1. INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

1.1. The two-body problem

Let $m_1, m_2 > 0$ be the masses of two bodies at positions $X_1(t) \neq X_2(t)$ in $\mathbb{R}^3$ at time t. Applying Newton's law[1] we have[2]

$$\left.\begin{aligned} m_1 \ddot{X}_1 &= \frac{\kappa m_1 m_2}{\|X_1 - X_2\|^2} \frac{X_2 - X_1}{\|X_2 - X_1\|}, \\ m_2 \ddot{X}_2 &= \frac{\kappa m_1 m_2}{\|X_1 - X_2\|^2} \frac{X_1 - X_2}{\|X_1 - X_2\|}. \end{aligned}\right\} \tag{1.1.1}$$

In fact Newton's law of gravity is telling us that the magnitude of the gravitation force between the two bodies is $\frac{\kappa m_1 m_2}{\|X_1 - X_2\|^2}$ and also that it is an attraction force; it means that the gravitation force exerted on the first body should be equal to the rhs of the first line of (1.1.1). On the other hand, Newton's law of motion is saying that the product of the mass and the acceleration is equal to the external force.

We define the center of mass as

$$\widetilde{X} = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}$$

and we see that, from (1.1.1), we have $(m_1+m_2)\ddot{\widetilde{X}} = m_1 \ddot{X}_1 + m_2 \ddot{X}_2 = 0$, so that the motion of the center of mass is a rectilinear uniform motion since $\ddot{\widetilde{X}} \equiv 0$ and thus $\widetilde{X}(t) = \widetilde{X}(0) + t\dot{\widetilde{X}}(0)$.

We define $X(t) = X_2(t) - X_1(t)$ and we get with $M = m_1 + m_2$,

$$\ddot{X} = \ddot{X}_2 - \ddot{X}_1 = -\frac{\kappa m_1}{\|X\|^3} X - \frac{\kappa m_2}{\|X\|^3} X = -\frac{\kappa(m_1 + m_2)}{\|X\|^3} X = -\frac{\kappa M}{\|X\|^3} X. \tag{1.1.2}$$

Remark. We can notice here that if $m_2 \ll m_1$, we have $\widetilde{X} \sim X_1$ and we can consider that X_1 is following a rectilinear motion. As a consequence for the two-body system Sun-Earth,

[1] Isaac Newton (1642–1727) is an english astronomer and mathematician who wrote in 1687 the masterpiece book entitled *Philosophiæ Naturalis Principia Mathematica.*

[2] We assume that the mappings $\mathbb{R} \ni t \mapsto X_j(t) \in \mathbb{R}^3$ are twice differentiable and we use the notations $\dot{X} = \frac{dX}{dt}$, $\ddot{X} = \frac{d^2X}{dt^2}$.

we can assume that the Sun is at the origin of coordinates and that the earth (or another planet) is located at X and follows (1.1.2) where M is (approximately) the mass of the sun; in particular the equation (1.1.2) is the same for the motions of all the planets in the solar system, naturally with different initial conditions.

We note then that $\frac{d}{dt}(\dot{X} \times X) = \ddot{X} \times X + \dot{X} \times \dot{X} = \ddot{X} \times X = 0$ from (1.1.2) so that

$$\dot{X}(t) \times X(t) = \dot{X}(0) \times X(0).$$

We shall assume that the vectors $X(0), \dot{X}(0)$ are independent, implying that the vector $N_0 = \dot{X}(0) \times X(0)$ is different from 0 and that the motion takes place in the plane through $X(0)$ with normal vector N_0: as a matter of fact, we have

$$\frac{d}{dt}\langle X(t) - X(0), N_0\rangle = \langle \dot{X}(t), \dot{X}(0) \times X(0)\rangle = \langle \dot{X}(t), \dot{X}(t) \times X(t)\rangle = 0,$$

and thus $\langle X(t) - X(0), N_0\rangle = 0 \quad$ for all t. As a consequence, we may start over the whole business and assume that $X(t)$ belongs to $\mathbb{R}^2 \backslash \{(0,0)\}$ and satisfies

$$\ddot{X} = -\frac{\kappa M}{\|X\|^3} X. \tag{1.1.3}$$

We define the total energy as

$$H = \frac{1}{2}\|\dot{X}\|^2 - \frac{\kappa M}{\|X\|}. \tag{1.1.4}$$

It is a *first integral* of the motion, i.e. it is constant along the *integral curves* $t \mapsto X(t)$. In fact we have, using (1.1.3-4)

$$\dot{H} = \langle \ddot{X}, \dot{X}\rangle + \frac{\kappa M}{\|X\|^2}\langle \frac{X}{\|X\|}, \dot{X}\rangle = \big(-\frac{\kappa M}{\|X\|^3} + \frac{\kappa M}{\|X\|^3}\big)\langle \dot{X}, X\rangle = 0,$$

and thus

$$\frac{1}{2}\|\dot{X}(t)\|^2 - \frac{\kappa M}{\|X(t)\|} = H_0. \tag{1.1.5}$$

On the other hand the *angular momentum* defined by

$$G(t) = \begin{vmatrix} x_1(t) & \dot{x}_1(t) \\ x_2(t) & \dot{x}_2(t) \end{vmatrix} \tag{1.1.6}$$

with $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, is also a first integral: the already seen fact that $\frac{d}{dt}(\dot{X} \times X) = 0$ (here we identify $X(t)$ wih the vector $\begin{pmatrix} x_1(t) \\ x_2(t) \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$) means that $\dot{G} = 0$. As a result, we have $G(t) = G_0$. Identifying $X(t)$ with the complex number $r(t)e^{i\theta(t)}$, we obtain

$$\left.\begin{aligned} \frac{1}{2}|\dot r e^{i\theta} + ir\dot\theta e^{i\theta}|^2 - \kappa M r^{-1} = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2) - \kappa M r^{-1} = H_0, \\ \operatorname{Im}\big(re^{-i\theta}(\dot r + ir\dot\theta)e^{i\theta}\big) = r^2\dot\theta = G_0. \end{aligned}\right\} \tag{1.1.7}$$

We assume now that $r = 1/u(\theta)$ and we get

$$\frac{1}{2}(\frac{u'^2}{u^4}\dot\theta^2 + \frac{\dot\theta^2}{u^2}) - \kappa M u = H_0, \quad \dot\theta = G_0 u^2,$$

yielding $\frac{G_0^2}{2}(u'^2 + u^2) = (\kappa M u + H_0)$ and thus $G_0^2(u'u'' + uu') = \kappa M u'$ so that

$$u'' + u = \kappa M G_0^{-2}. \tag{1.1.8}$$

Let us assume that at $t = 0$, we have $\theta = 0$ and $\langle \dot X(0), X(0)\rangle = 0$. Then $X(0) = r_0 > 0$ and we may also assume that $\dot X(0) = iv_0$ with $v_0 > 0$ so that $G_0 = r_0 v_0$. As a result, we get, assuming also $r'(0) = 0$,

$$\begin{aligned} \frac{1}{r} = u &= ({r_0}^{-1} - \kappa M G_0^{-2})\cos\theta + \kappa M G_0^{-2} \\ &= \kappa M G_0^{-2}\Big(1 + ({r_0}^{-1} - \kappa M G_0^{-2})\kappa^{-1}M^{-1}G_0^2\cos\theta\Big) \end{aligned}$$

that is

$$r = \frac{\kappa^{-1}M^{-1}G_0^2}{1 + e\cos\theta}, \quad e = ({r_0}^{-1} - \kappa M G_0^{-2})\kappa^{-1}M^{-1}G_0^2 = \frac{G_0^2}{\kappa M r_0} - 1 = \frac{r_0 v_0^2}{\kappa M} - 1.$$

Since $H_0 = \frac{1}{2}v_0^2 - \kappa M r_0^{-1}, G_0 = r_0 v_0$, we get that

$$\frac{r_0 H_0}{\kappa M} = \frac{1}{2}\frac{r_0 v_0^2}{\kappa M} - 1 = \frac{1+e}{2} - 1 = \frac{e-1}{2}$$

and with $E_0 = v_0^2/2, V_0 = -\kappa M/r_0, H_0 = E_0 + V_0$,

$$e = 1 + 2H_0 r_0 \kappa^{-1}M^{-1} = 1 - 2(E_0 + V_0){V_0}^{-1} = -2E_0 V_0^{-1} - 1$$

(1) If $r_0 v_0^2 = \kappa M$ i.e. $-2E_0 V_0^{-1} = 1$ i.e. $E_0 = |V_0|/2$, circle.
(2) If more generally $r_0 v_0^2 < 2\kappa M$ i.e. $E_0 < |V_0|$ ($H_0 = E_0 - |V_0| < 0$) we get an ellipse, with a semimajor axis a and eccentricity e so that

$$r_0 = a - c = a(1-e) = -a\frac{2r_0 H_0}{\kappa M}, \qquad \boxed{a = \frac{\kappa M}{-2H_0}},$$

$$e^2 = r_0^2 v_0^4 \kappa^{-2}M^{-2} - 2r_0 v_0^2 \kappa^{-1}M^{-1} + 1 = 1 + 2\kappa^{-2}M^{-2}r_0^2 v_0^2(2^{-1}v_0^2 - \kappa M r_0^{-1})$$

$$\boxed{e^2 = 1 + 2H_0 G_0^2 \kappa^{-2}M^{-2}}$$

If $|V_0| > E_0 > |V_0|/2$ the initial position is the perihelion: in that case $e = \frac{2E_0}{|V_0|} - 1 \in (0,1)$ and the equation is $r = p/(1 + e\cos\theta)$ with a positive e, so that the minimum of r (perihelion) is attained at $\theta = 0$ and we have indeed $r_0 = a - c = a(1-e)$.
If $E_0 < |V_0|/2$ the initial position is the aphelion: in this case $e = \frac{2E_0}{|V_0|} - 1 \in (-1,0)$ and the equation is $r = p/(1+e\cos\theta)$ with a negative e, i.e. $r = p/(1+|e|\cos(\theta+\pi))$ so that the minimum of r (perihelion) is attained at $\theta = \pi$ and the aphelion is attained at $\theta = 0$ so that $r_0 = a + c = a + |e|a = a(1-e)$, the same formula as before.

(3) If $H_0 = 0$ i.e. $E_0 = |V_0|$, we get a parabola.
(4) If $H_0 > 0$ i.e. $E_0 > |V_0|$, we get an hyperbola.

One of the most remarkable thing about this analysis[3] using differential equations, following Newton's laws established in 1687, is that we can recover the three Kepler's laws.[4] We assume that $H_0 < 0$.

(1) The trajectory is an ellipse (already seen).
(2) The rate of sweeping area is constant since $\frac{d\mathcal{A}}{dt} = \frac{1}{2}G$.
(3) The motion is periodic with a period T such that T^2/a^3 is constant (a is the half-principal axis of the ellipse).

In fact since the rate of sweeping area is constant, we get with $a^2 = b^2 + c^2$,

$$\text{Surface of the ellipse} = \pi ab = TG/2$$

so that

$$T = \frac{2\pi ab}{r_0 v_0} = \frac{2\pi a(a^2-c^2)^{1/2}}{r_0 v_0} = \frac{2\pi a^2(1-e^2)^{1/2}}{r_0 v_0}$$

and

$$\frac{T^2}{a^3} = \frac{4\pi^2 a^4(1-e^2)}{r_0^2 v_0^2 a^3} = \frac{4\pi^2 a(1-e)(1+e)}{r_0^2 v_0^2} = \frac{4\pi^2 a(1-e) r_0 v_0^2}{\kappa M r_0^2 v_0^2}$$
$$= \frac{4\pi^2 a(1-e)}{\kappa M r_0} = \frac{4\pi^2}{\kappa M} \qquad \text{since } a - c = r_0 = a(1-e).$$

1.2. First order scalar equation

Let $a, f : I \to \mathbb{R}$ be continuous functions defined on an interval of $\mathbb{R}$ containing 0. The unique solution of the initial value problem

$$\dot{x}(t) - a(t)x(t) = f(t), \qquad x(0) = x_0,$$

[3]Newton's reasoning in the *Principia* was purely geometrical and it is Leonard Euler (1707–1783) who solved the analytic problem.

[4]Johannes Kepler (1571-1630), a german astronomer, established the three laws that are now named after him; he was a student of the danish astronomer Tycho-Brahé (1546–1601). Tycho-Brahé made a large number of observations in his observatory of Uraniborg located in the island of Hveen, and published the *TabulæRudolphinæ*. Kepler's laws were grounded on the sole observations, and not on general principles.

is given by the formula

$$x(t) = x_0 e^{A(t)} + \int_0^t e^{A(t)-A(s)} f(s) ds, \quad \text{where } A(t) = \int_0^t a(s) ds.$$

1.3. Examples

Separation of variables. The ODE

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{1.3.1}$$

can be formally written as $g(y)dy = f(x)dx$ and if G (resp.F) is an antiderivative of g (resp.f), we get the implicit relation

$$dG = G'(y)dy = g(y)dy = f(x)dx = F'(x)dx = dF, \quad \text{i.e. } G(y) = F(x) + C.$$

In fact the above manipulations are not formal, since assuming that g is a non-vanishing continuous function near some point y_0, and f is a continuous function near some point x_0, the equation

$$g(y(x))y'(x) = f(x),$$

is simply $\frac{d}{dx}\Big(G\big(y(x)\big) - F(x)\Big) = 0$ i.e. the solutions of (1.3.1) should satisfy

$$G(y(x)) = F(x) + C.$$

For instance $\frac{dy}{dx} = y^2$ gives

$$\frac{dy}{y^2} = dx, \quad \frac{1}{y_0} - \frac{1}{y} = x - x_0, \quad y = \frac{y_0}{1 + x_0 y_0 - x y_0}.$$

Note that the solution is identically 0 if $y_0 = 0$ and is not defined at $x_1 = x_0 + \frac{1}{y_0}$ if $y_0 \neq 0$. In particular if $y_0 > 0$, the solution blows-up at a point $x_1 > x_0$, whereas if $y_0 < 0$, the solution blows-up at a point $x_1 < x_0$.

Orthogonal trajectories. Let us consider a family of curves in the plane, indexed by some real parameter t. The curve C_t is defined as

$$C_t = \{(x, y) \in \mathbb{R}^2, F(x, y; t) = 0\},$$

where F is some real-valued function of three real variables. We are looking for the orthogonal trajectories, that is for the curves $\mathbb{R} \ni t \mapsto \gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$ such that

$$\text{for all } t, \gamma(t) \in C_t \text{ i.e.} \quad F(x(t), y(t); t) = 0,$$
$$\text{the curve } \gamma \text{ is orthogonal to } C_t \text{ at } \gamma(t) \text{ i.e.} \quad \dot{\gamma}(t) \text{ is proportional to } d_{x,y}F(\gamma(t)).$$

We obtain the equations $F(x(t), y(t), t) = 0$,

$$\begin{vmatrix} \dot{x} & \partial_x F \\ \dot{y} & \partial_y F \end{vmatrix} = 0. \tag{1.3.2}$$

For instance for the family of homothetic ellipses

$$\frac{x^2}{2} + y^2 = t^2$$

the orthogonal trajectories with equation $y = \varphi(x)$ should satisfy that

$$\begin{vmatrix} 1 & x \\ y' & 2y \end{vmatrix} = 0, \quad \text{i.e. } x\frac{dy}{dx} = 2y, \quad \frac{dy}{2y} = \frac{dx}{x}$$

$\frac{1}{2} \ln |y| = \ln |x| + C, \quad |y| = \lambda x^2$, a family of parabolas.

Homogeneous equations. An homogeneous ODE is of the type,

$$\frac{dy}{dx} = F(y/x). \tag{1.3.3}$$

Defining $v(x) = y(x)/x$, we get $x\frac{dv}{dx} + v = F(v)$ which is the separable equation

$$\frac{dv}{F(v) - v} = \frac{dx}{x}.$$

Exact equations. We consider a *differential form* ω of degree 1 defined in $Q = I \times J \subset \mathbb{R}^2$, where I, J are some intervals of $\mathbb{R}$: it means that we are given two real-valued C^1 functions, a, b defined on Q and we note

$$\omega = adx + bdy.$$

We say that ω is exact when it coincides with the differential of a function F defined on Q, i.e. whenever

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy, \quad \text{i.e. } a = \frac{\partial F}{\partial x},\ b = \frac{\partial F}{\partial y}.$$

An iff condition for ω to be exact in Q is $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. To ω is naturally associated the ODE

$$a(x, y) + b(x, y)\frac{dy}{dx} = 0. \tag{1.3.4}$$

When the 1-form $\omega = dF$ is exact, we find that a solution of this ODE should satisfy $F(x, y(x)) = C$ since if $y(x)$ verifies (1.3.4), we get

$$\frac{d}{dx}\Big(F\big(x, y(x)\big)\Big) = \frac{\partial F}{\partial x}(x, y(x)) + \frac{\partial F}{\partial y}(x, y(x))y'(x) = a(x, y) + b(x, y)y' = 0.$$

For instance, to solve

$$2xy^3 + 3x^2y^2\frac{dy}{dx} = 0, \tag{1.3.5}$$

we note that $\omega = 2xy^3dx + 3x^2y^2dy$ is exact since $\frac{\partial a}{\partial y} = 6xy^2 = \frac{\partial b}{\partial x}$ so that $\omega = dF$ with $\partial_x F = 2xy^3$, i.e. $F = x^2y^3 + \varphi(y)$ and the condition $\partial_y F = 3x^2y^2$ gives φ constant. Finally the solutions of (1.3.5) satisfy the implicit condition

$$x^2y^3 = \lambda^3, \quad \text{i.e. } y = \lambda x^{-2/3}$$

which is well-defined outside of 0.

Integrating factor. It may happen that the 1-form ω is not exact but that $\mu\omega$ is indeed exact for some function μ. The function μ is called an integrating factor. For instance, the 1-form

$$\omega = (y^2 + xy)dx - x^2dy$$

is not exact since $\partial_y a = 2y + x, \partial_x b = -2x$ but

$$x^{-1}y^{-2}\omega = (x^{-1} + y^{-1})dx - xy^{-2}dy$$

is exact since $\partial_y(x^{-1} + y^{-1}) = -y^{-2} = \partial_x(-xy^{-2})$. We are reduced to the previous section.

We may ask the question of a way of determining an integrating factor. Generally speaking, with $\omega = adx + bdy$ given, we would have to solve the equations

$$\partial_y(\mu a) = \partial_x(\mu b), \quad \text{i.e.} \quad b\partial_x\mu - a\partial_y\mu = (\partial_y a - \partial_x b)\mu, \tag{1.3.6}$$

which is a *partial differential equation* (PDE for short) in μ. We shall study later this type of equation in full generality, but some examples are simple to tackle.

Let us find an integrating factor for

$$\omega = (3xy + y^2)dx + (x^2 + xy)dy.$$

According to the discussion above, the form $\mu\omega$ is exact when

$$(x^2 + xy)\partial_x\mu - (3xy + y^2)\partial_y\mu = (3x + 2y - 2x - y)\mu = (x + y)\mu.$$

It is not so easy to solve, but we may try a function μ independent of y: that gives

$$x(x + y)\mu'(x) = (x^2 + xy)\mu'(x) = (x + y)\mu$$

which is satisfied when $x\mu'(x) = \mu$ so that x is indeed an integrating factor for ω: we check

$$\partial_y\big(x(3xy + y^2)\big) = 3x^2 + 2yx = \partial_x\big(x(x^2 + xy)\big).$$

Riccati equations. The general form of the Riccati ODE is

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2. \tag{1.3.7}$$

If we assume that y_1 is a solution, we look for another solution of the form $y_2 = y_1 + \frac{1}{v}$. We then have

$$\begin{aligned}\frac{dy_2}{dx} - (q_0 + q_1 y_2 + q_2 y_2^2) &= -v^{-2}\frac{dv}{dx} + \frac{dy_1}{dx} - (q_0 + q_1 y_1 + q_2 y_1^2) - q_1 v^{-1} - q_2 v^{-2} - 2q_2 y_1 v^{-1} \\ &= -v^{-2}\frac{dv}{dx} - q_1 v^{-1} - q_2 v^{-2} - 2q_2 y_1 v^{-1},\end{aligned}$$

so that for y_2 to be a solution, we just need $v^{-2}\frac{dv}{dx} + q_1 v^{-1} + q_2 v^{-2} + 2q_2 y_1 v^{-1} = 0$, i.e.

$$\frac{dv}{dx} = -vq_1 - q_2 - 2vq_2 y_1 = -v(q_1 + 2q_2 y_1) - q_2$$

which is a linear equation that we can solve explicitly according to the section 1.2.

For instance the equation $y' = 1 + x^2 - 2xy + y^2$ has the solution $y_1 = x$, so that according to the previous discussion $y_2 = y_1 + 1/v$ is an other solution provided that v satisfies

$$v' = -v(-2x + 2x) - 1, \quad v = -x + \lambda, \quad y_2 = x + \frac{1}{\lambda - x}.$$

Bernoulli equations. A Bernoulli equation is an ODE of the type

$$y' + a(x)y = b(x)y^n. \tag{1.3.8}$$

We see that the function $v = y^{1-n}$ satisfies

$$v' = (1-n)y^{-n}y' = (1-n)y^{-n}(-ay + by^n) = (n-1)av + (1-n)b$$

which is a linear equation.

Lagrange equations. A Lagrange equation is an ODE which is linear in x and y, namely

$$a(y')x + b(y')y = c(y'). \tag{1.3.9}$$

Assuming that $x = x(p), y = y(p), \frac{dy}{dx} = p$

$$ax + by = c, \quad a'x + ax' + b'y + b = c', \quad a'x + ax' + b'\frac{c - xa}{b} + b = c'$$

which is a linear equation in x.

Note that for the Lagrange equation, the isocline curves of the ODE, i.e. the set of points

$$C_p = \{(x, y) \in \mathbb{R}^2, y = y(x),\ y \text{ is a solution of the ODE}, y'(x) = p\}$$

are lines. In fact, if the ODE is $F(x, y, y') = 0$ the isoclines are given by the points $(x, y(x))$ such that $F(x, y(x), p) = 0, y'(x) = p$. For the Lagrange equation, we get the line with equation $a(p)x + b(p)y = c(p)$.

Clairaut equations. The Clairaut equation is a particular case of the Lagrange equation:

$$y - xy' = a(y'). \tag{1.3.10}$$

For instance, let us solve $y = xy' - {y'}^2/4$. We assume that $x = x(p), y = y(p)$ and we get

$$y = px - p^2/4, \quad p\dot{x} = \dot{y} = x + p\dot{x} - p/2$$

so that $x = p/2, y = p^2/4$ i.e. $y = x^2$. We have also the affine solutions $x \mapsto \alpha x - \alpha^2/4$ since $\alpha x + \beta = y = x\alpha - \alpha^2/4$. If we consider the envelope of this family of lines, we get a curve $x(t), y(t)$ such that

$$y = tx - t^2/4, \quad \dot{y} = t\dot{x}$$

i.e. $t\dot{x} = \dot{y} = x + t\dot{x} - t/2$ which gives $x = t/2, y = t^2/4$, recovering the solution $y = x^2$.

1.A. Appendix of chapter 1

On the vector product in $\mathbb{R}^3$. We recall that the vector product in $\mathbb{R}^3$ is defined by the formula

$$\langle A \times B, C \rangle = \det(A, B, C), \quad A, B, C \in \mathbb{R}^3. \tag{1.A.1}$$

In particular the vector product $A \times B$ is orthogonal to A and to B since $\det(A, B, B) = \det(A, B, A) = 0$; moreover $A \times B = 0$ if A and B are dependent vectors, since then $\det(A, B, C) = 0$ for all C. Assuming that A, B are independent vectors, and e a unit vector orthogonal to the the plane (A, B) such that $\det(A, B, e) > 0$, we get $A \times B = \lambda e$

$$\begin{aligned}\lambda &= \langle A \times B, e \rangle = \det(A, B, e) \\ &= \text{volume of the parallelepiped } (A, B, e) = \text{area of the parallelogram } (A, B) \times 1.\end{aligned}$$

As a consequence, we recover the familiar definition of the vector product as the unique vector orthogonal to the plane (A, B) whose length is the area of the parallelogram (A, B) and such that $\det(A, B, A \times B) \geq 0$. On the analytic side, the formula (A.1.1) gives

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}. \tag{1.A.2}$$

Conics in polar coordinates. Let Δ be a given line in $\mathbb{R}^2$, F be a point outside Δ and $e \geq 0$. A conic $C(\Delta, F, e)$ with eccentricity e, focus F and directrix Δ is the set of points M such that

$$MF = eMH, \quad MH = \text{distance}(M, \Delta). \tag{1.A.3}$$

In fact assuming that $\Delta \equiv x = p/e$ $(p \geq 0)$ and F is the origin, we get

$$r^2 = e^2 (\frac{p}{e} - r \cos\theta)^2 = (p - er\cos\theta)^2,$$

i.e. $r(1 + e\cos\theta) = p$ or $r(1 - e\cos\theta) = -p$ i.e.

$$C(\Delta, F, e) = \{re^{i\theta}, r(1 + e\cos\theta) = p\}.$$

We note that

(1) For $e = 0$: circle of center F and radius p with equation $r = p$.
(2) For $0 < e < 1$: ellipse with equation $r(1 + e\cos\theta) = p$ (the quantity $(1 + e\cos\theta)$ is positive).
(3) For $e = 1$: parabola with equation $r(1 + \cos\theta) = p$.
(4) For $e > 1$: hyperbola with equation $r(1 + e\cos\theta) = p$.

Looking at the Cartesian coordinates equation from (1.A.3), we obtain for $F = c > 0, e = c/a$, $\Delta \equiv x = \frac{a^2}{c}$, the equation

$$(x-c)^2 + y^2 = \frac{c^2}{a^2}\big(x - \frac{a^2}{c}\big)^2$$

which is

$$x^2\big(\frac{a^2-c^2}{a^2}\big) + y^2 = a^2 - c^2.$$

(1) For $e = 0$, i.e. $c = 0$: circle of center 0 and radius a with equation $x^2 + y^2 = a^2$.
(2) For $0 < e < 1$, i.e. $0 < c < a$, $b^2 = a^2 - c^2$: ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(3) For $e = 1$, i.e. $F = p/2, \Delta \equiv x = -p/2$: parabola with equation $y^2 = 2px$.
(4) For $e > 1$, i.e. $c > a$, $b^2 = c^2 - a^2$: hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Envelopes. Let us consider a family of plane curves Γ_t given by some implicit equation $F(x, y, t) = 0$. We are looking for an envelope $\gamma(t) = (x(t), y(t))$ such that $\gamma(t) \in \Gamma_t$ and the tangent vectors are proportional, namely we have to solve $F(x(t), y(t), t) = 0 = \dot{x}\partial_x F + \dot{y}\partial_y F$ i.e. since $\frac{d}{dt}(F(x(t), y(t), t)) = \dot{x}\partial_x F + \dot{y}\partial_y F + \partial_t F$,

$$F(x, y, t) = 0 = \partial_t F(x, y, t). \tag{1.A.4}$$