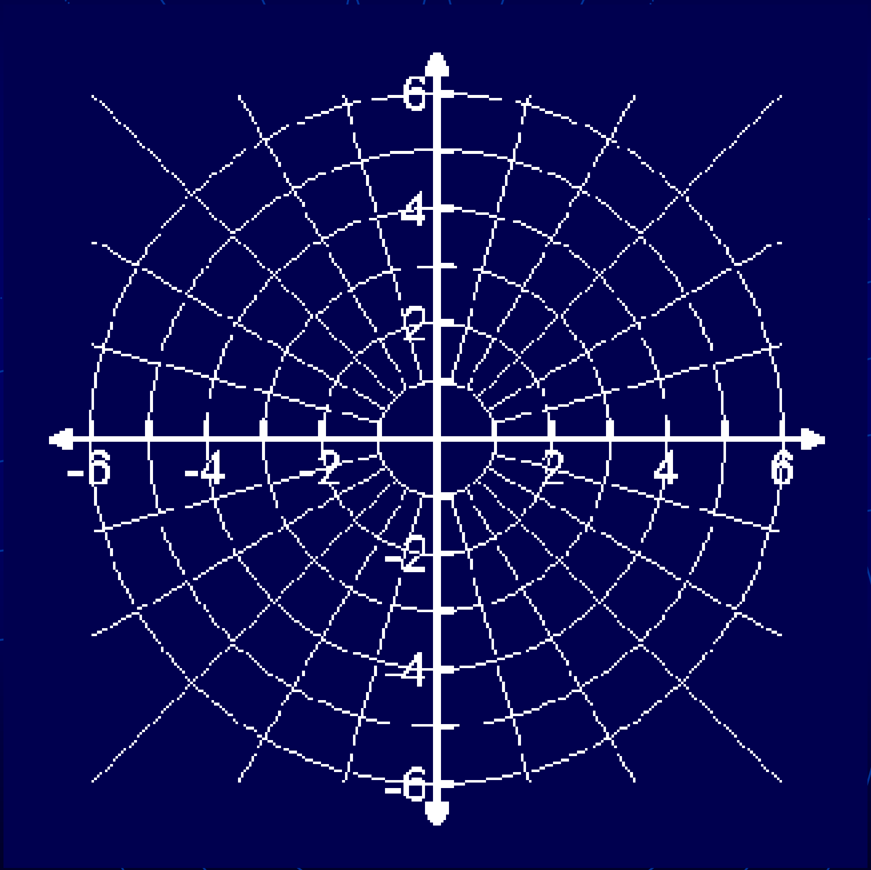
In general, the point (r, θ) can be represented as

$$(\mathrm{r}, \theta) = (r, \theta \pm 2n\pi) \text{ or } (\mathrm{r}, \theta) = (-r, \theta \pm (2n+1)\pi)$$

where n is any integer.

Try plotting the following points in polar coordinates and find three additional polar representations of the point:

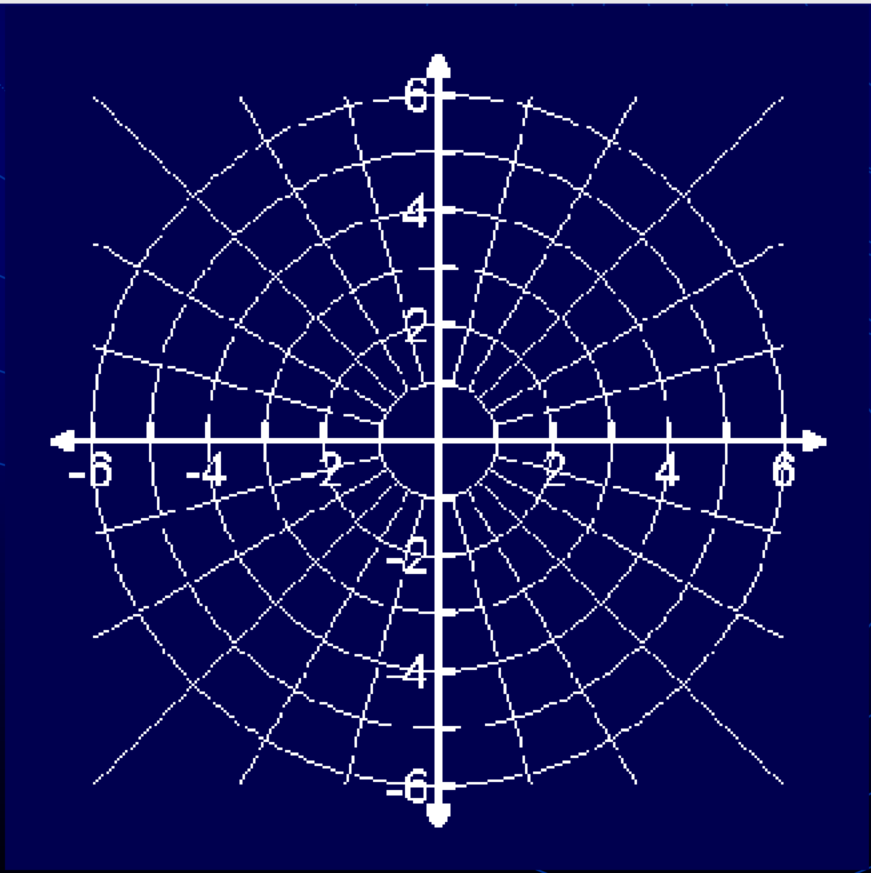
1. $\left(4, \frac{2\pi}{3}\right)$
2. $\left(5, -\frac{5\pi}{3}\right)$
3. $\left(-3, -\frac{7\pi}{6}\right)$
4. $\left(-\frac{7}{8}, -\frac{\pi}{6}\right)$



Graphing a Polar Equation

$$r = 4\sin\theta$$

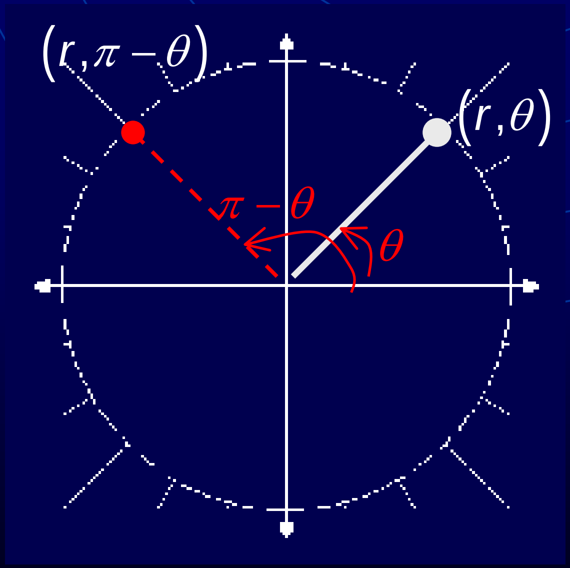
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0



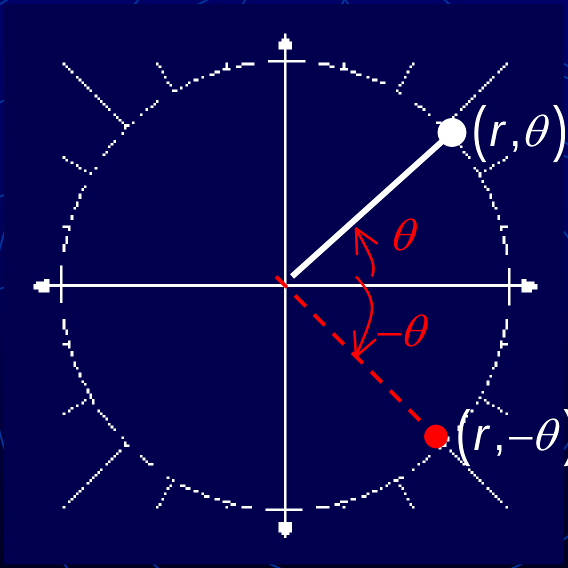
circle

Using Symmetry to Sketch a Polar Graph

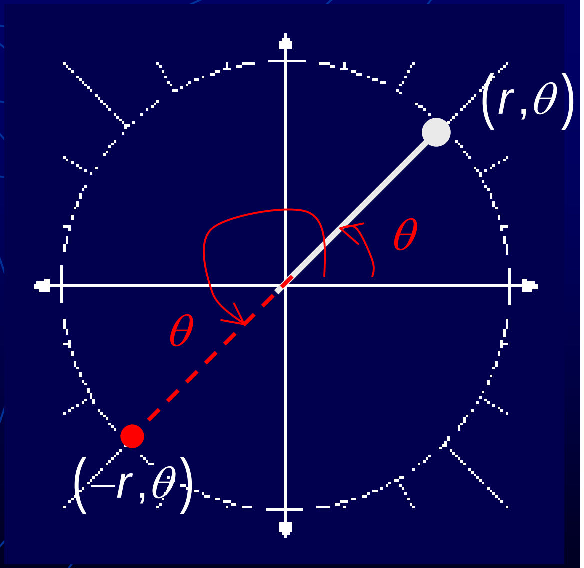
Symmetry with respect to the line $\theta = \frac{\pi}{2}$.



Symmetry with respect to the Polar Axis.



Symmetry with respect to the Pole



Tests for Symmetry

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

1. The line $\theta = \dfrac{\pi}{2}$: Replace (r, θ) with $(r, \pi - \theta)$.

2. The polar axis: Replace (r, θ) with $(r, -\theta)$.

3. The pole: Replace (r, θ) with $(-r, \theta)$.

Using Symmetry to Sketch

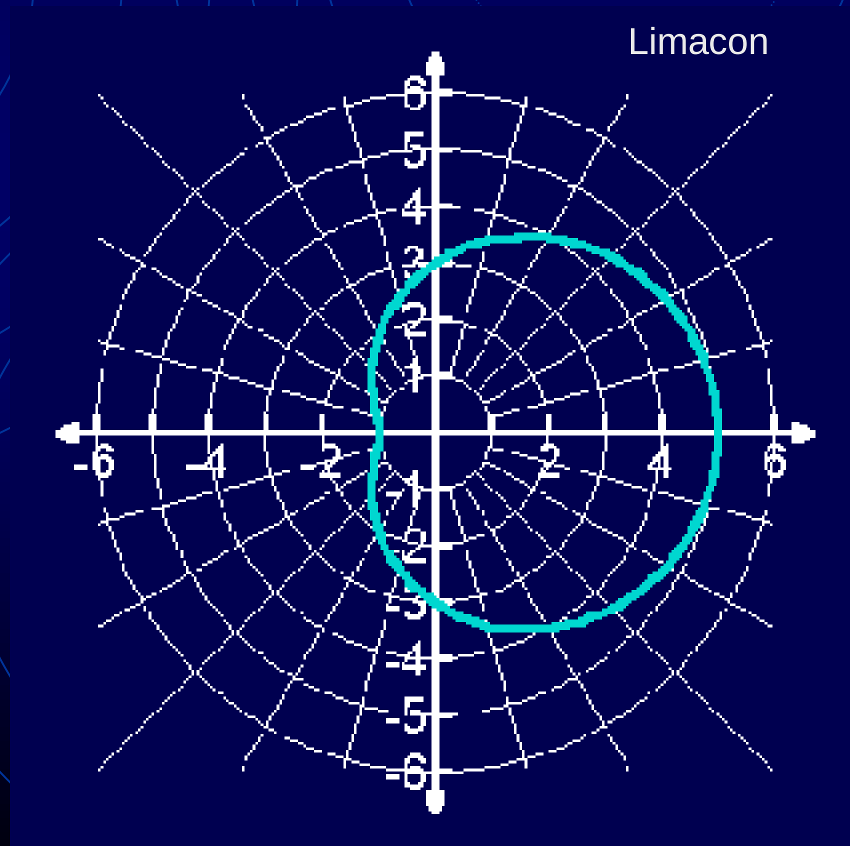
Graph: $r = 3 + 2\cos\theta$.

Replacing (r,θ) by $(r,-\theta)$ produces

$$r = 3 + 2\cos(-\theta)$$
$$= 3 + 2\cos\theta.$$

Thus, the graph is symmetric with respect to the polar axis, and you need only plot points from 0 to π.

θ	r
0	5
$\frac{\pi}{6}$	$3+\sqrt{3}$
$\frac{\pi}{3}$	4
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	2
$\frac{5\pi}{6}$	$3-\sqrt{3}$
π	1



Symmetry Test Fails

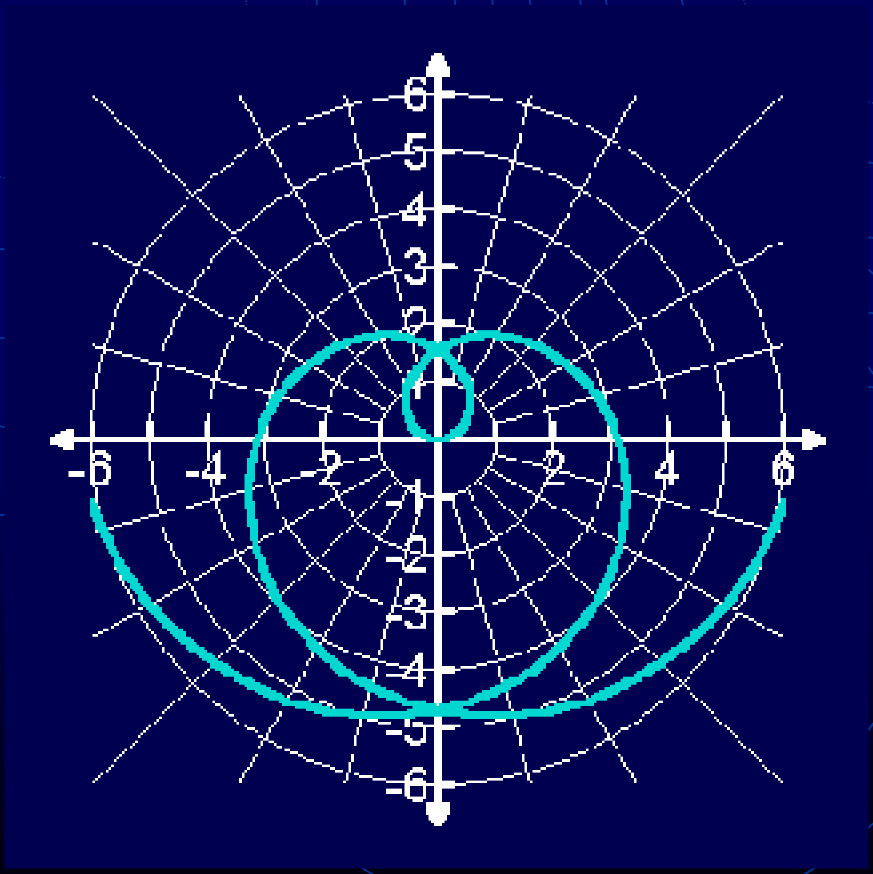
Unfortunately the tests for symmetry can guarantee symmetry, but there are polar curves that fail the test, yet still display symmetry. Let's look at the graph of $r=\theta+2\pi$. Try the symmetry tests. What happens?

Original Equation	*Replacement*	*New Equation*	
$r=\theta+2\pi$	(r,θ) with $(r,\pi-\theta)$	$r=-\theta+3\pi$	$\therefore$ Not symmetric about the line $\theta=\frac{\pi}{2}$.
$r=\theta+2\pi$	(r,θ) with $(r,-\theta)$	$r=-\theta+2\pi$	ॐ Not symmetric about the polar axis.
$r=\theta+2\pi$	(r,θ) with $(-r,\theta)$	$-r=\theta+2\pi$	ॐ Not symmetric about the pole.

All of the tests indicate that no symmetry exists. Now, let's look at the graph.

You can see that the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

$$r = \theta + 2\pi$$



Spiral of Archimedes

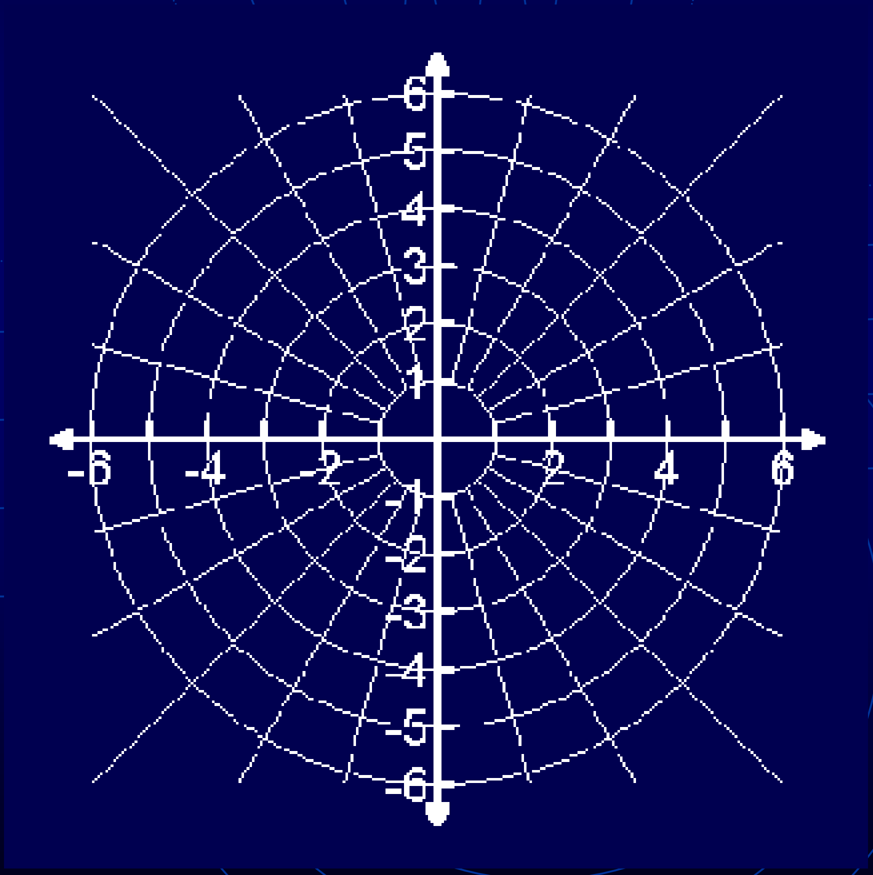
Quick Test for Symmetry

The graph of $r = a(\sin\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The graph of $r = a(\cos\theta)$ is symmetric with respect to the polar axis.

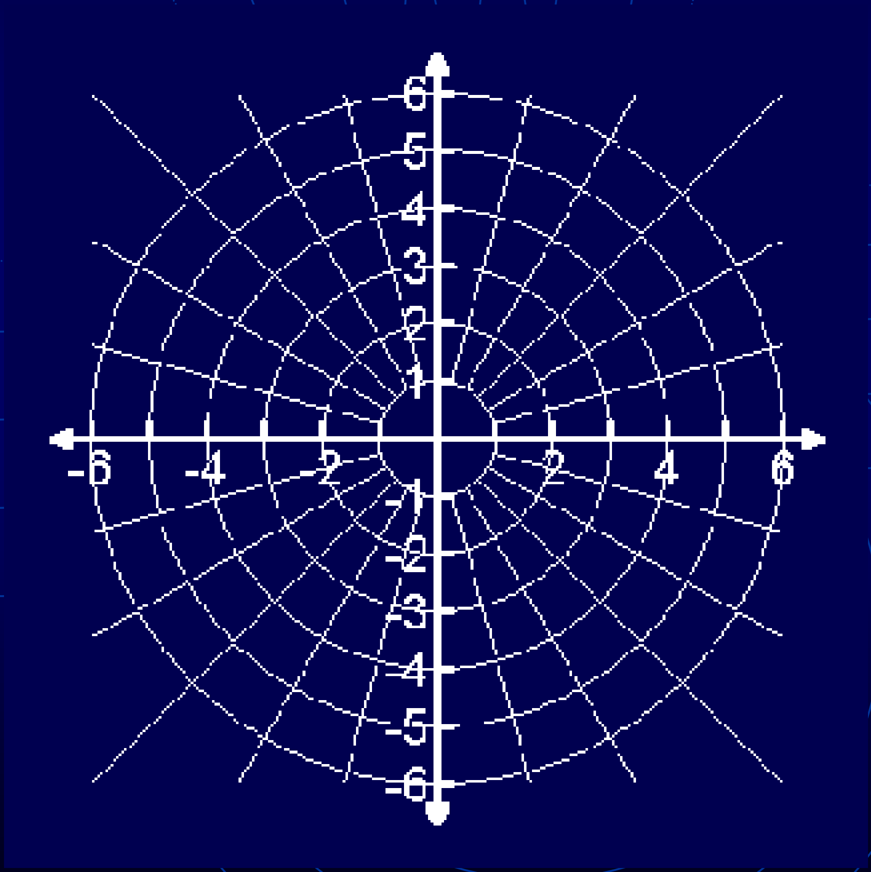
Determine the effect of "a" on the graph of $r = a\sin\theta$.

Click on the graph below.



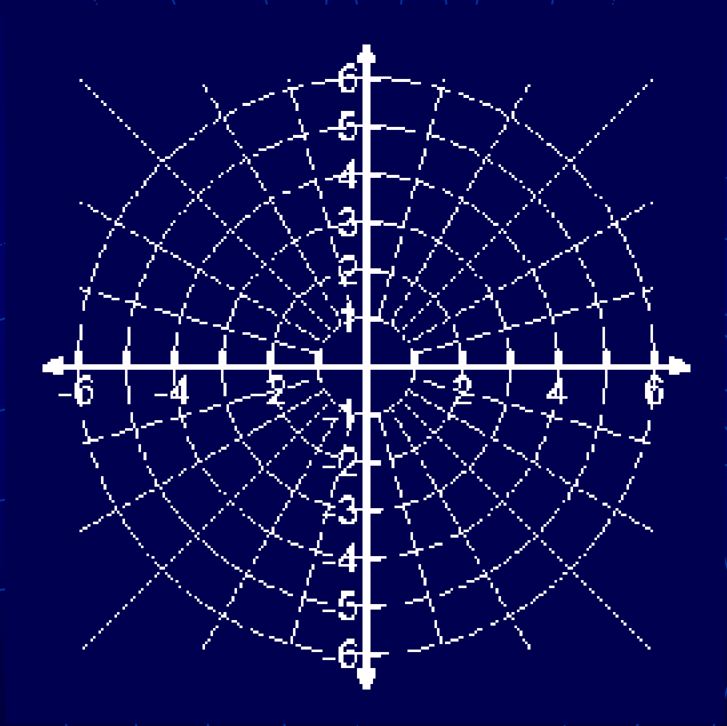
Determine the effect of “a” on the graph of $r = a\cos\theta$.

Click on the graph below.



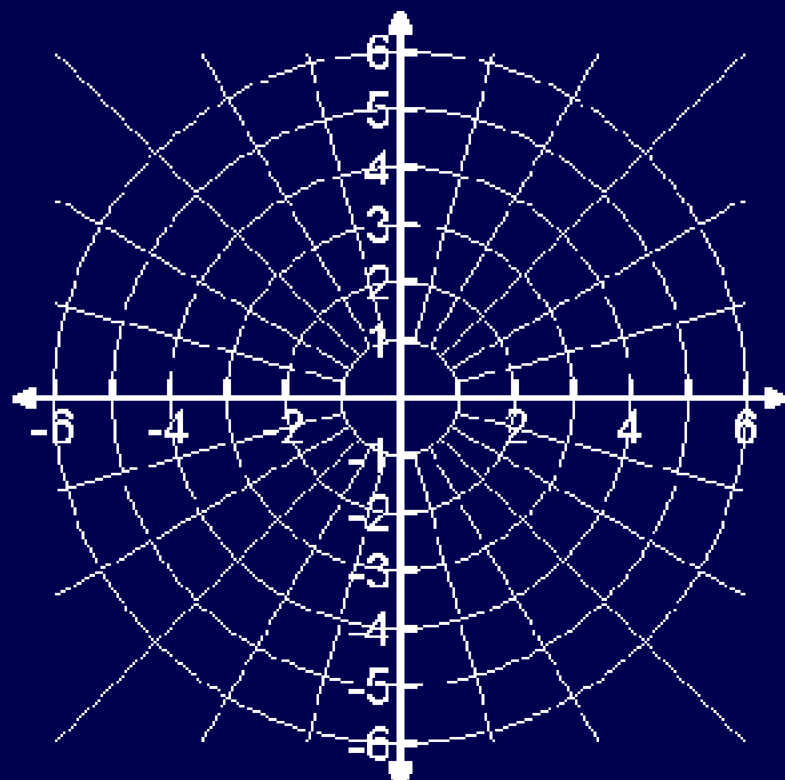
Determine the effect of "a" and "b" on the graph of $r = a \pm b\sin\theta$.

Click on the graph below.



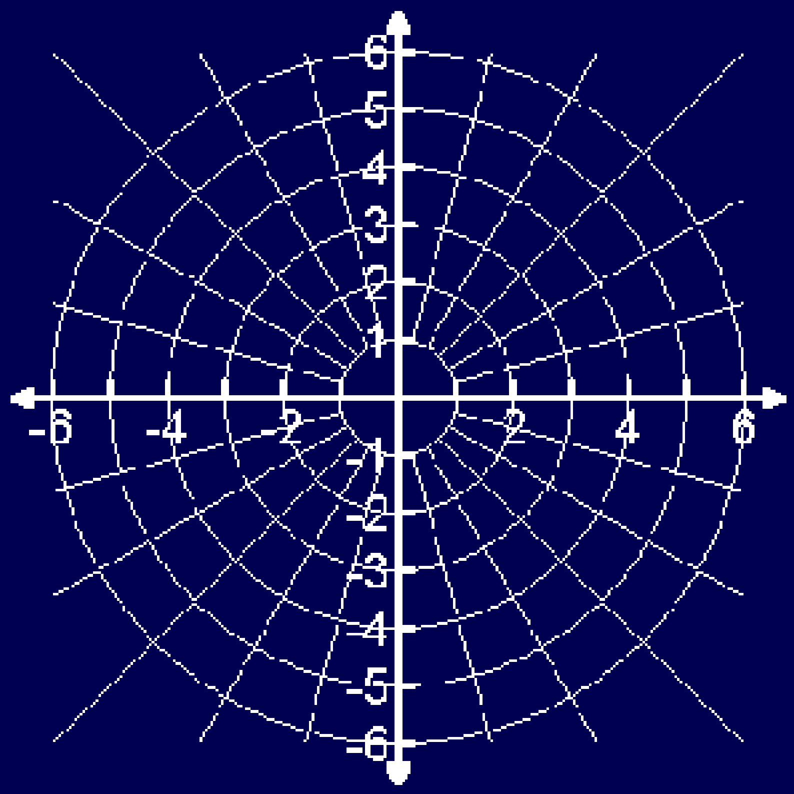
Determine the effect of "a" on the graph of $r = a\sin b\theta$.

Click on the graph below.



Determine the effect of "a" and "b' on the graph of $r = a\cos b\theta$.

Click on the graph below



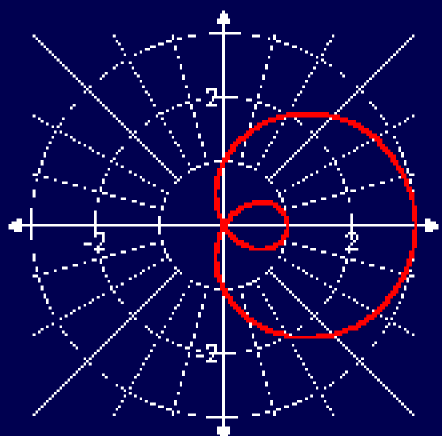
Summary of Special Polar Graphs

Limacons: $r = a \pm b\cos\theta$

$$r = a \pm b\sin\theta$$

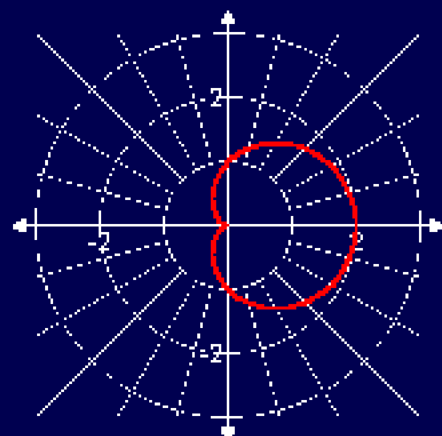
$$(a > 0, b > 0)$$

$\frac{a}{b} < 1$



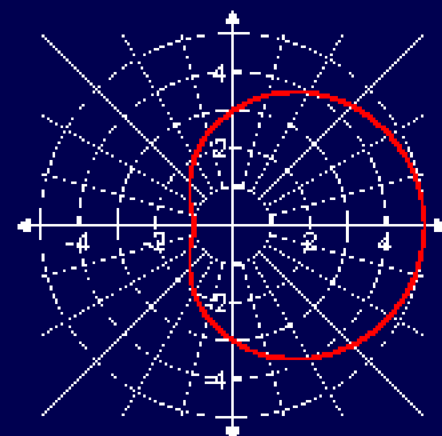
Limacon with
inner loop

$\frac{a}{b} = 1$



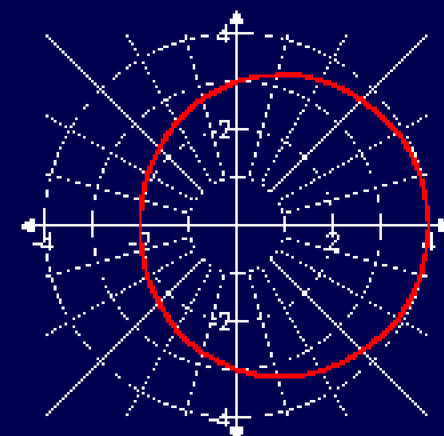
Cardiod
(heart-shaped)

$1 < \frac{a}{b} < 2$



Dimpled
Limacon

$\frac{a}{b} \geq 2$



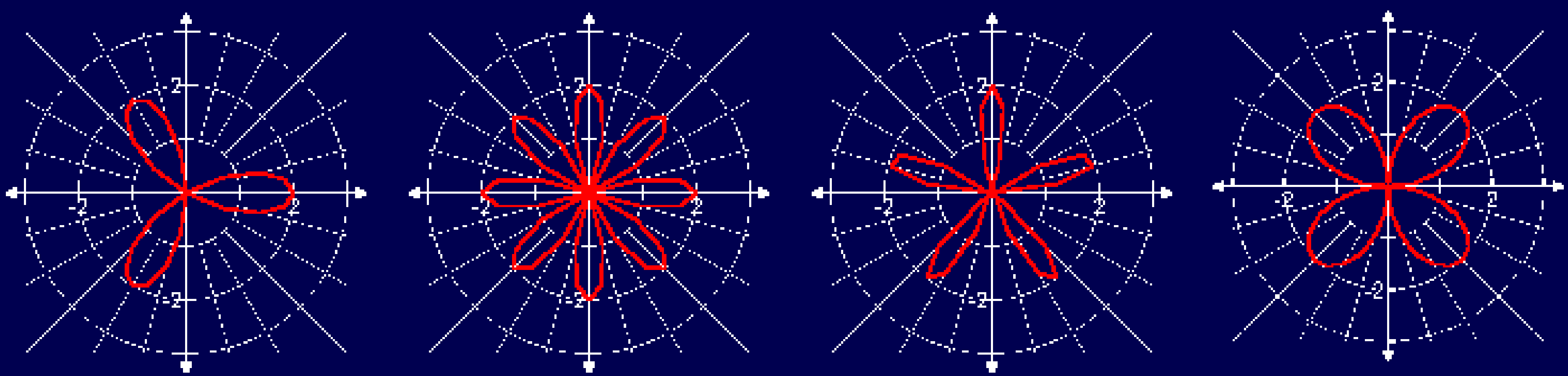
Convex
Limacon

Rose Curves: b petals if b is odd

$2b$ petals if b is even

$(b \geq 2)$

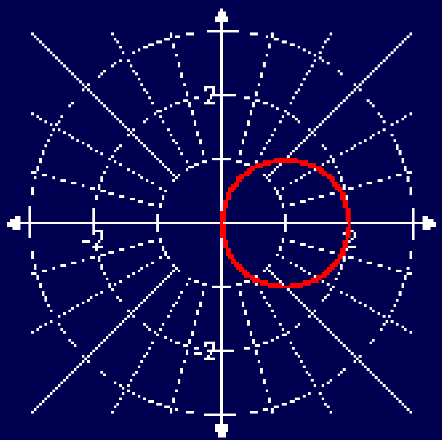
$r = a\cos b\theta$ $\quad$ $r = a\cos b\theta$ $\quad$ $r = a\sin b\theta$ $\quad$ $r = a\sin b\theta$



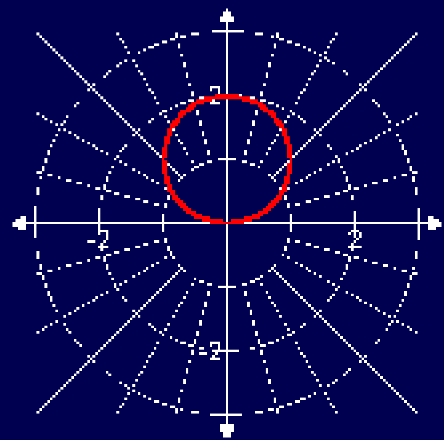
Circles and Lemniscates

Circles

$r = a\cos\theta$

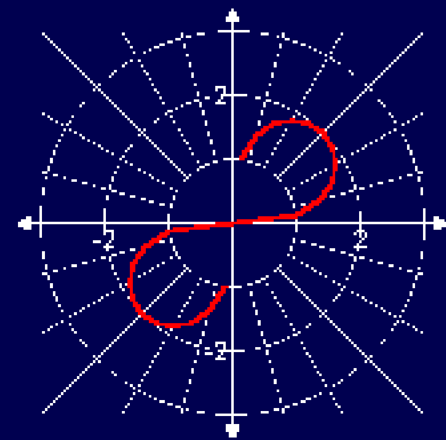


$r = a\sin\theta$

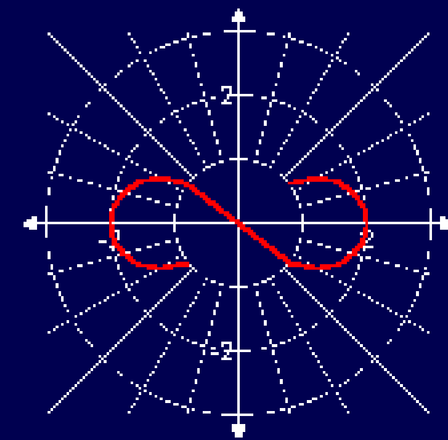


Lemniscates

$r^2 = a^2 \sin 2\theta$



$r^2 = a^2 \cos 2\theta$



Analyzing Polar Graphs

Analyze the basic features of $r = 3\cos 2\theta$.

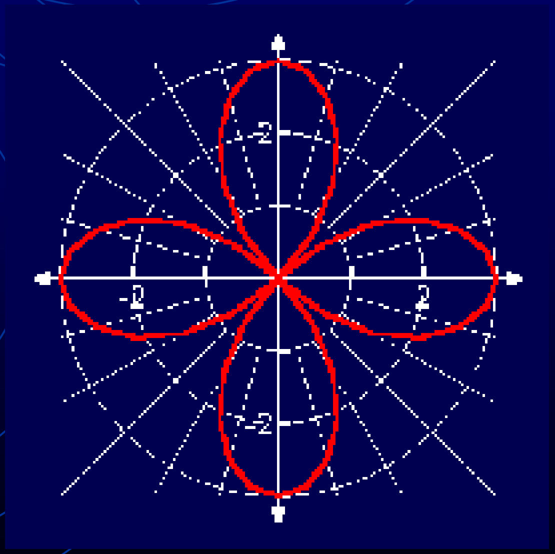
Type of Curve: Rose Curve with 2b petals = 4 petals

Symmetry: Polar axis, pole, and $\theta = \frac{\pi}{2}$.

Maximum Value of | r |: | r | = 3 when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Zeros of r: *r = 0 when* $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

You can use this same process to
Analyze any polar graph.



Analyze and Sketch the graph:

	$r = 2 + \cos\theta$	$r = 1 - \sin\theta$
Type of curve:	Convex Limacon	Cardioid Limacon
Symmetry:	sym. @ polar axis	sym. @ line $\theta = \frac{\pi}{2}$
Maximum | r |	$(3,0)$	$\left(2, \frac{3\pi}{2}\right)$
Zeros of r	none	$\left(0, \frac{\pi}{2}\right)$