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PERSPECTIVES

Why Market-Valuation-Indifferent Indexing Works

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By the end of the 20th century, even casual investors had become comfortable with the idea of index funds. The idea of a *better* index fund (see Arnott, Hsu, and Moore 2005), however, is mind-boggling. This article offers one man's view of why it will actually work. He defines market-valuation-indifferent (MVI) indexing to be indexing in which the index is built on any weights that avoid the problem with market capitalization.

The bad news about stock markets is that they price stocks imperfectly. The good news is that the mispricings are always relative. Not only will overpriced stocks be counterbalanced by underpriced stocks, but the distribution of error at any point in time will be symmetrical. We can picture this distribution as a bell-shaped curve with "error" on the horizontal axis and some measure of "frequency" on the vertical axis. Because it reflects both the number of companies and their size, aggregate value is the appropriate measure of frequency.

But which measure of aggregate value—true value or market value? If we use market value, then, alas, it will make bigger bets on overpriced stocks and smaller bets on underpriced stocks.

To get a handle on how much error, we begin by defining

u = relative error (expressed as a fraction of true value) and

$v(u)$ = amount of *true* value with error.

When we consider the thousands of stocks in the market, the randomness of particular stocks is submerged in a density function that associates a relatively stable amount of density function $v(u)$ with relative error u to satisfy

$$v(u) = v(-u). \tag{1}$$

But $1 + u$ is the *market* value of \$1.00 of *true* value with relative error u. So the amount of market value with error u is $(1 + u)v(u)$; then, the error distribution satisfies

$$(1+u)\,v(u) > (1-u)\,v(-u). \tag{2}$$

Unlike the distribution of the pricing error that uses true value, the error distribution for market values is skewed to the right. This lack of symmetry is the problem with capitalization weighting: By using market values to determine its weights, a cap-weighted index fund will invest more money in overpriced stocks than in underpriced stocks.

Consider a symmetrical distribution of market errors u around a mean error $\bar{u}$. For each stock whose error exceeds the mean by $u - \bar{u}$, there will tend to be a stock whose error falls short of the mean by $\bar{u} - u$. Expressed in terms of a frequency function $v(\ )$ of true values, the original symmetry condition is obviously satisfied by

$$v(u-\bar{u}) = v(\bar{u}-u), \tag{3}$$

because the second argument is indeed minus the first, as we specified. On the other hand, if we expect market errors to be symmetrical around a mean error of zero, we need to add the following condition: weighted by the true values, the mean of the errors in market price is zero. In terms of our symbols, we can express the new condition:

$$\sum uv(u) = 0. \tag{4}$$

Obviously, the sum over all stocks—underpriced and overpriced—is zero.

The Basic Equation

How does MVI indexing avoid cap-weighted indexing's problem? The key is a simple equation linking the covariances of portfolio weights with

- market price per share,
- true value per share, and
- errors in market price per share.

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If, as before, u is the relative error, then $1 + u$ is the ratio of market to true value v and

$$v(1+u) = v + vu \tag{5}$$

is market price. So, to a common divisor equal to the number of stocks, the covariance of portfolio weights w with share prices is

$$\begin{aligned} &\sum wv(1+u) - \sum w \sum u(1+u) \\ &\quad = \sum wv + \sum wvu - \sum w \sum v \\ &\quad = \left[\sum wv - \sum w \sum v\right] + \sum wvu. \end{aligned} \tag{6}$$

The expression in brackets is the covariance of portfolio weights with the true share values. Now, consider the covariance of portfolio weights with dollar errors in share price,

$$\begin{aligned} &\operatorname{cov} w(uv) = \sum wuv - \sum w \sum uv \\ &\sum wuv = \operatorname{cov} w(uv) + \sum w \sum uv. \end{aligned} \tag{7}$$

We see that the expression $\sum wuv$ equals this covariance plus the product $\sum w \sum uv$. But under our expanded symmetry condition, we have

$$\sum uv = \sum w \sum uv = 0. \tag{8}$$

So the first of the three covariances equals the algebraic sum of the second and third. The second is the covariance of portfolio weights with true values, and the third is the covariance of the weights with the dollar errors in prices.

Implications for MVI Indexing

One application of MVI indexing is weighting schemes in which the covariance of weights with market values is zero. In this case, to satisfy Equation 6, either the other two covariances must offset exactly—which is highly improbable—or both must be zero.

An extreme example is a portfolio with equal weights. On average, the number of overpriced stocks will be the same as the number of underpriced stocks. But if all the stocks are assigned the same weight, the investment in the overpriced segment will depend only on that number and the investment in the underpriced segment will depend only on that same number. So the two investments will tend to be equal—in contrast to the cap-weighted index fund, which pays more for the overpriced segment and less for the underpriced segment. Alas, a scheme that weights large-cap and small-cap stocks the same is going to have small-cap market bias, however, relative to many benchmark portfolios, hence more sensitivity to any systematic small-cap factor (as discussed in, for example, Fama and French 1973).

The equation relating the three covariances can be applied in other ways. For example, instead of demonstrating empirically that a given set of weights has zero covariance with market prices, we can appeal to *a priori* reasons why certain sets of weights will have a zero covariance with the errors. We have seen that if the portfolio gives the same weight to underpricing errors it gives to overpricing errors, the third covariance vanishes.

But then the other two covariances in the equation must be equal. So we can use market values, which are observable, rather than true values, which are not, to estimate the small-cap bias in such sets of weights.

Eliminating Small-Cap Bias

Is the constant-weight portfolio the best MVI indexing can do? Does it have the smallest tracking error *versus* a conventional cap-weighted index? Some weighting schemes will have less small-cap bias than others. Examples include weighting by number of employees, number of customers, or sales. And some schemes may actually weight large caps more heavily than the market indexes do. Suppose we used the number of corporate jets or corporate limousines. Readers are encouraged to give free rein to their imagination.

A different approach is to rank stocks by capitalization. Form cap-weighted portfolios that start with the biggest single stock, the biggest 2 stocks, etc., up to 500 stocks. Every one of these portfolios except the last will have a large-cap bias relative to the S&P 500 Index. But each MVI portfolio will have a small-cap bias relative to its corresponding cap-weighted counterpart. Thus there will always be a unique number of stocks for which the MVI portfolio has the same small-cap bias as the cap-weighted S&P 500. If this breakeven portfolio includes enough stocks, it can still be satisfactorily diversified.

We have still other ways to remove small-cap bias. Consider a cap-weighted portfolio of the 100 smallest companies in the Wilshire 5000 Index. It will have

- no alpha resulting from MVI indexing and
- lots of small-cap bias.

A short position in this portfolio will offset a lot of small-cap bias without reducing the MVI alpha.

An appropriate blend of any two schemes with opposite biases will always eliminate bias relative to any given benchmark. And if different clients have different benchmarks, the blend can be tailored to their benchmarks.

The Source of MVI's Advantage

Stocks in the MVI portfolio with a given true value may get a large weight or a small weight. Because they are as likely to be underpriced as overpriced, however, whatever weight the method assigns is as likely to contribute to the underpriced stock as to the overpriced stock. Averaged across all the stocks in the MVI portfolio, the aggregate dollar investments will tend to be the same.

Of course, at a point in time, real stocks won't oblige the author by falling into exactly counterbalancing pairs. But the easiest way to explain how MVI capitalizes on the tendency for pricing errors to be symmetric is to focus on such an idealized pair.

Because of the errors in market price, the corresponding underpriced or overpriced stocks in a *cap-weighted* portfolio will have different market values even if they have the same true values. Let the true values of those stocks be v, and let the aggregate pricing errors be $+e$ and $-e$.

If cap-weighted investors spend $v + e$ dollars on the former and $v - e$ dollars on the latter, they will spend a total of

$$(v + e) + (v - e) = 2v \tag{9}$$

dollars and get

$$(v + e)\left(\frac{v}{v + e}\right) + (v - e)\left(\frac{v}{v - e}\right) = 2v \tag{10}$$

worth of true value.

On the other hand, the MVI investors spend the same number of dollars on the underpriced as they spend on the overpriced stocks. But a dollar spent on overpriced securities buys less true value than a dollar spent on underpriced securities. For example, a dollar spent on a stock with true value v and market price $v + e$ buys $v/(v + e)$ of the true value; a dollar spent on a stock with true value v and market price $v - e$ buys $v/(v - e)$ of the true value. If the MVI investors spend v dollars on each stock, they make the same total investment as the cap-weighted investors and get

$$v\left(\frac{v}{v + e}\right) + v\left(\frac{v}{v - e}\right)$$

worth of true value, or

$$\begin{aligned} v\left[\frac{v(v - e) + v(v + e)}{v^2 - e^2}\right] &= v^2\left(\frac{2v}{v^2 - e^2}\right) \\ &= \left(\frac{v^2}{v^2 - e^2}\right)2v \\ &= \left[\frac{1}{1 - \left(\frac{e}{v}\right)^2}\right]2v, \end{aligned} \tag{11}$$

where the expression in brackets is always greater than zero. (The expression e/v is what we previously called u—price error relative to true value.)

Thus for the same total investment, the MVI investors own more true value than the cap-weighted investors, with a difference that depends only on the relative size of the aggregate pricing error. The gain for the whole market sums across errors occurring with a wide range of frequencies. If the frequency function is $f(e/v)$, then the gain can be expressed as

$$\int_{-\infty}^{\infty} \frac{f(e/v)\,d(e/v)}{1 - (e/v)^2}. \tag{12}$$

For small errors, we can approximate this integral by

$$\begin{aligned} &\int_{-\infty}^{\infty}\left[1 + \left(\frac{e}{v}\right)^2\right] f\left(\frac{e}{v}\right) d\left(\frac{e}{v}\right) \\ &= \int_{-\infty}^{\infty} f\left(\frac{e}{v}\right) d\left(\frac{e}{v}\right) + \int_{-\infty}^{\infty}\left(\frac{e}{v}\right)^2 f\left(\frac{e}{v}\right) d\left(\frac{e}{v}\right). \end{aligned} \tag{13}$$

The value of the first integral is 1. If, as we have assumed for the frequency distribution of true values, the mean of the errors is 0, then the second integral is the variance of the errors.

When stocks are accurately priced, the MVI portfolio realizes no gain relative to the price-weighted portfolio. But when the error in market prices is expressed as a fraction of the true value, then the gain from MVI is the square of the standard error, σ. **Table 1** displays a range of possible values of σ, σ^2, $1 + \sigma^2$, and (for reasons to be explained) $1/(1 - \sigma^2)$. MVI investors realize this benefit even if mispriced stocks never revert to their true values. If reversion occurs, it offers an additional benefit (see Appendix A).

To be sure, the correct integral is not as simply related to the standard error of stock prices as our crude approximation is. But in the event, small pricing errors will be much more frequent than large pricing errors. The reader can get some sense of how bad our approximation is by imagining that, instead of being sample averages, the numbers in

Table 1. MVI's Advantage for Indicated Standard Errors in Market Price

σ	σ^2	$1 + \sigma^2$	$\frac{1}{1 - \sigma^2}$
0.01	0.0001	1.0001	1.0001
0.02	0.0004	1.0004	1.0004
0.04	0.0016	1.0016	1.0016
0.08	0.0064	1.0064	1.0064
0.12	0.0144	1.0144	1.0146
0.14	0.0196	1.0196	1.0200
0.16	0.0256	1.0256	1.0263
0.18	0.0324	1.0324	1.0335
0.20	0.0400	1.0400	1.0417
0.22	0.0484	1.0484	1.0509
0.24	0.0576	1.0576	1.0611
0.26	0.0676	1.0676	1.0725
0.28	0.0784	1.0784	1.0851
0.30	0.0900	1.0900	1.0989
0.32	0.1024	1.1024	1.1141
0.34	0.1156	1.1156	1.1307
0.36	0.1296	1.1296	1.1489
0.38	0.1444	1.1444	1.1688
0.40	0.1600	1.1600	1.1905
0.42	0.1764	1.1764	1.2142
0.44	0.1936	1.1936	1.2401
0.46	0.2116	1.2116	1.2684
0.48	0.2304	1.2304	1.2994
0.50	0.2500	1.2500	1.3333

the "σ" column are price errors on a specific stock, in which that stock's contribution to the approximation error is the difference between the right-hand columns. It takes a 31 percent pricing error to produce a 1 percent error in such a stock's contribution to the integral. And all individual stock errors, small or large, positive or negative, cause the author's approximation to understate the true gain from MVI. But that's the only purpose in including the right-hand column. The author trusts nobody will think it is an estimate of the true value of the integral for the indicated variance.

Because we can't observe the market's pricing errors, we can't readily resolve debates about their magnitude. Eugene Fama has one view; Fischer Black had another. A 1 percent standard error in stock prices produces a gain relative to cap weighting of 0.0001—surely too small to warrant interest in MVI weighting. But the gain increases rapidly as the standard error increases, being 400 times as big for a 20 percent standard error. Can we afford to be wrong about our preconceptions?

Trading Costs

MVI portfolio managers trade more than managers of cap-weighted portfolios, although how much more depends on the price discrepancies the MVI managers choose to tolerate before trading back to the prescribed weights. The trade size will increase with $\sqrt{t}$, so volume will be proportional to

$$\frac{size}{t} = \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}} = \frac{1}{size}. \quad (14)$$

The cost of increasing the trigger size is departure from the portfolio proportions prescribed by MVI. Trading lags bring MVI closer to the cap-weighted result.

When all prices rise or fall in proportion to the MVI portfolio manager's weights, however, no trading is needed.

Conclusion

The author has argued that one doesn't need to know true values in order to avoid the problem with cap-weighted index funds. One can still enjoy all the benefits of an index fund—a high level of diversification and low trading costs—by investing randomly with respect to the market's pricing errors.

Appendix A: Reversion to True Value

The rate of return from the reversion of market value to true value depends on the *reversion rate*. Is the average time to reversion 1 year or 10 years? We do not know.

Presumably, resulting rates of return are also proportional to the initial pricing error. Assume over- and underpriced stocks have the same absolute error e; then, for an overpriced stock with true value v_1 and market price p_1, the rate of return is proportional to

$$\frac{v_1 - p_1}{p_1} = \frac{-e}{p_1}, \quad \text{(A1)}$$

and for an underpriced stock with true value v_2 and market price p_2, the rate of return is proportional to

$$\frac{v_2 - p_2}{p_2} = \frac{e}{p_2}. \quad \text{(A2)}$$

For the MVI investor with equal positions in the two stocks, the average return is

$$\begin{aligned}\frac{1}{2}\left(\frac{e}{p_2}-\frac{e}{p_1}\right)&=\frac{e}{2}\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\\&=\frac{e}{2}\left(\frac{p_1-p_2}{p_1\ p_2}\right)\frac{e}{2}\left(\frac{2e}{p_1\ p_2}\right)\\&=\frac{e^2}{p_1\ p_2}\approx\left(\frac{e}{v}\right)^2\end{aligned} \tag{A3}$$

before dividing by the effective reversion time.

For the whole portfolio, the return is

$$\frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{e}{v}\right)^2 f\left(\frac{e}{v}\right)d\left(\frac{e}{v}\right)=\frac{1}{T}\left(\operatorname{var}\frac{e}{v}\right), \tag{A4}$$

again assuming a mean of zero.

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