

STRONGLY TORSION GENERATED GROUPS FROM K-THEORY OF REAL C^* -ALGEBRAS

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ABSTRACT. We pursue the program initiated in [7], which consists of an attempt by means of K -theory to construct a strongly torsion generated group with prescribed center and integral homology in dimensions two and higher. Using algebraic and topological K -theory for real C^* -algebras, we realize such a construction up to homological dimension five. We also explore the limits of the K -theoretic approach.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

A group G is *strongly n -torsion generated* for some integer $n \geq 2$, if it possesses a *normal generator* of order n , in other words, an element g_n of order n such that the normal closure of $\{g_n\}$ in G is equal to G . This amounts to saying that every element of G is a product of conjugates of g_n . If this holds for every $n \geq 2$, G is called *strongly torsion generated*. The most familiar examples of such groups are the infinite alternating group A_∞ and the infinite special linear groups $\mathrm{SL}_\infty(\mathbb{Z})$ and $\mathrm{SL}_\infty(K)$, with K a field or a skew field. (For further examples, see [8].) In fact, one of the key ideas in our method is the intimate link of this notion with algebraic K -theory embodied by the fact that *for a unital ring R , the group of infinite elementary matrices $E(R)$ and the infinite Steinberg group $\mathrm{St}(R)$ are strongly torsion generated*. This is established in [3, Lemma 1 and proof of Thm A].

The project we have started in [7] goes as follows. Given n abelian groups A and A_2, \dots, A_n , for some $n \geq 2$, we aim at constructing, as explicitly as possible, a strongly torsion generated group S having, up to isomorphism, A as center and A_q as q th (integral) homology group $H_q(S)$ with q ranging from 2 to n . The reason why there is no occurrence of the first homology of S is that it has to vanish; more precisely, *a strongly torsion generated group is perfect*, see [9, Lemma 7]. For a background motivation for such a mathematical quest, we refer to the Introduction of the companion paper [7], where this program is realized for $n = 3$, using only K -theory. Here, to achieve such a construction of S for $n = 5$, we mix K -theoretical methods with other ideas borrowed from [9], which are based on [2], and which combine combinatorial group theory, topology, K -theory and number theory (in connection with class groups of Dedekind domains, see [9, 13, 20]). Thus, our approach informs the fundamental question of which sequences of abelian groups can be the higher K -groups of a ring [4], [6]. In principle, the present note can be read completely independently of [7].

Our goal here is to establish the following result.

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Theorem 1.1. *Let A, B, C, D and E be five abelian groups. Then, there exists a group S with the following properties:*

- (i) *S is strongly torsion generated;*
- (ii) *the center of S is isomorphic to A , that is, $Z(S) \cong A$;*
- (iii) *S is perfect, that is, $H_1(S) = 0$;*
- (iv) *the higher homology of S , up to dimension five, is given by*

$$H_2(S) \cong B, \quad H_3(S) \cong C, \quad H_4(S) \cong D \quad \text{and} \quad H_5(S) \cong E.$$

In Section 2, we discuss K -theory of C^* -algebras; and thereby, in the following section, given an abelian group M , we construct a suitable real C^* -algebra whose topological K -theory is intimately related to M . This is one of the main constituents in the recipe to construct the group S of Theorem 1.1. We build the group S and establish Theorem 1.1 in Section 4. Finally, in Section 5, we show that our essentially K -theoretical approach to the construction of strongly torsion generated groups with prescribed center and higher homology groups is in a sense best possible. Although we are able to push the method as far as dimension ten, it is only up to dimension five that the homology groups may be arbitrarily prescribed.

2. TOPOLOGICAL AND ALGEBRAIC K -THEORY OF C^* -ALGEBRAS

To prove Theorem 1.1, we need some results on both algebraic and topological K -theory of C^* -algebras that are presented in this section. As general references for this material, we refer to [16, 23, 24, 27, 31, 32, 35]. We in fact only need *real* topological K -theory, but we also quote the results for *complex* topological K -theory, because they probably look more familiar to the less expert reader.

To begin with, there is a delicate issue about the notation “ $\widehat{\otimes}$ ” we use in this paper: when \mathcal{C} and \mathcal{D} are C^* -algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then $\mathcal{C} \widehat{\otimes} \mathcal{D}$ denotes the (completed) *minimal* tensor product over \mathbb{K} .

For a real (resp. complex) C^* -algebra \mathcal{A} (resp. \mathcal{B}), we denote by $KO_*^{\text{top}}(\mathcal{A})$ (resp. $K_*^{\text{top}}(\mathcal{B})$) its topological K -theory. Recall that topological K -theory satisfies *Bott periodicity*; that is, there are canonical and natural isomorphisms

$$KO_*^{\text{top}}(\mathcal{A}) \cong KO_{*+8}^{\text{top}}(\mathcal{A}) \quad \text{and} \quad K_*^{\text{top}}(\mathcal{B}) \cong K_{*+2}^{\text{top}}(\mathcal{B}).$$

For example, one has $K_{2n}^{\text{top}}(\mathbb{C}) \cong \mathbb{Z}$ and $K_{2n+1}^{\text{top}}(\mathbb{C}) = 0$ for $n \in \mathbb{Z}$, and

$n \pmod{8}$	0	1	2	3	4	5	6	7	8
$KO_n^{\text{top}}(\mathbb{R})$	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	\mathbb{Z}	0	0	0	0

Topological K -theory is also *continuous*, in the sense that the functors KO_*^{top} and K_*^{top} commute with filtered colimits (of real or complex C^* -algebras on one side and of abelian groups on the other). Furthermore, topological K -theory is also *Morita invariant*, in the sense that for each $n \geq 1$, there are canonical and natural isomorphisms

$$KO_*^{\text{top}}(\mathcal{A} \widehat{\otimes} M_n(\mathbb{R})) \cong KO_*^{\text{top}}(\mathcal{A}) \quad \text{and} \quad K_*^{\text{top}}(\mathcal{B} \widehat{\otimes} M_n(\mathbb{C})) \cong K_*^{\text{top}}(\mathcal{B}).$$

For $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we let $\mathcal{K}_{\mathbb{K}} \cong \text{colim}_n M_n(\mathbb{K})$ denote the algebra of compact operators on the separable real (resp. complex) Hilbert space $\ell_{\mathbb{K}}^2(\mathbb{N})$. Note that from Morita invariance and continuity, we get canonical isomorphisms

$$KO_*^{\text{top}}(\mathcal{K}_{\mathbb{R}}) \cong KO_*^{\text{top}}(\mathbb{R}) \quad \text{and} \quad K_*^{\text{top}}(\mathcal{K}_{\mathbb{C}}) \cong K_*^{\text{top}}(\mathbb{C}).$$

More generally, topological K -theory is also *stable*, in the sense that there are canonical and natural isomorphisms

$$KO_*^{\text{top}}(\mathcal{A} \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \cong KO_*^{\text{top}}(\mathcal{A}) \quad \text{and} \quad K_*^{\text{top}}(\mathcal{B} \widehat{\otimes} \mathcal{K}_{\mathbb{C}}) \cong K_*^{\text{top}}(\mathcal{B}).$$

Indeed, the above isomorphisms follow from Morita invariance of K_*^{top} and KO_*^{top} and their continuity. In fact, here one can also replace $\mathcal{K}_{\mathbb{K}}$ by $\mathcal{K}(\ell_{\mathbb{K}}^2(L))$ for any nonempty (finite or infinite) set L ; namely, by the C^* -algebra (over \mathbb{K}) of compact operators on the Hilbert space over \mathbb{K} of ℓ^2 -summable sequences, indexed by L , in \mathbb{K} . Finally, topological K -theory is *additive* in the sense that if \mathcal{A}_1 and \mathcal{A}_2 are two real C^* -algebras, and if \mathcal{B}_1 and \mathcal{B}_2 are two complex C^* -algebras, then there are canonical and natural isomorphisms

$$KO_*^{\text{top}}(\mathcal{A}_1 \times \mathcal{A}_2) \cong KO_*^{\text{top}}(\mathcal{A}_1) \oplus KO_*^{\text{top}}(\mathcal{A}_2)$$

and

$$K_*^{\text{top}}(\mathcal{B}_1 \times \mathcal{B}_2) \cong K_*^{\text{top}}(\mathcal{B}_1) \oplus K_*^{\text{top}}(\mathcal{B}_2).$$

For convenience, we now introduce the following definition.

Definition 2.1. Suppose that I is an interval of the real line, having its endpoints in $\mathbb{Z} \amalg \{-\infty, \infty\}$. A real C^* -algebra \mathcal{A} will be called *I -karoubian* if the canonical map

$$K_q^{\text{alg}}(\mathcal{A} \widehat{\otimes} \mathcal{K}(\ell_{\mathbb{R}}^2(L))) \longrightarrow KO_q^{\text{top}}(\mathcal{A} \widehat{\otimes} \mathcal{K}(\ell_{\mathbb{R}}^2(L))) \cong KO_q^{\text{top}}(\mathcal{A})$$

is an isomorphism for every integer $q \in I$ and for every infinite set L . We will say that \mathcal{A} is *karoubian* if it is $(-\infty, \infty)$ -karoubian. We introduce the same terminology for complex C^* -algebras, using complex topological K -theory K_*^{top} , and complex compact operators $\mathcal{K}(\ell_{\mathbb{C}}^2(L))$, in place of KO_*^{top} and $\mathcal{K}(\ell_{\mathbb{R}}^2(L))$ respectively.

The reason for this definition is the Karoubi Conjecture, proved by Suslin-Wodzicki in [28, 29], which says, using this terminology, that *any complex C^* -algebra is karoubian* (to be explicit again, using the *minimal* tensor product for complex C^* -algebras); in fact, Karoubi showed it is $[-\infty, n]$ -karoubian for $n = 0$ in [17] and for $n = 2$ in [18]. It turns out that *any real C^* -algebra is also karoubian* (again, for the *minimal* tensor product, over \mathbb{R}), as shown in Rosenberg [26, Thm 1.4] using Suslin-Wodzicki [28, 29] and Higson [15]. To be very meticulous, all these results are established with L countable (concretely, with $L := \mathbb{N}$) in the notation of Definition 2.1, but the proofs adapt *mutatis mutandis* to the case of an arbitrary infinite set L (merely by choice of an injection of \mathbb{N} in L).

The *real topological suspension* of a real C^* -algebra \mathcal{A} is the real C^* -algebra $S_{\mathbb{R}}^{\text{top}}(\mathcal{A}) := \mathcal{A} \widehat{\otimes} C_0(\mathbb{R} \rightarrow \mathbb{R})$. The *complex topological suspension* of a complex C^* -algebra \mathcal{B} is the complex C^* -algebra $S_{\mathbb{C}}^{\text{top}}(\mathcal{B}) := \mathcal{B} \widehat{\otimes} C_0(\mathbb{R} \rightarrow \mathbb{C})$. The *real* and *complex suspension isomorphisms* are canonical and natural isomorphisms

$$KO_*^{\text{top}}(S_{\mathbb{R}}^{\text{top}}(\mathcal{A})) \cong KO_{*+1}^{\text{top}}(\mathcal{A}) \quad \text{and} \quad K_*^{\text{top}}(S_{\mathbb{C}}^{\text{top}}(\mathcal{B})) \cong K_{*+1}^{\text{top}}(\mathcal{B}).$$

We set $(S_{\mathbb{K}}^{\text{top}})^j := S_{\mathbb{K}}^{\text{top}} \circ (S_{\mathbb{K}}^{\text{top}})^{j-1}$ for $j \geq 1$, with $(S_{\mathbb{K}}^{\text{top}})^0$ standing for the identity functor.

The next result is a straightforward application of Morita invariance and is well-known; we present it as a lemma for later reference. To state it, fix a nonempty set L , an integer $n \geq 0$, and an injection ι of the set $\{1, 2, \dots, n\}$ into L . Consider the corresponding rank n projector $P_n(\iota) \in \mathcal{K}(\ell_{\mathbb{R}}^2(L))$ (thought of as a “diagonal matrix” depending on ι), with 1 as each diagonal entry $\iota(1), \dots, \iota(n)$ and 0 everywhere else; as a matter of convention, $P_0(\iota)$ will stand for the zero operator.

Lemma 2.2. *Let \mathcal{A} be a real C^* -algebra; and, for $n \geq 0$, let*

$$\iota : \{1, 2, \dots, n\} \hookrightarrow L$$

be an injection with corresponding rank n projector $P_n(\iota)$. Then, after completion, the \mathbb{R} -linear map

$$\vartheta_n^{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathbb{R}} \mathcal{K}(\ell_{\mathbb{R}}^2(L)), \quad a \longmapsto a \otimes P_n(\iota)$$

induces a morphism of real C^* -algebras $\widehat{\vartheta}_n^{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{A} \widehat{\otimes} \mathcal{K}_{\mathbb{R}}$ such that the composition

$$KO_*^{\text{top}}(\mathcal{A}) \longrightarrow KO_*^{\text{top}}(\mathcal{A} \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \xrightarrow{\cong} KO_*^{\text{top}}(\mathcal{A})$$

is multiplication by n . A similar result holds for complex C^* -algebras. \square

3. REALIZATION OF GROUPS BY K -THEORY

Before stating the main result of this section, we recall some notation.

Notation 3.1. Given an abelian group M and an integer $n \geq 2$, we set

$$M/n := M/nM \quad \text{and} \quad {}_nM := \{x \in M \mid nx = 0\}.$$

There are isomorphisms $M/n \cong M \otimes_{\mathbb{Z}} \mathbb{Z}/n$ and ${}_nM \cong \text{Tor}(M, \mathbb{Z}/n)$; the former is canonical and natural and, when $n = 2$, so is the latter (see [21, pp. 150–151]).

The following result is proved at the end of the present section.

Proposition 3.2. *Let M be an abelian group. Then, there exists a real C^* -algebra \mathcal{E}_M with the following topological K -theory:*

$n \pmod{8}$	0	1	2	3	4	5	6	7
$KO_n^{\text{top}}(\mathcal{E}_M)$	M	0	0	0	M	$M/2$	$\Omega_2(M)$	${}_2M$

where $\Omega_2(M)$ is a suitable abelian group sitting in a short exact sequence

$$0 \longrightarrow M/2M \longrightarrow \Omega_2(M) \longrightarrow {}_2M \longrightarrow 0.$$

Remarks 3.3.

- (i) By construction, the C^* -algebra \mathcal{E}_M is *not* unital.
- (ii) We do not know how to make the dependence of the real C^* -algebra \mathcal{E}_M in the abelian group M functorial. It would certainly be of independent interest to obtain a functorial construction of it.
- (iii) In [5], we study the abelian group $\Omega_2(M)$ (and also various types of generalizations of it) systematically and with many more details. In particular, we provide another description of it, using algebraic K -theory (for discrete rings, i.e. without C^* -algebras). The abelian group $\Omega_2(M)$ *a priori* seems to depend on the choice of a presentation for M of the form

$$0 \longrightarrow \mathbb{Z}[J] \longrightarrow \mathbb{Z}[I] \longrightarrow M \longrightarrow 0,$$

that is, with given basis sets I, J of the two occurring free abelian groups (compare with the proof of Proposition 3.2 below). (However, see (v) below.) In [5], we show that whenever M is cyclic (finite or infinite), there are canonical isomorphisms

$$\Omega_2(M) \cong K_2^{\text{alg}}(\mathbb{Z}; M) \cong KO_2^{\text{top}}(\mathbb{R}; M)$$

the latter two groups being K -theory with coefficients in M . In particular, this implies that $\Omega_2(\mathbb{Z}/2) \cong \mathbb{Z}/4$. As a consequence, the short exact sequence of Proposition 3.2, with $\Omega_2(M)$ as middle term, does *not* in general split.

- (iv) An alternative approach to this result is provided by work of (chronologically) Bousfield [12], Hewitt [14] and Boersema [10], [11] on CRT-modules and united K -theory. (Such modules comprise three \mathbb{Z} -graded parts: the 2-periodic complex part modelled on complex K -theory, the 8-periodic real part modelled on real K -theory, and the 4-periodic part modelled on self-conjugate K -theory; each pair of graded modules is related by a long exact sequence. The essential information is given by the core that can be described by just the complex and real parts.) Theorem 8.4.4 and the Classification Theorem of [14] show that the displayed groups form the real part

of a CRT-module whose complex part is M in even dimensions, 0 otherwise. Then the Classification Theorem of [14] shows that such an abstract core must be the core of some exact/acyclic CRT-module. In the case when M is countable, the main result of [11] asserts that this CRT-module is then the united K -theory of some real C^* -algebra (indeed, Kirchberg algebra), whose real K -theory is as desired. J. Boersema (private communication) states that a similar argument can be applied in the uncountable case.

- (v) Lemma 8.4.3 of [14] exploits the CR-structure of the K -theory of \mathcal{E}_M to characterize $\Omega_2(M)$ (there labelled $J(M)$) as the unique group extension above such that the obvious composite

$$\Omega_2(M) \twoheadrightarrow {}_2M \hookrightarrow M \twoheadrightarrow M/2 \twoheadrightarrow \Omega_2(M)$$

is multiplication by 2. An advantage of this characterization is that it reveals $\Omega_2(M)$ to be independent of choice of presentation for M .

Before we pass to the proof of Proposition 3.2, we derive a consequence, which shows that one can nearly prescribe the topological K -theory for real C^* -algebras in degrees 0, 1, 2 and 3 modulo 8 (compare [32, Ex. 9.H, pp. 173–174]).

Corollary 3.4. *Let A_0, \dots, A_7 be eight abelian groups. Then, there exists a real C^* -algebra \mathcal{A} whose topological K -theory is given by*

$$KO_n^{\text{top}}(\mathcal{A}) \cong A_{[n]} \oplus A_{[n+4]} \oplus {}_2A_{[n+5]} \oplus \Omega_2(A_{[n+6]}) \oplus A_{[n+7]}/2,$$

where $[q] \in \{0, 1, \dots, 7\}$ denotes the reduction modulo 8 of the integer q .

Proof. It suffices to take

$$\mathcal{A} := \prod_{j=0}^7 (S_{\mathbb{R}}^{\text{top}})^{[j+4]} \mathcal{E}_{A_j},$$

where $[j+4] \in \{0, 1, \dots, 7\}$ is the reduction modulo 8 of $j+4$. \square

Corollary 3.5. *Let B_0, B_1, B_2, B_3 be four abelian groups. Then there exists a real C^* -algebra \mathcal{B} whose topological K -theory is given by:*

$$\begin{array}{c|c|c|c|c} n \pmod{8} & 0 & 1 & 2 & 3 \\ \hline KO_n^{\text{top}}(\mathcal{B}) & B_0 \oplus {}_2B_1 & B_1 & B_2 & B_2/2 \oplus B_3 \end{array}.$$

Proof. In the previous corollary choose $B_0 = A_4$, $B_1 = A_5$, $B_2 = A_2$, $B_3 = A_3$, with the remaining A_k all zero. \square

Further in this vein, the following interesting result is due to an anonymous referee.

Proposition 3.6. *Any three prescribed abelian groups may be realized as the consecutive KO -groups of a real C^* -algebra.*

Proof. For any abelian group M , consider the abelian group $M \oplus \psi^{-1}M$ with involution ψ^{-1} that interchanges summands (in CRT-theory, ψ^{-1} corresponds to complex conjugation in K -theory) [12] (3.5). There is then a CRT-module whose complex part is $M \oplus \psi^{-1}M$ in even dimensions, and zero otherwise [14] Theorem 6.0.5, Example 6.0.6(iii). It follows from the usual exact sequences (or [14] Theorem 8.4.2) that the real part consists of the fixed subgroup under ψ^{-1} , and is therefore isomorphic to M in even dimensions, zero otherwise. Then by Boersema's realization theorem (extended, if necessary, to the uncountable case), there is a real C^* -algebra \mathcal{F}_M say, such that $KO_*^{\text{top}}(\mathcal{F}_M)$ is M in even dimensions, zero otherwise.

Hence, given three abelian groups C_0, C_1, C_2 , the real C^* -algebra

$$\mathcal{C} = \mathcal{E}_{C_0} \times S\mathcal{F}_{C_1} \times S^6\mathcal{E}_{C_2}$$

has $KO_n^{\text{top}}(\mathcal{C}) = C_n$ for $n = 0, 1, 2$. \square

Note that a contrasting result is presented as Example 5.1 below.

Proof of Proposition 3.2. For a given abelian group M , we proceed almost exactly as in [32, Ex. 9.H, pp. 173–174], but correcting a critical mistake (and without implicitly assuming that I and J are countable). For a general and systematic approach in a much broader framework, we refer to [5]. So, to begin with, we consider a presentation of M as an abelian group, say

$$0 \longrightarrow \mathbb{Z}[J] \xrightarrow{\psi} \mathbb{Z}[I] \xrightarrow{\pi} M \longrightarrow 0$$

with J and I some sets providing (unordered) bases for $\mathbb{Z}[J]$ and $\mathbb{Z}[I]$, respectively. Denote by $N = (n_{ij})_{i \in I, j \in J} \in M_{I \times J}(\mathbb{Z})$ the corresponding matrix of ψ . As explained in *loc. cit.*, we can always find such a “based presentation” such that N has nonnegative entries, and we make this assumption.

Next, we fix an infinite set L (understood in the notation from now on) and two injections $J \hookrightarrow L$ and $\mathbb{N} \hookrightarrow L$ (considered as inclusions for simplicity); for example, when J is nonempty with its basepoint chosen, the set $L = J \times \mathbb{N}$ with the obvious two inclusions. To remain short and to preserve readability, we set

$$\mathcal{K} := \mathcal{K}(\ell_{\mathbb{R}}^2(L))$$

and we fix a bijection $L \xrightarrow{\cong} L \times L$, which induces, by conjugation, an isometric isomorphism of real Hilbert spaces $\ell_{\mathbb{R}}^2(L \times L) \xrightarrow{\cong} \ell_{\mathbb{R}}^2(L)$. Note that there is a canonical (up to choosing left or right) isometric isomorphism of real Hilbert spaces $\widehat{\bigoplus}_L \ell_{\mathbb{R}}^2(L) \xrightarrow{\cong} \ell_{\mathbb{R}}^2(L \times L)$, where $\widehat{\bigoplus}_L \ell_{\mathbb{R}}^2(L)$ is a Hilbert sum of real Hilbert spaces. We hence get a composition $\nabla: \widehat{\bigoplus}_L \mathcal{K} \rightarrow \mathcal{K}$ of morphisms of real C^* -algebras:

$$\begin{aligned} \nabla: \widehat{\bigoplus}_L \mathcal{K} &= \widehat{\bigoplus}_L \mathcal{K}(\ell_{\mathbb{R}}^2(L)) \hookrightarrow \mathcal{K}\left(\widehat{\bigoplus}_L \ell_{\mathbb{R}}^2(L)\right) \\ &\xrightarrow{\cong} \mathcal{K}(\ell_{\mathbb{R}}^2(L \times L)) \xrightarrow{\cong} \mathcal{K}(\ell_{\mathbb{R}}^2(L)) = \mathcal{K}, \end{aligned}$$

where the completed direct sum $\widehat{\bigoplus}_L \mathcal{K}$ is defined as the colimit, in the category of real C^* -algebras and over the poset of nonempty finite subsets F of L , of the corresponding finite cartesian products of real C^* -algebras $\prod_F \mathcal{K}$. Let F be a nonempty finite subset of L ; there is an obvious injective morphism of real C^* -algebras $j_F: \prod_F \mathcal{K} \hookrightarrow \widehat{\bigoplus}_L \mathcal{K}$, and a “diagonal inclusion” $i_F: \mathcal{K} \hookrightarrow \prod_F \mathcal{K}$, namely the unique morphism whose composition with any of the $|F|$ canonical projections $\prod_F \mathcal{K} \twoheadrightarrow \mathcal{K}$ is the identity. It is clear from the construction (compare Lemma 2.2) that for each nonempty finite subset F of L , the composition

$$\iota_F: \mathcal{K} \xrightarrow{i_F} \prod_F \mathcal{K} \xrightarrow{j_F} \widehat{\bigoplus}_L \mathcal{K} \xrightarrow{\nabla} \mathcal{K}$$

induces multiplication by $|F|$ on the level of KO_*^{top} -theory; correspondingly, if F is empty, we let $\iota_F: \mathcal{K} \rightarrow \mathcal{K}$ be the zero morphism. When $F = \{1, \dots, n\}$ for some integer $n \in \mathbb{N}$ (including $n = 0$), we write ι_n for ι_F .

Now, consider the (well-defined) morphism of real C^* -algebras

$$\alpha: \widehat{\bigoplus}_J \mathcal{K} \longrightarrow \widehat{\bigoplus}_I \mathcal{K}$$

given by the composition (of morphisms of real C^* -algebras)

$$\widehat{\bigoplus}_J \mathcal{K} \xrightarrow{\widehat{\bigoplus}_J (\iota_n \bullet j)} \widehat{\bigoplus}_J \widehat{\bigoplus}_I \mathcal{K} \xrightarrow{\cong} \widehat{\bigoplus}_I \widehat{\bigoplus}_J \mathcal{K} \hookrightarrow \widehat{\bigoplus}_I \widehat{\bigoplus}_L \mathcal{K} \xrightarrow{\widehat{\bigoplus}_I \nabla} \widehat{\bigoplus}_I \mathcal{K}$$

where, for each $j \in J$, the map $(\iota_{n_{\bullet j}}): \mathcal{K} \longrightarrow \widehat{\bigoplus_I \mathcal{K}}$ denotes the unique morphism of real C^* -algebras whose postcomposition by the i th projection map, namely

$$\mathcal{K} \xrightarrow{(\iota_{n_{\bullet j}})} \widehat{\bigoplus_I \mathcal{K}} \xrightarrow{\text{pr}_i} \mathcal{K},$$

is for each $i \in I$ the map $\iota_{n_{ij}}$ defined above using ∇ and the inclusion of the finite (possibly empty) set $\{1, \dots, n_{ij}\}$ into L . The fact that α is well-defined follows from the column-finiteness of the matrix N , see details in [5]. Observe that at the level of KO_*^{top} -theory, the morphism α really induces an abelian group homomorphism given, in the obvious way, by the integral matrix N : see Lemma 2.2 again and recall from the beginning of the present section that topological K -theory is additive and continuous, hence “converts” a completed direct sum “ $\widehat{\bigoplus}$ ” into the corresponding direct sum “ \bigoplus ” of abelian groups. (Of course, it is in this construction of α and to achieve this latter property of $KO_*^{\text{top}}(\alpha)$ that we need that the entries of N are nonnegative.)

Note. The mistake in [32, Ex. 9.H, pp. 173–174] alluded to earlier is that the map α therein is the wrong one; indeed, it is *not* multiplicative, *i.e.* *not* a morphism of \mathbb{R} -algebras. This traces back to the fact that there is no coproduct in the categories of (nonunital) rings and \mathbb{R} -algebras, and even more precisely, that the coproduct of the underlying additive abelian groups is *not* a coproduct of rings because the universal maps — *i.e.* given by the universal property, with the coproduct as domain — are *not* multiplicative in general (the coproduct additive group being equipped with the obvious termwise multiplication, *i.e.* being viewed as a subring of the corresponding cartesian product of rings).

Next, for readability, we set

$$\mathcal{A}_J := \widehat{\bigoplus_J \mathcal{K}}, \quad \mathcal{A}_I := \widehat{\bigoplus_I \mathcal{K}} \quad \text{and} \quad \mathcal{B}_I := S_{\mathbb{R}}^{\text{top}}(\mathcal{A}_I),$$

and we define the real C^* -algebra $\mathcal{C}_\alpha = \mathcal{C}_\alpha(M)$ as the *mapping cone* of α , that is,

$$\mathcal{C}_\alpha := \{(a, f) \in \mathcal{A}_J \times C[0, 1] \rightarrow \mathcal{A}_I \mid f(0) = 0, f(1) = \alpha(a)\}.$$

The long exact sequence in K -theory associated to the extension of real C^* -algebras

$$0 \longrightarrow \mathcal{B}_I \longrightarrow \mathcal{C}_\alpha \longrightarrow \mathcal{A}_J \longrightarrow 0$$

is a 24-term cyclic exact sequence, reading

$$\cdots \rightarrow KO_1^{\text{top}}(\mathcal{A}_J) \rightarrow \underbrace{KO_0^{\text{top}}(\mathcal{B}_I)}_{\cong KO_1^{\text{top}}(\mathcal{A}_I)} \rightarrow KO_0^{\text{top}}(\mathcal{C}_\alpha) \rightarrow KO_0^{\text{top}}(\mathcal{A}_J) \rightarrow \underbrace{KO_1^{\text{top}}(\mathcal{B}_I)}_{\cong KO_0^{\text{top}}(\mathcal{A}_I)} \rightarrow \cdots$$

Using the real suspension isomorphism and the value of $KO_*^{\text{top}}(\mathcal{K}) \cong KO_*^{\text{top}}(\mathbb{R})$ recalled earlier in this section, and applying Lemma 2.2 (more specifically, the fact that the homomorphism $KO_*^{\text{top}}(\alpha)$ is given by the matrix N), one readily obtains that

$n \pmod{8}$	0	1	2	3	4	5	6	7
$KO_n^{\text{top}}(\mathcal{C}_\alpha)$	$M/2$	$\Omega_2(M)$	$2M$	M	0	0	0	M

exploiting the fact that $\pi \otimes \text{id}_{\mathbb{Z}/2}$ is surjective and has kernel canonically isomorphic to $\text{Tor}(M, \mathbb{Z}/2) \cong {}_2M$. Finally, it suffices to take $\mathcal{E}_M := (S_{\mathbb{R}}^{\text{top}})^3(\mathcal{C}_\alpha)$, *i.e.* the 3-fold real topological suspension of the mapping cone real C^* -algebra \mathcal{C}_α . \square

4. CONSTRUCTION OF S AND PROOF OF THEOREM 1.1

In the present section, we make the construction of the group S of Theorem 1.1 explicit using results of [9], and then we establish the theorem.

First, letting A, B, C, D and E be fixed abelian groups, by [9, Thm 1] there exists a *strongly torsion generated* group $S' = S'(B, C, D, E, 0, \dots)$ with *trivial center* and with reduced integral homology given by

$$\begin{array}{c|c|c|c|c|c|c|c|c} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \hline H_n(S') & 0 & B & C & D & E & 0 & 0 & \cdots \end{array}$$

Next, as the group $S = S(A, B, C, D, E)$, we take the cartesian product

$$S := \text{St} \left((S^{\text{alg}})^{10} (\widehat{\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}}) \right) \times S'(B, C, D, E, 0, \dots).$$

Some explanations are in order. Firstly, $\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}$ is the real C^* -algebra, considered as a mere nonunital ring, obtained by stabilizing the real C^* -algebra \mathcal{E}_A of Proposition 3.2. Secondly, for a nonunital ring \mathcal{I} , $\widetilde{\mathcal{I}}$ is the *minimal unitalization* of \mathcal{I} , i.e. the unital ring given, as a \mathbb{Z} -module, by the direct sum $\widetilde{\mathcal{I}} := \mathcal{I} \oplus \mathbb{Z}$, and equipped with the multiplication defined by

$$(x, \lambda) \cdot (x', \lambda') := (xx' + \lambda x' + \lambda' x, \lambda \lambda'), \quad \text{for } x, x' \in \mathcal{I} \text{ and } \lambda, \lambda' \in \mathbb{Z}.$$

Thirdly, for a unital ring R , $S^{\text{alg}}(R) := R \otimes_{\mathbb{Z}} S(\mathbb{Z})$ denotes the *algebraic suspension* of R , which is a unital ring as well. Here, $S(\mathbb{Z}) := C(\mathbb{Z})/M(\mathbb{Z})$ is the *suspension of the ring of integers*, obtained as the quotient of the cone $C(\mathbb{Z})$ of \mathbb{Z} (given by the infinite matrices $(a_{ij})_{i,j \in \mathbb{N}}$ with only finitely many nonzero integer-valued entries in each row and in each column), by the twosided ideal $M(\mathbb{Z}) = \bigcup_{n \geq 1} M_n(\mathbb{Z})$ of all finite integral square matrices. Naturally, $(S^{\text{alg}})^{10}(R)$ denotes the 10-fold algebraic suspension of the unital ring R , and finally, $\text{St}(R)$ designates its *infinite Steinberg group*.

Proof of Theorem 1.1. Let Λ be the unital ring $(S^{\text{alg}})^{10} (\widehat{\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}})$. As already mentioned, by [3], the group $\text{St}(\Lambda)$ is strongly torsion generated. Furthermore, it is proved in [8], among other things, that a (possibly infinite) cartesian product of strongly torsion generated groups is strongly torsion generated again. Therefore, by our choice of S' , the product group $S = \text{St}(\Lambda) \times S'$ is strongly torsion generated. By [22, Thm 5.1], the center of $\text{St}(\Lambda)$ is naturally isomorphic to $K_2^{\text{alg}}(\Lambda)$. It follows that $\mathcal{Z}(S) \cong K_2^{\text{alg}}(\Lambda)$.

Let us now determine the K -groups $K_n^{\text{alg}}(\Lambda)$ with $n \leq 5$. First, recall that \mathcal{E}_A , as any real C^* -algebra, is karoubian – see Section 2. Thus, we have isomorphisms

$$K_*^{\text{alg}}(\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \cong KO_*^{\text{top}}(\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \cong KO_*^{\text{top}}(\mathcal{E}_A),$$

and the latter groups are 8-periodic and fully described in Proposition 3.2 with M standing for our A . Next, letting \mathcal{I} denote the nonunital ring $\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}$, we have isomorphisms

$$K_*^{\text{alg}}(\widetilde{\mathcal{I}}) \cong K_*^{\text{alg}}(\widetilde{\mathcal{I}}, \mathcal{I}) \oplus K_*^{\text{alg}}(\mathbb{Z})$$

and, for $n \geq 0$,

$$K_{-n}^{\text{alg}}(\widetilde{\mathcal{I}}) \cong \begin{cases} K_{-n}^{\text{alg}}(\mathcal{I}) & \text{if } n > 0 \\ K_0^{\text{alg}}(\mathcal{I}) \oplus \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Indeed, this uses the long exact sequence in algebraic K -theory, see [25, Thm 3.3.4], the split short exact sequence of nonunital rings

$$0 \longrightarrow \mathcal{I} \longrightarrow \widetilde{\mathcal{I}} \xrightarrow{\sim} \mathbb{Z} \longrightarrow 0,$$

see [25, Def. 1.5.7], the fact that K_{-n}^{alg} satisfies excision for $n \geq 0$, see [25, Thm 1.5.9 and Def. 3.3.1], and the regularity of the ring of integers \mathbb{Z} , which implies that its negative K -groups all vanish, see [25, Ex. 3.1.2(4) and Def. 3.3.1]. By the *suspension isomorphism* in algebraic K -theory, to the effect that

$$K_*^{\text{alg}}((S^{\text{alg}})^\ell(R)) \cong K_{*-\ell}^{\text{alg}}(R),$$

for any unital ring R and any $\ell \geq 1$, we obtain that for $n \leq 9$,

$$\begin{aligned} K_n^{\text{alg}}(\Lambda) &\cong K_{n-10}^{\text{alg}}(\widehat{\mathcal{E}_A \otimes \mathcal{K}_{\mathbb{R}}}) \cong K_{n-10}^{\text{alg}}(\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \\ &\cong KO_{n-10}^{\text{top}}(\mathcal{E}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \cong KO_{n-10}^{\text{top}}(\mathcal{E}_A). \end{aligned}$$

In particular, for $n \in \{2, 3, 4, 5\}$, we have

$$K_n^{\text{alg}}(\Lambda) \cong \begin{cases} A & \text{if } n = 2, \\ 0 & \text{if } n = 3, 4, 5. \end{cases}$$

Now, for $n \leq 6$, combining [25, Thms 5.2.2 and 5.2.7; Cor. 5.2.8] with the Hurewicz Theorem [30, Thm 10.25], we have canonical isomorphisms

$$H_n(\text{St}(\Lambda)) \cong H_n(B\text{St}(\Lambda)^+) \cong \pi_n(B\text{St}(\Lambda)^+) \cong \begin{cases} 0 & \text{if } n \leq 5, \\ K_6^{\text{alg}}(\Lambda) & \text{if } n = 6. \end{cases}$$

In total, it follows that $\mathcal{Z}(S) = \mathcal{Z}(\text{St}(\Lambda)) \cong A$ and that $H_n(\text{St}(\Lambda)) = 0$, for $n \in \{1, 2, 3, 4, 5\}$. From the Künneth Theorem [30, Thm 13.31], we deduce that

$$H_n(S) \cong H_n(S')$$

whenever n belongs to $\{1, 2, 3, 4, 5\}$, and this completes the proof. \square

Remark 4.1. Keeping notation as in the proof, in dimension 6 we have

$$H_6(S) \cong H_6(\text{St}(\Lambda)) \cong K_6^{\text{alg}}(\Lambda) \cong K_{-4}^{\text{alg}}(\widehat{\mathcal{E}_A \otimes \mathcal{K}_{\mathbb{R}}}) \cong KO_{-4}^{\text{top}}(\mathcal{E}_A) \cong A.$$

5. THE POTENTIAL LIMITS OF THE METHOD

Here, we explain to what extent our method could possibly be pushed further to get stronger results, namely, to prescribe, besides the center, extra homology groups. The following example, due to an anonymous referee, illustrates the difficulty.

Example 5.1. *If, for a real C^* -algebra \mathcal{B} , both $KO_0^{\text{top}}(\mathcal{B})$ and $KO_3^{\text{top}}(\mathcal{B})$ are odd torsion groups, then*

$${}_2KO_1^{\text{top}}(\mathcal{B}) = 0 = KO_2^{\text{top}}(\mathcal{B})_{(2)}/2.$$

The reason for this is that the long exact sequence relating real and complex K -theory [16] gives short exact sequences such as

$$\begin{array}{ccccc} KO_3 & \rightarrow & KU_3 & \xrightarrow{\quad} & KO_1 \\ & & \downarrow \cong & \nearrow r & \\ & & KU_1 & & \end{array} \quad \text{and} \quad KO_0 \rightarrow KO_1 \xrightarrow{c} KU_1,$$

where r is realization, c is complexification. Hence in this case both r and c are monomorphisms on 2-torsion subgroups in dimension 1; however, there is the usual relation $rc = 2$. This shows that $KO_1^{\text{top}}(\mathcal{B})$ is 2-torsion-free. Similarly, in dimension 2 both r and c are epimorphisms after 2-localization, so that (again because $rc = 2$) the 2-localization of $KO_2^{\text{top}}(\mathcal{B})$ is 2-divisible.

In general, a significant constraint arises from the fact that the K -theory of a real C^* -algebra \mathcal{A} is a module over $KO_*^{\text{top}}(\mathbb{R})$ [19, (2.5)]. Since the \mathbb{Z} -graded ring $KO_*^{\text{top}}(\mathbb{R})$ contains the torsion-free generator $\xi \in KO_4^{\text{top}}(\mathbb{R})$ such that $\xi^2 = 4\beta$, where multiplication by $\beta \in KO_8^{\text{top}}(\mathbb{R})$ gives the period 8 Bott periodicity isomorphism, it follows that multiplication by ξ induces a period 4 isomorphism on $KO_*^{\text{top}}(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}]$. Moreover, in the general case, as we have already seen, 2-torsion tends to cloud the issue. Therefore, to simplify matters, we restrict attention to uniquely 2-divisible groups.

Theorem 5.1. *Given ten uniquely 2-divisible abelian groups*

$$A \text{ and } A_2, \dots, A_{10},$$

there exists a group S with the following properties :

- (i) *S is strongly torsion generated;*
- (ii) *the center of S is isomorphic to A , that is, $\mathcal{Z}(S) \cong A$;*
- (iii) *S is perfect, that is, $H_1(S) = 0$;*
- (iv) *the higher homology of S , up to dimension ten, is given by the tableau*

$$\begin{array}{c|c|c|c|c|c|c|c} n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline H_n(S) & A_2 & A_3 & A_4 & A_5 & A \oplus A_6 & A_7 & A_8 \end{array}$$

$$\begin{array}{c|c} 9 & 10 \\ \hline (A \otimes A_3) \oplus A_9 \oplus \text{Tor}(A, A_2) & H \oplus (A \otimes A_4) \oplus A_{10} \oplus \text{Tor}(A, A_3) \end{array}$$

where H is an abelian group (described in the proof below) mapping onto $A/3$.

Proof. First, by Proposition 3.2, there exists a real C^* -algebra $\mathcal{L} = \mathcal{E}_{\mathbb{Z}[\frac{1}{2}]}$ such that

$$KO_q^{\text{top}}(\mathcal{L}) = \begin{cases} \mathbb{Z}[\frac{1}{2}] & q \equiv 0 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider L and $\mathcal{K} := \mathcal{K}(\ell_{\mathbb{R}}^2(L))$ as in the proof of Proposition 3.2, using a fixed “based presentation” of an arbitrary uniquely 2-divisible abelian group M , with matrix $N = (n_{ij})$ having nonnegative integral coefficients. Now, we mimic that proof; this time with, in place of \mathcal{A}_J , \mathcal{A}_I and α , the real C^* -algebras

$$\mathcal{A}'_J := \widehat{\bigoplus_J \mathcal{L} \widehat{\otimes} \mathcal{K}} \quad \text{and} \quad \mathcal{A}'_I := \widehat{\bigoplus_I \mathcal{L} \widehat{\otimes} \mathcal{K}}$$

and the morphism of real C^* -algebras

$$\alpha' := \text{id}_{\mathcal{L} \widehat{\otimes} \mathcal{K}} \alpha : \mathcal{A}'_J \rightarrow \mathcal{A}'_I,$$

respectively; in other words, (with $i \in I$ and $j \in J$) we replace each real C^* -algebra endomorphism $\iota_{n_{ij}} : \mathcal{K} \rightarrow \mathcal{K}$ by the endomorphism $\text{id}_{\mathcal{L} \widehat{\otimes} \mathcal{K}} \iota_{n_{ij}} : \mathcal{L} \widehat{\otimes} \mathcal{K} \rightarrow \mathcal{L} \widehat{\otimes} \mathcal{K}$. One then similarly deduces that for every uniquely 2-divisible abelian group M there exists a real C^* -algebra \mathcal{L}_M whose topological K -theory is given by

$$KO_q^{\text{top}}(\mathcal{L}_M) = \begin{cases} M & q \equiv 0 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define Λ to be the ring $(S^{\text{alg}})^{10}(\widetilde{\mathcal{L}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}})$. Therefore, we have isomorphisms

$$\begin{aligned} K_n^{\text{alg}}(\Lambda) &\cong K_{n-k}^{\text{alg}}(\widetilde{\mathcal{L}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}}) \cong K_{n-k}^{\text{alg}}(\mathcal{L}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \\ &\cong KO_{n-k}^{\text{top}}(\mathcal{L}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}}) \cong KO_{n-k}^{\text{top}}(\mathcal{L}_A) \end{aligned}$$

for any n with $2 \leq n \leq 10$.

Now, we claim that the homology of the group $\text{St}(\Lambda)$ is given by the tableau

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|} n & 1 & 2 & \cdots & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline H_n(\text{St}(\Lambda)) & 0 & 0 & \cdots & 0 & A & 0 & 0 & {}_2A & H \end{array}$$

where H is described below. Indeed, this is clear for $n = 1$ and $n = 2$. For $3 \leq n \leq 6$, by the Hurewicz Theorem applied to the CW-complex $B\text{St}(\Lambda)^+$, we have

$$H_n(\text{St}(\Lambda)) \cong K_n^{\text{alg}}(\Lambda) \cong KO_{n-10}^{\text{top}}(\mathcal{L}_A) \cong \begin{cases} 0 & \text{if } 3 \leq n \leq 5, \\ A & \text{if } n = 6. \end{cases}$$

Now, Whitehead's exact sequence for the 5-connected CW-complex $X := B\text{St}(\Lambda)^+$ reads

$$\cdots \rightarrow \underbrace{\pi_8(X)}_{\cong K_8^{\text{alg}}(\Lambda)} \rightarrow \underbrace{H_8(X)}_{\cong H_8(\text{St}(\Lambda))} \rightarrow \Gamma_7(X) \rightarrow \underbrace{\pi_7(X)}_{\cong K_7^{\text{alg}}(\Lambda)} \rightarrow \underbrace{H_7(X)}_{\cong H_7(\text{St}(\Lambda))} \rightarrow \underbrace{\Gamma_6(X)}_{=0},$$

see [34]. Since $K_n^{\text{alg}}(\Lambda) \cong KO_{n-10}^{\text{top}}(\mathcal{L}_A)$ vanishes for $7 \leq n \leq 9$, it follows that

$$H_7(\text{St}(\Lambda)) = 0$$

and, for $8 \leq n \leq 9$,

$$H_n(\text{St}(\Lambda)) \cong \Gamma_{n-1}(X).$$

By Whitehead [33], we have $\Gamma_7(X) \cong \pi_7(X)/2 = 0$. It follows from [1, Cor. (3), p. 170] that there is a short exact sequence

$$0 \longrightarrow \pi_7(X)/2 \longrightarrow \Gamma_8(X) \longrightarrow {}_2\pi_6(X) \longrightarrow 0,$$

so that $\Gamma_8(X) \cong {}_2K_6^{\text{alg}}(\Lambda) \cong {}_2A = 0$ in our situation. By [1, Cor. (5.2), p. 178], there is an exact sequence

$$\text{Ker}(\eta^1) \xrightarrow{d_2} \pi_6(X)/3 \oplus P \longrightarrow \Gamma_9(X) \longrightarrow {}_2\pi_7(X)/\text{Im}(\eta^1) \longrightarrow 0,$$

where $\eta^1: \pi_6(X)/2 \longrightarrow \pi_7(X)$ is induced by a certain Hopf map, and P is a pushout of abelian groups of the form

$$\begin{array}{ccc} \pi_6(X)/2 & \longrightarrow & \pi_8(X)/2 \\ \downarrow & \lrcorner & \downarrow \\ \pi_6(X)/8 & \longrightarrow & P \end{array}$$

all of which vanish by 2-divisibility of A . Therefore,

$$\Gamma_9(X) \cong A/3.$$

Thus, $H := H_{10}(\text{St}(\Lambda))$ fits into Whitehead's exact sequence as:

$$\cdots \rightarrow \underbrace{\pi_{10}(X)}_{\cong A} \longrightarrow H \longrightarrow \underbrace{\Gamma_9(X)}_{\cong A/3} \longrightarrow \underbrace{\pi_9(X)}_{=0}$$

Now, as the group S , we take

$$S := \text{St} \left((S^{\text{alg}})^k(\widetilde{\mathcal{L}_A \otimes \mathcal{K}_{\mathbb{R}}}) \right) \times S'(A_2, \dots, A_{10}, 0, \dots),$$

where the group $S' = S'(A_2, \dots, A_{10}, 0, \dots)$ is provided by [9, Thm 1], and is *strongly torsion generated*, with *trivial center*, and with reduced integral homology given by

$$\begin{array}{c|c|c|c|c|c|c|c|} n & 1 & 2 & \cdots & 10 & 11 & 12 & \cdots \\ \hline H_n(S') & 0 & A_2 & \cdots & A_{10} & 0 & 0 & \cdots \end{array}$$

Since by [3] $\text{St}(\Lambda)$ is strongly torsion generated, so too by [8] is the cartesian product $S = \text{St}(\Lambda) \times S'$. Clearly, the center of S is isomorphic to that of $\text{St}(\Lambda)$, where $\Lambda := (S^{\text{alg}})^{10}(\mathcal{L}_A \widehat{\otimes} \mathcal{K}_{\mathbb{R}})$; this center is given by

$$\mathcal{Z}(\text{St}(\Lambda)) \cong K_2^{\text{alg}}(\Lambda) \cong KO_{-8}^{\text{top}}(\mathcal{L}_A) \cong A.$$

Finally, it suffices to apply the Künneth Theorem (including its non-natural splitting) to conclude. \square

Now we see that our K -theoretical approach to construct the desired group S as a product of a group $S' = S'(A_2, A_3, \dots)$ from [9] with an infinite Steinberg group $\text{St}(\Lambda)$ — where Λ is a unital ring suitably obtained from an appropriate real C^* -algebra — will inevitably introduce in the group $H_6(S)$ a version of A . Here, A is at the same time isomorphic to $\mathcal{Z}(\text{St}(\Lambda)) \cong K_2^{\text{alg}}(\Lambda)$ and to $K_6^{\text{alg}}(\Lambda)$ (by the hidden real Bott periodicity “transported” by a Karoubi isomorphism). This shows exactly the limits we alluded to earlier. It is also worth noting that the arguments of Remark 3.3 (iv) reveal that there is no gain in the apparent generalization to consideration of real Banach algebras (even if karoubian) instead of real C^* -algebras; for, the K -theory of such a real Banach algebra can always be attained by a real C^* -algebra.

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