CYCLES IN GRAPHS AND DERANGEMENTS

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1. Introduction

My intention in writing this paper is hidden in the following combinatorial relations:

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}, \tag{1.1}$$

$$w_n = (n-2)!\sum_{i=0}^{n-2}\frac{1}{i!} \qquad (n\geq 2). \tag{1.2}$$

Where d_n is the number of derangements of n distinct objects (see [1,2,3]), and w_n is the number of distinct paths between any pair of vertices in a complete graph on n vertices. Considering the well-known expansion $e^x=\sum_{i=0}^{\infty}\frac{x^i}{i!}$ for $x=-1, x=1$ respectively, we can get the following approximate formulas:

$$d_n \approx \frac{n!}{e} \qquad w_n \approx e(n-2)!.$$

Undoubtedly, the following question has come to mind:

Are there any closed form formulas related to (1.1) and (1.2)?

Here, we give an affirmative answer to this question by the following interesting formulas.

2. Two Interesting Formulas About e

Theorem 2.1. *For every positive integer $n\geq 1$,*

$$\sum_{i=0}^{n}\frac{n!}{i!} = \lfloor en! \rfloor \tag{2.1}$$

Proof. Since $n\geq 1$,

$$\frac{1}{n!} > \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots = \sum_{i=0}^{\infty}\frac{1}{i!}-\sum_{i=0}^{n}\frac{1}{i!} > 0$$

and so

$$0 < en! - \sum_{i=0}^{n}\frac{n!}{i!} < 1.$$

Since $\sum_{i=0}^{n}\frac{n!}{i!}$ is an integer, the irrationality of e and the truth of the theorem both follow.

Theorem 2.2. *For every positive integer* $n \geq 1$ *and* $m \in [\frac{1}{3}, \frac{1}{2}]$,

$$d_n = \lfloor \frac{n!}{e} + m \rfloor. \tag{2.2}$$

Proof. We know

$$d_n = n!(1 - \frac{1}{1!} + \cdots + \frac{(-1)^n}{n!}) = \frac{n!}{e} + (-1)^n(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \cdots),$$

so

$$|d_n - \frac{n!}{e}| < \frac{1}{n+1}, \qquad (\forall n \in \mathbb{N}).$$

If n is even, $d_n > \frac{n!}{e}$ and $d_n = \lfloor \frac{n!}{e} + m \rfloor$ provided $\frac{1}{n+1} \leq m \leq 1$. If n is odd, $d_n < \frac{n!}{e}$ and $d_n = \lfloor \frac{n!}{e} + m \rfloor$ provided $0 < \frac{1}{n+1} + m \leq 1$. So we require $\frac{1}{3} \leq m \leq 1$ and $0 \leq m \leq \frac{1}{2}$.

Note and Problem 1. From Theorems (2.1) and (2.2), we obtain

$$\begin{aligned} n!\sum_{i=0}^{n} \frac{1^i}{i!} &= \lfloor en! \rfloor \\ n!\sum_{i=0}^{n} \frac{(-1)^i}{i!} &= \lfloor e^{-1}n! + m \rfloor, \quad m \in [\frac{1}{3}, \frac{1}{2}]. \end{aligned}$$

Now, suppose x is a real number, is there interval I_x, such that

$$n!\sum_{i=0}^{n} \frac{x^i}{i!} = \lfloor e^x n! + m \rfloor, \quad m \in I_x?$$

for example we know $0 \in I_1$ and $[\frac{1}{3}, \frac{1}{2}] \subseteq I_{-1}$. Relations (2.1) and (2.2) are useful for counting. In next theorems you can see some of their usefulness.

3. Paths and Cycles In Complete Graphs

Theorem 3.1. *The number of paths between every pair of vertices in a complete graph on* n *vertices* $(n > 2)$, *is:*

$$w_n = \lfloor e(n-2)! \rfloor,$$

and sum of their lengths is:

$$L_w(n) = 1 + (n-2)\lfloor e(n-2)! \rfloor.$$

Proof. Suppose u and v are two vertices in a complete graph on n vertices. For counting number of paths between u and v, we classify them according their length; a path of length i has the following form:

$$u\overbrace{\bigcirc\bigcirc\bigcirc\cdots\bigcirc}^{(i-1)-\text{vertices}}v.$$

So, the number of paths of length i is:

$$w(i) = \frac{(n-2)!}{(n-1-i)!}$$

and since $1 \leq i \leq n-1$ and applying Theorem (2.1) with $n > 2$, we obtain

$$w_n = \sum_{i=1}^{n-1} w(i) = \sum_{i=1}^{n-1} \frac{(n-2)!}{(n-1-i)!} = \sum_{i=0}^{n-2} \frac{(n-2)!}{i!} = \lfloor e(n-2)! \rfloor.$$

Also

$$L_w(n) = \sum_{i=1}^{n-1} iw(i) = \sum_{i=1}^{n-1} \frac{i(n-2)!}{(n-1-i)!} = 1+(n-2)\sum_{i=0}^{n-2} \frac{(n-2)!}{i!} = 1+(n-2)\lfloor e(n-2)! \rfloor.$$

This completes the proof.

Similarly we can prove next theorem.

Theorem 3.2. *The number of cycles through every vertex in a complete graph on n vertices $(n > 2)$, is:*

$$c_n = \lfloor e(n-1)! \rfloor - n,$$

and sum of their lengths is:

$$L_c(n) = \lfloor en! \rfloor - \lfloor e(n-1)! \rfloor - 2n + 1.$$

Note and Problem 2. We know that a complete graph on n vertices is regular of degree $n-1$. Can we have similar formulas about k-regular graphs?

4. A New Formula For Number of Derangements

Theorem 4.1. *The number of derangements of n distinct objects is:*

$$d_n = \left\lfloor \frac{n! + 1}{e} \right\rfloor.$$

Proof. Apply theorem (2.2) with $m = \frac{1}{e}$.

Obviously , we have other formulas for the number of derangements. For example we know that d_n is the nearest integer to $\frac{n!}{e}$ and the following formula holds.

$$d_n = \begin{cases} \lfloor \frac{n!}{e} \rfloor & n \text{ is odd,} \\ \lfloor \frac{n!}{e} \rfloor + 1 & n \text{ is even.} \end{cases}$$

These all follow from (1.1) and (2.2). Also since $|d_n - \frac{n!}{e}| < \frac{1}{n+1}$ we have $d_n - \frac{n!}{e} \to 0$, that $d_n = \lfloor \frac{n!}{e} + \epsilon \rfloor$ for sufficiently large n for any $\epsilon > 0$.
But it seems that $d_n = \lfloor \frac{n!+1}{e} \rfloor$ is the best closed formula for derangements.

Acknowledgment. I would like to express my gratitude to M. Bayat, H. Teimoori, D. Hebri and referee for their valuable guidance and comments.

References

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