Area in Grassmann Geometry

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Abstract. In this survey paper we give the basic properties of Grassmann algebras, present a generalised theory of area from a Grassmann algebra perspective, present a version for Grassmann algebras of the Buchberger algorithm, and give examples of computation and deduction in Grassmann geometry.

The automatic proof of geometry theorems using the powerful algorithms of Wu and Buchberger is the most impressive achievement to date in automated theorem proving. It is our view, though, that progress in the automation of geometry requires something more than the invention and refinement of algorithms. What we have in mind is the creation of algebraic structures that internalize the rich variety of geometric concepts in ways that are amenable to computation. In this survey paper we present such a structure, along with its variant of the Buchberger algorithm.

Grassmann algebras are appropriate many-sorted algebraic structures for affine geometry, well-suited to deduction and to computation of quantities such as areas. The basic geometric objects are points. Vectors and less familiar geometric entities such as bipoints and bivectors are generated as we will describe. To illustrate the invariant coordinate-free flavour of Grassmann algebra we consider a general notion of 'area' enclosed by a closed curve in a space of arbitrary dimension, which in two-dimensional spaces reduces to a scale-free version of the familiar concept. The general notion supports the extension of many plane geometry theorems involving area to higher dimensions.

White [28], Sturmfels and Whitely [27] and others have developed a similar approach to projective geometry theorem proving, based on the Cayley Algebra, and using Cayley factorization as the basic algorithmic tool. Hestenes and Ziegler in [17] present an extensive study of the projective model of affine geometry using Grassmann-Cayley algebra. Also interesting is the work of Chou, Gao and Zhang ([5], [6] and [7]) in which theorems of two and three dimensional geometry are proved using a formalism based on areas and volumes which allows higher level interpretation of the resultant proofs. Our way is in a sense more elementary and its development has been partly motivated by the desire to transparently incorporate elementary geometric reasoning and computation in systems with wider reasoning capabilities.

1 Grassmann Algebras

A *Grassmann algebra* $\Omega[\mathcal{K},\mathcal{P}]$ is a ring (associative and with unit element 1) which is generated by (the union of) disjoint distinguished subsets $\mathcal{K}$ and $\mathcal{P}$ such that

GA1 $\mathcal{K}$ is a field (under the ring operations);
GA2 $\mathcal{P}$ is an affine space over $\mathcal{K}$ (under the ring operations);
GA3 $aA = Aa$ for every $a \in \mathcal{K}$, $A \in \mathcal{P}$;
GA4 $BA = -AB$ for every A,B$\in \mathcal{P}$.
GA5 for $P_1, P_2, \ldots, P_k$ in $\mathcal{P}$,

$$P_1 P_2 \cdots P_k = 0 \Rightarrow P_1, P_2, \ldots, P_k \text{ dependent (over } \mathcal{K}).$$

The meaning of **GA2** is that $\mathcal{P}$ is closed under *affine combinations*:

$$A, B \in \mathcal{P}, a, b \in \mathcal{K} \text{ and } a + b = 1 \Rightarrow aA + bB \in \mathcal{P}.$$

For the real case, all geometric interpretations of expressions and equations between expressions follow from the single fundamental *semantic rule*:

> if A and B are points and a and b positive real numbers with $a + b = 1$ then $aA + bB$ may be interpreted as the point P which divides the line segment from A to B in the ratio b to a.

Elements of $\mathcal{K}$ are called *numbers*, elements of $\mathcal{P}$, *points*, and elements of the set $\mathcal{V} = \mathcal{P} - \mathcal{P}$, *vectors*. These sets are disjoint, except that 0 is both a number and a vector. Throughout this papaer we use the letters A, ..., P for points, and U, ..., Z for vectors.

The implication **GA5** is in fact an equivalence:

$$P_1 P_2 \cdots P_k = 0 \iff P_1, P_2, \ldots, P_k \text{ are dependent.}$$

For if $P_1, P_2, \ldots, P_k$ are dependent then one of the P_j is expressible as a linear combination of the others and hence, clearly, the product $P_1 P_2 \cdots P_k$ is zero. This condition is often easier to handle than the standard definition of (linear) dependence. The equivalence continues to hold, as one may show, even if some or all of the P_j are replaced by vectors.

Theorem 1 (*The Instantiation Theorem*). *For points $P_1, \ldots, P_k$ in a Grassmann algebra,*

$$P_1 P_2 \cdots P_k = 0 \iff \begin{array}{l} \textit{either } P_1 P_2 \cdots P_{k-1} = 0 \\ \textit{or } P_k \textit{ is spanned by } P_1, P_2, \ldots, P_{k-1}. \end{array}$$

and this remains valid if any or all of the points are replaced by vectors.

Proof. Suppose that $P_1P_2\cdots P_k = 0$. According to **GA5** there exist scalars $p_1, p_2, \ldots, p_k$, not all zero, such that

$$p_1P_1 + p_2P_2 + \cdots p_kP_k = 0.$$

If p_k is non-zero then the second of the stated alternatives holds. Otherwise one of $P_1, P_2, \ldots, P_{k-1}$ is a linear combination of the rest and hence $P_1P_2\cdots P_{k-1} = 0$.

The converse implication is obvious. □

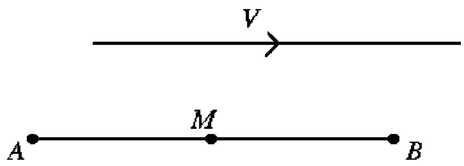

Fig. 1. $V = B - A$ and $AM + MB = AB$.

Note that $AB = 0 \iff B = A$ and $AV = 0 \iff V = 0$. Also, if we define $\mathsf{midpoint}(A, B) = \frac{1}{2}(A + B)$ then

$$AM = MB \iff M = \mathsf{midpoint}(A, B),$$

giving a precise interpretation to traditional notation.

For $k = 3$ the *Instantiation Theorem* is:

$$ABP = 0 \iff \text{either } AB = 0 \text{ or } P = aA + bB \text{ for some } a, b.$$

Hence for non-coincident points A and B we define

$$\mathsf{collinear}(A, B, C) \iff ABC = 0.$$

Similarly, for non-collinear points A, B and C

$$\mathsf{coplanar}(A, B, C, D) \iff ABCD = 0.$$

Theorem 2 (*The Boundary Theorem*)**.** *Let P be a point and $V_1, V_2, \ldots, V_k$ vectors in a Grassmann algebra. Then*

$$PV_1\cdots V_k = 0 \iff V_1\cdots V_k = 0$$

Proof. Use the *Instantiation Theorem*, together with the fact that a point P cannot be a linear combination of vectors. □

Theorem 3 (*The Exchange Theorem*)**.** *If*

$$A_1\cdots A_nP_j = 0 \textit{ for } j = 1 \textit{ to } n+1,$$

then either $A_1\cdots A_n = 0$ or $P_1\cdots P_{n+1} = 0$.

Proof. Suppose that $A_1 \cdots A_n P_j = 0$ for $j = 1$ to $n+1$ and that $A_1 \cdots A_n$ is not 0. According to the *Instantiation Theorem*, each P_j is expressible as a linear combination of $A_1, \ldots, A_n$. Now when the product $P_1 \cdots P_{n+1}$ is expanded each term in the resultant sum is a product of $n+1$ elements of the set $\{A_1, \ldots, A_n\}$ and hence vanishes. □

The *Exchange Theorem* clearly remains valid if some or all of the A_j and P_j are replaced by vectors. It is the Grassmann geometry analogue of the "no zero divisors" property for fields.

2 Existence and Uniqueness

Let Ω be a Grassmann algebra. Either there exists a finite maximal set of independent points $O_0, O_1, \ldots, O_n$ in Ω or not.

If there does then we say that Ω is *finite-dimensional* and that $(O_0, O_1, \ldots, O_n)$ is an *affine coordinate system* for Ω and that $(O, X_1, \ldots, X_n)$, where

$$O = O_0, X_1 = O_1 - O_0, \ldots, X_n = O_n - O_0,$$

is the corresponding *Cartesian coordinate system*. And then, also, both $(O_0, O_1, \ldots, O_n)$ and $(O, X_1, \ldots, X_n)$ are bases for the linear space Ω_0 spanned by $\mathcal{P}$, and $(X_1, \ldots, X_n)$ is a basis for $\mathcal{V}$.

Theorem 4 (*The Dimension Theorem*)**.** *For a finite-dimensional Grassmann algebra, all coordinate systems Ω have the same number of elements.*

The number of elements in a coordinate system of a finite-dimensional Grassmann algebra, less one, is called its *dimension*.

The next theorem pins down the structure of a finite-dimensional Grassmann algebra Ω. For each natural number k, let Ω_k be the linear subspace of Ω that is spanned by $\mathcal{P}^{k+1}$; let $\Omega_{-1} = \mathcal{K}$. Elements of Ω_k are called *chains of dimension* k (or of *degree* $k+1$), or, briefly, *k-chains*. In particular, -1-chains are numbers, and 0-chains are points, multiples of points or vectors.

Theorem 5 (*The Structure Theorem*)**.** *Let Ω be an n-dimensional Grassmann algebra. As a linear space Ω is the direct sum of the subspaces $\Omega_{-1}, \Omega_0, \ldots, \Omega_n$; moreover, for each k, Ω_k has dimension $\binom{n+1}{k+1}$ and (hence) Ω has dimension 2^{n+1}.*

Using the 2^{n+1} basis elements 1, O_0, O_1, …, O_n, O_0O_1, …, O_0O_n, O_1O_2, …, O_1O_n, …, $O_{n-1}O_n$, …, $O_0O_1 \cdots O_n$ corresponding to an affine coordinate system $O_0, \ldots, O_n$, one obtains a multiplication table for the n-dimensional algebra Ω. Since an algebra is determined by its dimension together with the multiplication table for a basis, the uniqueness (up to isomorphism) of the Grassmann algebra of dimension n is established.

If $O, X_1, \ldots, X_n$ is the cartesian coordinate system associated with $(O_0, O_1, \ldots, O_n)$ (meaning that $O = O_0$, $X_1 = O_1 - O$, …, $X_n = O_n - O$) then it is easy to see that 1, O, X_1, ..., X_n, OX_1, ...,OX_n, X_1X_2, ..., X_1X_n, ..., $X_{n-1}X_n$, ..., $OX_1 \cdots X_n$ is also a basis for Ω, with the property that its elements of dimension k form a basis for Ω_k.

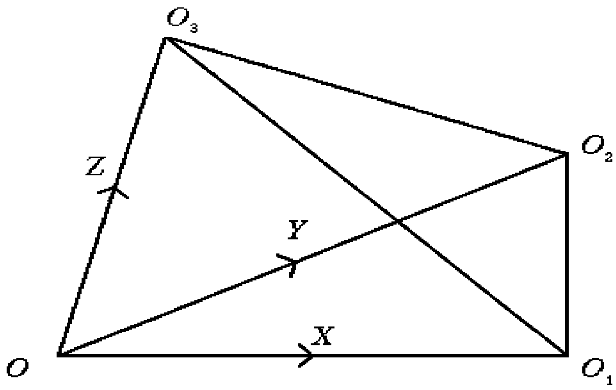

Fig. 2. A coordinate system.

To take an important example, in the 3-dimensional Grassmann algebra Ω every element ϕ of is uniquely expressible in the form

$$\phi = \phi_{-1} + \phi_0 + \phi_1 + \phi_2 + \phi_3$$

where each ϕ_i is a chain of dimension i. Moreover if (O, X, Y, Z) is a cartesian coordinate system for Ω, these components are themselves uniquely expressible, respectively, as

$$\begin{aligned}
\phi_{-1} &= k, \\
\phi_0 &= oO + xX + yY + zZ, \\
\phi_1 &= aOX + bOY + cOZ + uXY + vYZ + wZX \\
\phi_2 &= eOXY + fOYZ + gOZX + tXYZ \\
\phi_3 &= dOXYZ,
\end{aligned}$$

the coefficients $k, o, x, \ldots, d$ being numbers.

Theorem 6 (*The Existence Theorem*)**.** *For each natural number n there exists a Grassmann algebra of dimension n over* $\mathcal{K}$.

Proofs of the above theorems are given in [11].

3 Multipoints and Multivectors

A product AB of two distinct points is called a *bipoint*, a product ABC of three non-collinear points is called a *tripoint* and, in general, a non-zero product

$A_1 \cdots A_k$ of points is called a *multipoint* (of *degree* k) — it is the *exterior product* of the points.

The following easily proved equivalence shows how a bipoint is to be interpreted.

$$CD = kAB \iff \mathsf{collinear}(A, B, C), \mathsf{collinear}(A, B, D), D - C = k(B - A).$$

In particular the bipoint from C to D equals the bipoint from A to B if and only if all four points are collinear and the vector from C to D equals the vector from A to B. Accordingly a bipoint is sometimes called a *line vector*.

One easily sees that $kAB = AP$ for some point P. Hence the set of all bipoints is closed under multiplication by non-zero scalars. We say that two bipoints are *projectively equivalent* if one is a non-zero scalar multiple of the other. This is an equivalence relation. Writing

$$[AB] = \{kAB : \ k \text{ non-zero}\},$$

we see that there is a one-to-one correspondence between the set of such equivalence classes of bipoints (modulo multiplication by non-zero scalars) and the set of all lines. Accordingly we may unambiguously view a line as an algebraic object $[AB]$ rather than as a set of points. In a similar fashion, a plane may be viewed as an equivalence class of tripoints $[ABC]$. This is the Grassmann algebra approach to projective geometry.

A non-zero product of k vectors is called a *multivector* of *degree* k; in particular, a multivector of degree 2 is called a *bivector* and one of degree 3 is called a *trivector*. Thus, if U, V and W are independent vectors then VW is a bivector and UVW is a trivector.

For independent points A, B, C, D the 1-chain $AB+CD$ cannot be a bipoint or a bivector, since squares of bipoints and bivectors are obviously 0.

Two multivectors are said to be *parallel* if they are dependent; this just means that each is a non-zero multiple of the other. By the *Structure Theorem*, parallel multivectors must have the same degree. Parallelism is an equivalence relation on the set of all multivectors.

For vectors V and W we define

$$\mathsf{parallel}(V, W) \iff VW = 0.$$

Theorem 7. *Bivectors VW and $V'W'$ are parallel if and only if $span(V, W) = span(V', W')$.*

This extends in the obvious way to multivectors of arbitrary degree. Thus there is a one-to-one correspondence between the projective equivalence classes of multivectors and the linear subspaces of V; these are the points, lines, planes and hyperplanes at infinity.

4 Linear Maps

Let Ω and Λ be Grassmann algebras (over the same field $\mathcal{K}$) with point spaces $\mathcal{P}$ and $\mathcal{Q}$ and vector spaces $\mathcal{V}$ and W respectively.

A ring homomorphism $\mathbf{T} : \Omega \to \Lambda$ is called a *linear map* (or *Grassmann algebra homomorphism*), if it leaves numbers fixed and maps points to points. A linear map $\mathbf{T}$ preserves lines and ratios of distances along lines, because for any a, b in $\mathcal{K}$ and A, B in $\mathcal{P}$

$$\mathbf{T}(aA + bB) = a\mathbf{T}(A) + b\mathbf{T}(B).$$

Also $\mathbf{T}$ preserves vectors, because, for any A, B in $\mathcal{P}$,

$$\mathbf{T}(B - A) = \mathbf{T}(B) - \mathbf{T}(A),$$

and $\mathbf{T}$ acts linearly on vectors.

One easily shows that

$$\mathsf{parallelogram}(A, B, C, D) \Rightarrow \mathsf{parallelogram}(\mathbf{T}(A), \mathbf{T}(B), \mathbf{T}(C), \mathbf{T}(D));$$

that

$$M = \mathsf{centroid}(A, B, C) \Rightarrow \mathbf{T}(M) = \mathsf{centroid}(\mathbf{T}(A), \mathbf{T}(B), \mathbf{T}(C));$$

and that $\mathbf{T}$ preserves parallelism of multivectors.

To specify a linear map on a finite-dimensional Grassmann algebra one need only give its action on points, as the following theorem shows.

Theorem 8. *Let $\mathbf{T} : \mathcal{P} \to Q$ be a function such that*

$$\mathbf{T}(aA + bB) = a\mathbf{T}(A) + b\mathbf{T}(B)$$

for all A, B in $\mathcal{P}$ and a, b in $\mathcal{K}$ with $a + b = 1$. Then $\mathbf{T}$ extends uniquely to a linear map $\mathbf{T} : \Omega \to \Lambda$.

Proof. Let $(O_0, O_1, \ldots, O_n)$ be an affine coordinate system for Ω. Every element of Ω is uniquely expressible as a linear combination of products of the form $O_i O_j \cdots O_k$ with $i < j < \cdots < k$; extend $\mathbf{T}$ to Ω by first defining $\mathbf{T}(1) = 1$ and

$$\mathbf{T}(O_i O_j \cdots O_k) = \mathbf{T}(O_i)\mathbf{T}(O_j) \cdots \mathbf{T}(O_k)$$

for these basic products and then extending linearly to arbitrary elements. Obviously the extended $\mathbf{T}$ preserves points and leaves numbers fixed, and it is straightforward to check that it is a ring homomorphism. And uniqueness follows from the fact that any other such extension must agree with $\mathbf{T}$ on the basic products and hence on arbitrary elements. □

The *translation* $\mathbf{T} : \mathcal{P} \to \mathcal{P}$ defined by a vector U is given by

$$\mathbf{T}(P) = P + U \text{ for every } P \in \mathcal{P}.$$

$\mathbf{T}$ preserves affine combinations of points, since for $a + b = 1$ and A, B in P

$$\begin{aligned}\mathbf{T}(aA + bB) &= aA + bB + U \\ &= a(A + U) + b(B + U) \\ &= a\mathbf{T}(A) + b\mathbf{T}(B).\end{aligned}$$

Hence $\mathbf{T}$ extends to a linear map $\mathbf{T} : \Omega \to \Omega$. Note that $\mathbf{T}$ leaves all vectors (and hence all multivectors) fixed:

$$\mathbf{T}(B - A) = \mathbf{T}(B) - \mathbf{T}(A) = (B + U) - (A + U) = B - A.$$

A linear map is determined by its action on a single point A, together with its action on vectors.

If ABC and PQR are tripoints in a two-dimensional Grassmann algebra Ω then $PQR = kABC$ for some number k. Hence, applying the linear map $\mathbf{T} : \Omega \to \Omega$ to this equation

$$\mathbf{T}(PQR) = k\mathbf{T}(ABC),$$

and we write

$$\frac{\mathbf{T}(PQR)}{\mathbf{T}(ABC)} = \frac{PQR}{ABC},$$

or, equivalently,

$$\frac{\mathbf{T}(PQR)}{PQR} = \frac{\mathbf{T}(ABC)}{ABC}.$$

This ratio is a number (since $\mathbf{T}(ABC) = \mathbf{T}(A)\mathbf{T}(B)\mathbf{T}(C)$) is a multiple of ABC) that depends only on $\mathbf{T}$; it is the *determinant* of $\mathbf{T}$:

$$\mathbf{det}(\mathbf{T}) = \frac{\mathbf{T}(ABC)}{ABC}.$$

From this one may give basis-free proofs of the properties of determinants. In particular, if $\mathbf{T} : \Omega \to \Omega$ and $\mathbf{S} : \Omega \to \Omega$ be linear maps then $\mathbf{det}(\mathbf{TS}) = \mathbf{det}(\mathbf{T})\mathbf{det}(\mathbf{S})$. The definition and properties of the determinant function $\mathbf{det}$ extend easily to higher dimensions.

There is a standard construction, not needed for our purposes, by which any finite-dimensional linear space may be embedded in a Grassmann algebra as the space spanned by its points. From this it follows, for example, that one consequence of the *Dimension Theorem* is that all bases of a finite-dimensional linear space have the same number of elements. This may be proved without using the Grassmann multiplication, but the extra structure makes the proof simpler.

The new kinds of quantities discussed above are not only of geometric interest. In physics, the invariance properties of angular momentum, for example, are different from those of momentum (see [14], pp. 52-5) and it is incorrect to represent both as being vectors, though that is what is usually done — angular momentum is, more correctly, a bivector. Similarly, a force which is constrained to act along a line is properly represented as what we have called a bipoint or line vector. Hestenes [16] advocates a Clifford algebra approach to physics in which such matters are treated correctly.

5 Closed Curves

A polygon has area, a polyhedron has volume. Area and volume have simple properties by which they may be characterized. Of these the most important, perhaps, is additivity — if a polygon is made by glueing together two other polygons then its area is the sum of their areas. In this section, we consider these matters. But first we must pin down what the entities are that have area or volume.

We define an (*oriented*) *edge* to be an ordered pair of distinct points (A, B); A is called its *initial vertex*and B its *final vertex*. An (*oriented*) *closed curve* (or *closed polygonal arc*) is a finite (non-empty) set of edges with the property that each of its points occurs once as the initial vertex of an edge and once as the final vertex of an edge. A closed curve is called *minimal* if no proper subset of it is a closed curve.

A closed curve may be viewed as a permutation of a finite set of points which leaves no point fixed. (All permutations would be obtained if we allowed degenerate edges (P, P).) Minimal curves are the cyclic permutations, and corresponding to the fact that every permutation is a composite of unique cyclic permutations, every closed curve is the union of a unique family of minimal closed curves called its *components*.

We usually denote a minimal closed curve

$$\{(A_1, A_2), (A_2, A_3), \ldots, (A_{k-1}, A_k), (A_k, A_1)\}$$

by $\langle A_1, A_2, \ldots, A_k\rangle$. We call $\langle A_k, A_{k-1}, \ldots, A_1\rangle$ the *opposite* of $\langle A_1, A_2, \ldots, A_k\rangle$ and extend this notion to general closed curves in the obvious way.

A curve is called a *segment* if it has 2 vertices and an (*oriented*) *triangle* if it has 3 vertices. While there are two oriented triangles associated with a set of 3 points $\{A, B, C\}$, namely $\langle A, B, C\rangle$ and its opposite $\langle C, B, A\rangle$, a segment and its opposite are identical.

To motivate what follows, readers will find it instructive to compute the value of the quantity $AB + BC + CD + DA$ for a square $\langle A, B, C, D\rangle$, where $B = A + X, C = B + Y, D = C - X$ — it turns out to be $2XY$. An equally simple calculation shows that the value of the quantity $AB + BC + CA$ for a triangle $\langle A, B, C\rangle$, where $A = O, B = O + aX, C = O + cX + dY$ is $adXY$; note that in a rectangular coordinate system, a is the base length of the triangle, and d is its height.

With a minimal closed curve $\langle A_1, A_2, \ldots, A_k\rangle$ we may associate the quantity

$$\mathsf{m}\langle A_1, A_2, \ldots, A_k\rangle = A_1A_2 + A_2A_3 + \cdots + A_{k-1}A_k + A_kA_1,$$

called its (generalized, oriented) *area*. We extend m to general closed curves additively.

Observe that the area of the opposite of a curve is the negative of its area and that $\mathsf{m}\langle A, B\rangle = 0$.

One easily shows that for any point P

$$\mathsf{m}\langle A_1, A_2, \ldots, A_k\rangle = (A_1 - P)(A_2 - P) + \cdots \\ +(A_{k-1} - P)(A_k - P) + (A_k - P)(A_1 - P).$$

Hence the quantity $\mathsf{m}\langle A_1, A_2, \ldots, A_k\rangle$ is *vectorial*, meaning that it is expressible in terms of just vectors. For example,

$$\mathsf{m}\langle A, B, C, D\rangle = AB + BC + CD + DA = (A - C)(B - D),$$

and so the area of the quadrilateral $\langle A, B, C, D\rangle$ is zero if and only if $\mathsf{parallel}(A - C, B - D)$.

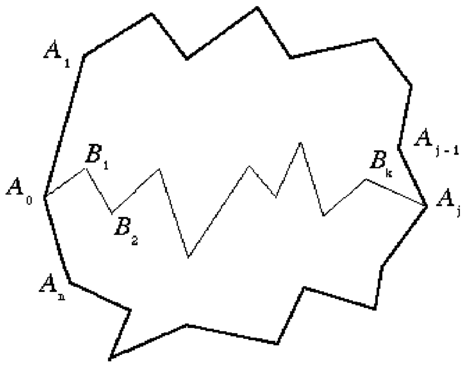

Fig. 3. The area property **A1**.

Although m is dimension-free it shares the key properties of one-dimensional oriented area. In particular, a trivial algebraic identity implies that m is *additive* and *translation-invariant* in the following sense.

Theorem 9.

A1 $\mathsf{m}\langle A_0, A_1, \ldots, A_n\rangle =$
$\mathsf{m}\langle A_0, \ldots, A_j, B_k, \ldots, B_1\rangle + \mathsf{m}\langle A_0, B_1, \ldots, B_k, A_j, \ldots, A_n\rangle$;
A2 $\mathsf{m}\langle A_0 + V, A_1 + V, \ldots, A_n + V\rangle = \mathsf{m}\langle A_0, A_1, \ldots, A_n\rangle$.

We also have the characteristic properties of area of triangles:

Theorem 10.

T1 $\mathsf{m}\langle A + a(C - B), B, C\rangle = \mathsf{m}\langle A, B, C\rangle$;

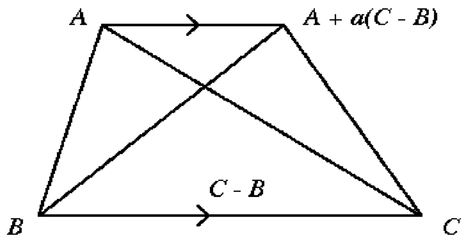

Fig. 4. The area property **T1.**

T2 $\mathsf{m}\langle A, B, B + c(C - B)\rangle = c\,\mathsf{m}\langle A, B, C\rangle$;
T3 $\mathsf{m}\langle A, B, C\rangle = 0 \iff \mathsf{collinear}(A, B, C)$;

The area identity

$$\mathsf{m}\langle A, B, C\rangle + \mathsf{m}\langle C', B', A'\rangle$$
$$+\,\mathsf{m}\langle A, A', B', B\rangle + \mathsf{m}\langle B, B', C', C\rangle + \mathsf{m}\langle C, C', A', A\rangle = 0;$$

is what lies behind the following theorem. Note that if any one of the hypotheses is swapped with the conclusion, then another valid theorem is obtained.

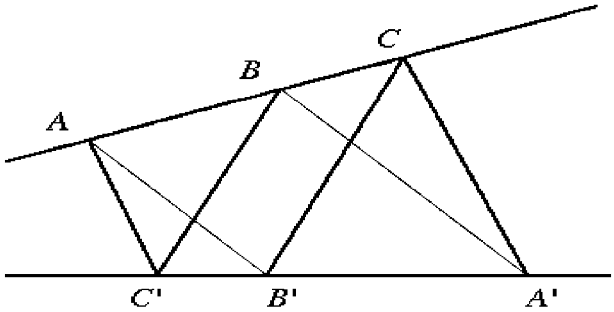

Fig. 5. The Parallel Pappus Theorem.

Geometry Theorem 1 (*The Parallel Pappus Theorem*).

hyp$_1$ $\mathsf{collinear}(A, B, C)$
hyp$_2$ $\mathsf{collinear}(A', B', C')$
hyp$_3$ $\mathsf{parallel}(B - C', C - B')$
hyp$_4$ $\mathsf{parallel}(C - A', A - C')$
conc $\mathsf{parallel}(A - B', B - A')$

6 Plane Area

We define an (*oriented*) *polygon* (or *polygonal curve*) to be a minimal curve whose vertices are coplanar. It is called an (*oriented*) *quadrilateral* if it has 4 vertices, a *pentagon* if it has 5, and so on.

Standard oriented area, as usually defined, has the properties **T1**, **T2** and **T3**, and hence, according to the following, must be an oriented area in our invariant sense, unique up to a scalar multiple.

Theorem 11. *Let m and m' be area functions for triangles in a plane. Then there exists a nonzero number k such that $\mathsf{m}' = k\,\mathsf{m}$.*

Proof. Let $\langle A, B, C\rangle$ be a fixed non-degenerate triangle in the plane and let

$$k = \frac{\mathsf{m}'\langle A, B, C\rangle}{\mathsf{m}\langle A, B, C\rangle}.$$

Let $\langle A', B', C'\rangle$ be any non-degenerate triangle in the plane. There exist numbers b and c such that

$$B' = A + b(B - A) + c(C - A) = B_1 + c(C - A),$$

where $B_1 = A + b(B - A)$. Hence, using properties **T1** and **T2** of m and m', we have

$$\begin{aligned}\frac{\mathsf{m}'\langle A, B', C\rangle}{\mathsf{m}'\langle A, B, C\rangle} &= \frac{\mathsf{m}'\langle A, B', C\rangle}{\mathsf{m}'\langle A, B_1, C\rangle}\frac{\mathsf{m}'\langle A, B_1, C\rangle}{\mathsf{m}'\langle A, B, C\rangle}\\ &= b\\ &= \frac{\mathsf{m}\langle A, B', C\rangle}{\mathsf{m}\langle A, B_1, C\rangle}\frac{\mathsf{m}\langle A, B_1, C\rangle}{\mathsf{m}\langle A, B, C\rangle}\\ &= \frac{\mathsf{m}\langle A, B', C\rangle}{\mathsf{m}\langle A, B, C\rangle}.\end{aligned}$$

Using the analagous equations involving A' and C' instead of B', we have

$$\begin{aligned}\frac{\mathsf{m}'\langle A', B', C'\rangle}{\mathsf{m}'\langle A, B, C\rangle} &= \frac{\mathsf{m}'\langle A', B', C'\rangle}{\mathsf{m}'\langle A', B', C\rangle}\frac{\mathsf{m}'\langle A', B', C\rangle}{\mathsf{m}'\langle A', B, C\rangle}\frac{\mathsf{m}'\langle A', B, C\rangle}{\mathsf{m}'\langle A, B, C\rangle}\\ &= \frac{\mathsf{m}\langle A', B', C'\rangle}{\mathsf{m}\langle A', B', C\rangle}\frac{\mathsf{m}\langle A', B', C\rangle}{\mathsf{m}\langle A', B, C\rangle}\frac{\mathsf{m}\langle A', B, C\rangle}{\mathsf{m}\langle A, B, C\rangle}\\ &= \frac{\mathsf{m}\langle A', B', C'\rangle}{\mathsf{m}\langle A, B, C\rangle}\end{aligned}$$

and so

$$\mathsf{m}'\langle A', B', C'\rangle = k\,\mathsf{m}\langle A', B', C'\rangle;$$

moreover, property **T3** ensures that this holds also for degenerate triangles $\langle A', B', C'\rangle$. □

Let (O, O_1, O_2) be an affine coordinate system for our plane and (O, X, Y) the associated cartesian coordinate system. Note that

$$\mathsf{m}\langle O, O_1, O_2\rangle = OO_1 + O_1O_2 + O_2O = XY.$$

Consider a polygon $\langle A_1, A_2, \ldots, A_k\rangle$ in the plane. Observing that $(A_i - O)(A_{i+1} - O)$, being a product of vectors spanned by X and Y is a multiple of XY, we see that

$$\begin{aligned}\mathsf{m}\langle A_1, A_2, \ldots, A_k\rangle &= A_1A_2 + \cdots + A_{k-1}A_k + A_kA_1 \\ &= (A_1 - O)(A_2 - O) + \cdots \\ &\quad + (A_{k-1} - O)(A_k - O) + (A_k - O)(A_1 - O) \\ &= 2aXY, \\ &= 2a\,\mathsf{m}\langle O, O_1, O_2\rangle\end{aligned}$$

for some number a, the basis-dependent *scalar oriented area* of the polygon. We call the absolute value of a the *scalar area* of the polygon and we call its sign the *orientation* of the polygon relative to the basis $\langle X, Y\rangle$. If $\mathsf{m}(\sigma) = 0$ then we say that σ has orientation 0. Here we must assume that the underlying field is equipped with an absolute value function (like the real numbers or the complex numbers). Scalar area and orientation are separately translation invariant.

Typically of Grassmann geometry, the following theorem generalizes in an obvious way from paired triangles to paired closed curves.

Geometry Theorem 2.

hyp$_1$ parallel$(A - B', A' - B)$
hyp$_2$ parallel$(B - C', B' - C)$
hyp$_3$ parallel$(C - A', C' - A)$
conc $\mathsf{m}\langle A', B', C'\rangle = \mathsf{m}\langle A, B, C\rangle$

The invariant treatment of area we are advocating renders some affine geometry theorems transparently provable by algebraic arguments that are easily automated.

The proof of the following in [8] (p. 55) uses two auxiliary points and two lemmas. Note too that our formulation contains more information than a traditional one, since it asserts that two orientations are the same, as well as two scalar areas.

Geometry Theorem 3.

hyp$_1$ collinear(A, D, P)
hyp$_2$ collinear(B, C, P)
hyp$_3$ $M =$ midpoint(A, C)
hyp$_4$ $N =$ midpoint(B, D)
conc $\mathsf{m}\langle A, B, C, D\rangle = 4\,\mathsf{m}\langle M, N, P\rangle$

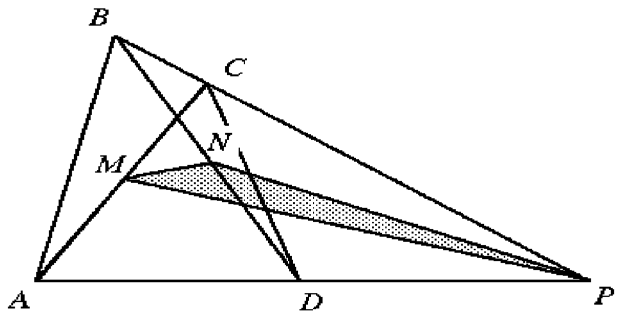

Fig. 6. Geometry Theorem 3.

Proof. We have:

$$\begin{aligned}
&4(PM + MN + NP)\\
&= 2P(A + C) + (A + C)(B + D) + 2(B + D)P\\
&= 2(AD + DP + PA) - 2(BC + CP + PB) + (AB + BC + CD + DA)\\
&= AB + BC + CD + DA.
\end{aligned}$$

□

Suppose that the vertices of a polygon $\langle A_1, \ldots, A_n\rangle$ are given by

$$A_i = O + x(t_i)X + y(t_i)Y,$$

where x and y are functions from the field $\mathcal{K}$ to itself. Then it is straightforward to show that the scalar oriented area of the polygon relative to $\langle X, Y\rangle$ is

$$\frac{1}{2}\sum(x(t_i)y'(t_i) - y(t_i)x'(t_i))(t_{i+1} - t_i),$$

where $x'(t_i)) = x(t_{i+1}) - x(t_i)/(t_{i+1} - t_i)$ (and $\sum$ is a cyclic sum over $i = 1$ to n). In the limit this becomes the line integral formula for area.

Consider a triangle $\langle A, B, C\rangle$ in a plane having scalar oriented area a relative to $\langle X, Y\rangle$. From

$$AB + BC + CA = 2aXY,$$

we see, using the fact that $AXY = OXY$ (because A lies in the plane of (O, X, Y)), that

$$ABC = 2aOXY,$$

and hence (unambiguously)

$$\mathsf{m}\langle A, B, C\rangle = \frac{ABC}{2OXY}.$$

Thus we have an alternative representation of area for triangles in a plane as a ratio of tripoints, and

$$\frac{\mathsf{m}\langle A', B', C'\rangle}{\mathsf{m}\langle A, B, C\rangle} = \frac{A'B'C}{ABC}.$$

Grassmann geometry gives new meaning to traditional Euclidean geometry notations. For example, the following propositions have obvious interpretions in terms of absolute area and orientation.

Geometry Theorem 4.

hyp parallelogram(A, B, C, D)
conc $ACD = ABC$

Geometry Theorem 5.

hyp parallel$(B - A, C - D)$
conc $ABC = ABD$

Since

$$\begin{aligned}\mathbf{det}(\mathbf{T}) &= \frac{\mathbf{T}(A)\mathbf{T}(B)\mathbf{T}(C)}{ABC} \\ &= \frac{\mathsf{m}\langle \mathbf{T}(A), \mathbf{T}(B), \mathbf{T}(C)\rangle}{\mathsf{m}\langle A, B, C\rangle}\end{aligned}$$

the determinant of a linear transformation $\mathbf{T}$ of the plane is the (constant) ratio by which $\mathbf{T}$ transforms areas.

7 The Boundary Operator

Let $(O, X_1, ..., X_n)$ be a cartesian coordinate system for a Grassmann algebra Ω. Every element ϕ of Ω may be written uniquely in the form

$$\begin{aligned}\phi = x + yO + x_1X_1 + ... + x_nX_n + ... + y_1OX_1 + ... + y_nOX_n \\ ... + x_{12}X_1X_2 + ... + x_{1n}X_1X_n + ... + x_{n-1,n}X_{n-1}X_n \\ ... + y_{12}OX_1X_2 + ... + x_{123}X_1X_2X_3 + ...\end{aligned}$$

The *boundary operator* is the linear function $\partial : \Omega \to \Omega$ given by

$$\partial\phi = y + y_1X_1 + ... + y_nX_n ... + y_{12}X_1X_2 + ... + y_{123}X_1X_2X_3 + ...;$$

∂ simply strips O from those basis elements $OX_i...X_k$ containing it and annihilates those that don't. Obviously $\partial^2 = 0$.

If $A = O + \Sigma a_iX_i$ and $B = O + \Sigma b_jX_j$ then

$$\begin{aligned}\partial(AB) &= \partial(\Sigma b_jOX_j - \Sigma a_iOX_i + \Sigma a_ib_jX_iX_j) \\ &= \Sigma b_jX_j - \Sigma a_iX_i \\ &= B - A.\end{aligned}$$

One easily shows that $\partial(ABC) = (B - A)(C - A) = BC - AC + AB$ and $\partial(ABCD) = (B - A)(C - A)(D - A) = BCD - ACD + ABD - ABC$ and, by induction,

$$\begin{aligned}\partial(A_1A_2...A_k) &= (A_2 - A_1)...(A_k - A_1) \\ &= A_2A_3...A_k - A_1A_3...A_k + ...\end{aligned}$$

Since every element of a Grassmann algebra is a linear combination of products of points, we see that, despite the definition, ∂ is independent of the coordinate system.

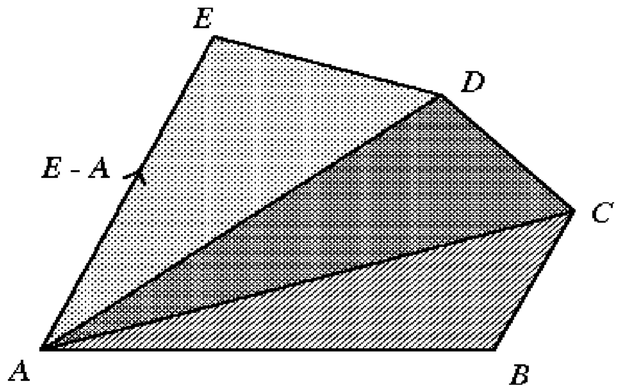

Fig. 7. Typical boundary calculations.

Typical boundary calculations (see Fig. 7) give:

$$\partial(AB + BC + CD + DE) = E - A,$$
$$\partial(AB + BC + CD + DE + EA) = 0,$$
$$\partial(ABC + ACD + ADE) = AB + BC + CD + DE + EA.$$

Observe that for any point A, $\partial(A) = 1$, while for any vector V, $\partial(V) = 0$; and, in general, $\partial(\Sigma a_i A_i) = \Sigma a_i$.

Recall that elements $\Sigma a_i A_i$ of the vector space Ω_0 spanned by all points are called *chains of dimension 0* or, briefly, *0-chains*; thus every 0-chain is either a point or a vector or a multiple of a point. Elements $\Sigma a_{ij} A_i A_j$ of the vector space Ω_1 spanned by all bipoints are called *1-chains*; these include bipoints and bivectors. In general, elements $\Sigma a_{i_0 \ldots i_k} A_{i_0} \ldots A_{i_k}$ of the vector space Ω_k spanned by all k-dimensional multipoints are called k-*chains*; for consistency, numbers are called -1-*chains*, and we write $\Omega_{-1} = \mathcal{K}$.

Some of our earlier proofs become simpler if we use the boundary operator. For example, the fact that

$$CD = AB \Rightarrow D - C = B - A$$

is obtained immediately by just applying ∂ to both sides of the hypothesis equation.

A k-chain ϕ is called a (*global*) k-*boundary* if it belongs to the range $ran(\partial)$ (that is to say, has the form $\partial(\psi)$ for some $k+1$-chain ψ) and a k-*cycle* if it belongs to the kernel $ker(\partial)$ (that is to say, satisfies $\partial(\phi) = 0$). From the fact that $\partial^2 = 0$ it follows that every k-boundary is a cycle. It is easy to see that the converse also holds. For example, if the 1-chain $\phi = \Sigma y_i OX_i + \Sigma x_{i,j} X_i X_j$ is a cycle, then $0 = \partial(\phi) = \Sigma y_i X_i$, and so every $y_i = 0$ and ϕ has the form $\phi = \Sigma x_{i,j} X_i X_j = \partial \Sigma x_{i,j} OX_i X_j$, and this argument evidently works for k-chains in general. Thus the k-cycles (or k-boundaries) are precisely the pure vectorial k-chains.

In particular, areas of closed curves are 1-cycles, and for any P

$$\mathsf{m}\langle A_0, A_1, \ldots, A_n\rangle = \partial(A_0A_1P + A_1A_2P + \ldots A_{n-1}A_nP + A_nA_0P),$$

a formula that offers additional insight into our generalized notion of area.

Note that ∂ does not preserve products and so is not a ring homomorphism; however there is a simple interaction with multiplication:

Theorem 12. *For multipoints ϕ and ψ*

$$\partial(\phi\psi) = (\partial\phi)\psi + (-1)^{degree(\phi)}\phi\partial\psi.$$

This is a special case of Theorem 16 below.

8 Grassmann Polynomials

The following is a typical affine geometry theorem.

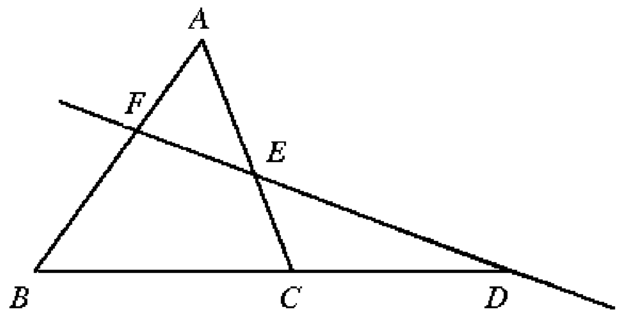

Fig. 8. The Menelaus theorem.

Geometry Theorem 6 (*The Menelaus Theorem*).

hyp$_1$ $D = aB + a'C$
hyp$_2$ $E = bC + b'A$
hyp$_3$ $F = cA + c'B$
hyp$_4$ $a'b'c' = -abc$
conc collinear(D, E, F)

The hypotheses are constraints on some points. We may think of drawing a diagram. Choose a point D on the line through B and C. Choose E on the line through C and A. Choose F on the line through A and B in such a way that the fourth condition is satisfied. The conclusion is another constraint that the three points must satisfy. The universe in which all this is interpreted is just a plane containing the points. What about the proof? Given our definition of collinearity, it is a simple algebraic calculation:

$$DEF = abcBCA + a'b'c'CAB = (abc + a'b'c')ABC = 0$$

Associated with the Menelaus theorem we have an algebra of Grassmann expressions, satisfying the usual rules for numbers and points. Generalising, we define the *Grassmann polynomial algebra* over the field $\mathcal{K}$, $\mathcal{K}[n,m]$, to be the (unique up to isomorphism) associative algebra over $\mathcal{K}$ freely generated by the n point indeterminates $P_1, P_2, \ldots, P_n$ and m number indeterminates $x_1, x_2, \ldots, x_m$, subject to the relations

1. $P_i P_j = -P_j P_i$, $1 \leq i, j \leq n$,
2. $x_i x_j = x_j x_i$, $1 \leq i, j \leq m$, and
3. $P_i x_j = x_j P_i$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Elements of $\mathcal{K}[n,m]$ are called *(Grassmann) polynomials.* Sometimes we use $A, B, C, \ldots$ and $a, b, c, \ldots$ to signify point and number variables respectively. In $\mathcal{K}[n,m]$, two polynomials are equal if and only if their equality represents an identity involving up to n points and m numbers which holds in every Grassmann algebra $\Omega[\mathcal{K},\mathcal{P}]$; that is, elements of $\mathcal{K}[n,m]$ are equal if and only if they define the same *polynomial function* on every Grassmann algebra over $\mathcal{K}$.

$\mathcal{K}[n,0]$ is a copy of the Grassmann algebra $\Omega[\mathcal{K},\mathcal{P}]$ with basis $\{P_1, P_2, \ldots, P_n\}$, while $\mathcal{K}[0,m]$ is just the familiar commutative ring of polynomials in $x_1, x_2, \ldots, x_m$ over $\mathcal{K}$. For this latter ring, the Gröbner basis algorithm of Buchberger provides a powerful algorithmic tool: see [2]. But much work has appeared in the literature generalising this algorithm to other sorts of structure, including to a broad class which includes Grassmann algebras: see Apel [1]. The Grassmann case is treated in isolation in [26], and we present below a streamlined version of that treatment which uses results from [1] and which incorporates the boundary map.

The Hilbert Basis theorem extends easily to $\mathcal{K}[n,m]$, since it is a finite-dimensional extension of a standard polynomial ring. Hence every left ideal, right ideal and two-sided ideal is finitely generated.

Throughout the remainder of this paper, f, g, h, k will stand for elements of $\mathcal{K}[n,m]$, F, G for finite subsets of $\mathcal{K}[n,m]$, r, s, t, u for elements of $T_{n,m}$, and $a, b, c, \alpha, \beta, \gamma$ for numbers in $\mathcal{K}$. Additionally, we will often use $A, B, C, \ldots$ for point indeterminates.

We define the *standard order* on the indeterminates of $\mathcal{K}[n,m]$ as follows:

$$P_1 < P_2 < \ldots < P_n < x_1 < x_2 < \ldots < x_m.$$

A non-zero product of the indeterminates in which they appear in ascending order is called a *term*. For example, $x_2^3 x_4 P_3 P_7$ is a term, but $x_4 P_7 P_3$ is not. Also 1 is a term. We denote by $T_{n,m}$ the set of all terms in $\mathcal{K}[n,m]$. Each polynomial in $\mathcal{K}[n,m]$ is uniquely expressible (modulo re-ordering of the summand terms) as a linear combination of terms $T_{n,m}$, obtained by expanding in the usual way. We call this a *canonical form* of the element. The set of terms occurring in this linear combination for a given $p \in \mathcal{K}[n,m]$ is denoted by $T(p)$. If $|T(p)| = 1$, then we denote the single element of $T(p)$ by $T(p)^*$.

For $t \in T_{n,m}$, let $P(t)$ denote the set of point variables appearing in t. For a set S of point variables, $T(S)$ denotes the unique $t \in T_{n,0}$ such that $P(t) = S$.

9 Ideals and Geometry Theorems

Let $\Omega = \Omega(\mathcal{K}, \mathcal{P})$ be a Grassmann algebra. As we have seen, affine geometry theorems may be expressed in terms of Grassmann algebra. The hypotheses of the theorem correspond to certain Grassmann expressions in the points and numbers mentioned in the theorem being zero, and so does the one (or possibly more) conclusion, and we can say the theorem is true for Ω if and only if all substitutions of points from $\mathcal{P}$ for the point variables mentioned in the Grassmann geometry theorem statement which satisfy the hypotheses also satisfy the conclusion. So, if the hypotheses are expressed in the form $f_1 = 0, f_2 = 0, \ldots, f_k = 0$ and the conclusion in the form $g = 0$ where all $f_i, g \in \mathcal{K}[n,m]$, then we say the f_i are the *hypothesis polynomials* and g is the *conclusion polynomial* of the *possible theorem*

$$f_1 = 0, f_2 = 0, \ldots, f_k = 0 \Rightarrow g = 0.$$

The *consequence space* associated with $F \subseteq \mathcal{K}[n,m]$ is defined to be $\mathcal{C}(F) = \{f : f \in \mathcal{K}[n,m], f(a) = 0 \text{ for all } a \in \Omega^n \times \mathcal{K}^m \text{ that satisfy } g(a) = 0 \text{ for all } g \in F\}$. Thus $\mathcal{C}(F)$ is the set of polynomials which vanish whenever all polynomials in F do. Thus the possible theorem $f_1 = 0, f_2 = 0, \ldots, f_k = 0 \Rightarrow g = 0$ is true for Ω if and only if $g \in \mathcal{C}(\{f_1, f_2, \ldots, f_k\})$.

For $F \subseteq \mathcal{K}[n,m]$, we denote the *ideal generated by* F by (F), so that

$$(F) = \{\sum_i p_i f_i q_i : p_i, q_i \in \mathcal{K}[n,m], f_i \in F\}.$$

Similarly, we denote the *left ideal generated by* F by $(F)_L$, so that $(F)_L = \{\sum_i p_i f_i | p_i \in \mathcal{K}[n,m], f_i \in F\}$. It is easy to see that $\mathcal{C}(F)$ is actually an ideal of $\mathcal{K}[n,m]$ for all $F \subseteq \mathcal{K}[n,m]$.

In earlier examples we used the fact that for any $F \subseteq \mathcal{K}[n,m]$, $(F) \subseteq \mathcal{C}(F)$. Our method often boiled down to doing some *equational reasoning*. For example, consider the simple theorem:

Geometry Theorem 7.

hyp parallelogram(A, B, C, D)
conc parallelogram(B, C, D, A).

Equivalently, we have the hypothesis $B - A - C + D = 0$ and conclusion $C - B - D + A = 0$, so the conclusion polynomial is just -1 times the hypothesis

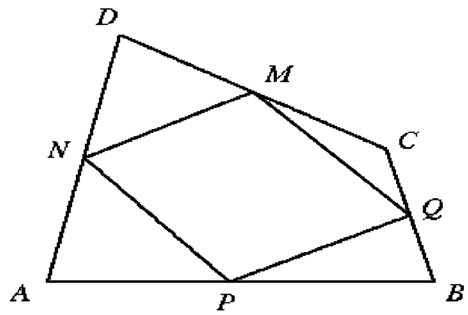

Fig. 9. A property of quadrilaterals.

polynomial, so $C - B - D + A \in (B - A - C + D)$ (in the appropriate Grassmann polynomial algebra).

Another example

Geometry Theorem 8 .

hyp$_1$ $P = \mathsf{midpoint}(A, B)$
hyp$_2$ $Q = \mathsf{midpoint}(B, C)$
hyp$_3$ $M = \mathsf{midpoint}(C, D)$
hyp$_4$ $N = \mathsf{midpoint}(D, A)$
conc $\mathsf{parallelogram}(P, Q, M, N)$

(see Fig. 9) may be re-written as

hyp$_1$ $f_1 = 0$
hyp$_2$ $f_2 = 0$
hyp$_3$ $f_3 = 0$
hyp$_4$ $f_4 = 0$
conc $g = 0$

where $f_1 \equiv 2P - A - B$, $f_2 \equiv 2Q - B - C$, $f_3 \equiv 2M - C - D$, $f_4 \equiv 2N - D - A$ and $g \equiv Q - P - M + N$, It is easy to see that

$$g = \frac{1}{2}f_2 - \frac{1}{2}f_1 - \frac{1}{2}f_3 + \frac{1}{2}f_4$$

and hence $g \in (f_1, f_2, f_3, f_4)$.

Geometry Theorem 4, expressing a familiar fact about parallelograms, follows from the fact that

$$ABC - ACD = (-AC)(B - A - C + D)$$

so again the ideal generated by the hypothesis polynomial contains the conclusion polynomial. The same happens in the Menelaus theorem, the first example we see in which number variables occur in an essential way. The reader should be convinced that any algebraic proof which makes use of substitution and simplification is really just showing that the conclusion polynomial is in the ideal generated by the hypothesis polynomials.

We would like to know exactly when a theorem of Grassmann geometry is true; that is, we would like a method of checking whether $f \in \mathcal{C}(F)$. In the case where there are no point variables, it is clear that if some *power* of f is in (F), then $f \in \mathcal{C}(F)$. Conversely, we have the famous

Theorem 13 (*The Hilbert Nullstellensatz*). *Suppose $\mathcal{K}$ is an algebraically closed field. For any $F \subseteq \mathcal{K}[0, m]$, $f \in \mathcal{C}(F)$ if and only if there exists some integer $\rho > 0$ such that $f^\rho \in (F)$.*

The Nullstellensatz implies the following very useful

Corollary 14. *Suppose $\mathcal{K}$ is an algebraically closed field. For any $F \subseteq \mathcal{K}[0, m]$ and $f \in \mathcal{K}[0, m]$, $f \in \mathcal{C}(F)$ if and only if $1 \in (F \cup \{fx_{m+1} - 1\})$ in $\mathcal{K}[0, m+1]$.*

This provides a necessary and sufficient condition for determining whether a geometry theorem is true over the complex numbers if $n = 0$, and has been exploited with great success: see [21] and [22] where this *refutational* approach is discussed in detail, as well as [4] where the useful notion of *genericity* is given a full and elegant treatment.

Unfortunately there is no known analog of the Nullstellensatz for Grassmann polynomials, over any kinds of field. Moreover, the notion of genericity seems unable to be captured in terms of ideals only. The sufficient condition $f \in (F)$ is quite useful as we have seen, but is not necessary, even if $m = 0$. A simple counterexample is

$$ABC = 0 \Rightarrow AB + BC + CA = 0.$$

Clearly $AB + BC + CA \notin (ABC)$. Nevertheless, the implication holds, since $\partial(ABC) = AB + BC + CA$.

We may define the boundary map on $\mathcal{K}[n, m]$ in the obvious manner. Linearity of ∂ implies that for all $F \subseteq \mathcal{K}[n, m]$, $\mathcal{C}(F)$ is closed under taking boundaries: $\partial(f) \in \mathcal{C}(F)$ for all $f \in \mathcal{C}(F)$. We call an ideal of $\mathcal{K}[n, m]$ closed under taking boundaries a *∂-ideal.*

It makes sense to talk about the smallest ∂-ideal containing a subset F of $\mathcal{K}[n, m]$ — the ∂-ideal *generated by F* — essentially because $\mathcal{K}[n, m]$ together with the unary operation ∂ is a *multi-operator group*: see [23]. Notation: $(F)_\partial$. Thus for $F \subseteq \mathcal{K}[n, m]$, $(F) \subseteq (F)_\partial \subseteq \mathcal{C}(F)$.

Ideals of the form $\mathcal{C}(F)$ have another important algebraic property. We say that $f \in \mathcal{K}[n, m]$ is *point homogeneous of degree* $p \geq 0$ (usually shortened to just *homogeneous*) if $T(f) \subseteq \{t \in T_{n,m} \mid |P(t)| = p\}$; thus f is homogeneous of degree p if every term in f contains a product of p point variables. The set of homogeneous polynomials of degree p in $\mathcal{K}[n, m]$, $p < n$, will be denoted by $\mathcal{K}[n, m]^{(p)}$. Clearly $\mathcal{K}[n, m] = \sum_{p=1}^{n} \mathcal{K}[n, m]^{(p)}$, so that every $\phi \in \mathcal{K}[n, m]$ may be uniquely expressed as the sum of its *homogeneous components.*

A *homogeneous left ideal* I of $\mathcal{K}[n,m]$ is a left ideal for which $I = \sum_{p=1}^{n}(I \cap \mathcal{K}[n,m]^{(p)})$. A *homogeneous subset* of $\mathcal{K}[n,m]$ is a subset all elements of which are homogeneous.

We note that $\mathcal{K}[n,m]$ is a graded ring with respect to the grading determined by the degree of homogeneity. The above definition of 'homogeneous' is consistent with that used for graded rings generally.

The following results will be useful when we come to consider algorithms. The next one was proved in [26].

Theorem 15. *In $\mathcal{K}[n,m]$, a homogeneous left ideal is an ideal.*

We can now easily prove the following useful product rule.

Theorem 16. *For $\phi \in \mathcal{K}[n,m]^{(p)}$ and $\psi \in \mathcal{K}[n,m]^{(q)}$,*

$$\partial(\phi\psi) = (\partial\phi)\psi + (-1)^p \phi\partial\psi.$$

Proof. Let $\phi = \sum_i \alpha_i s_i$, $\psi = \sum_j \beta_j t_j$, where for all i, j, α_i and β_j are numbers, s_i is a term of degree p and t_j a term of degree q. Then

$$\phi\psi = \sum_i \alpha_i s_i \sum_j \beta_j t_j = \sum_{i,j} \alpha_i \beta_j s_i t_j,$$

so

$$\begin{aligned}
\partial(\phi\psi) &= \partial(\sum_{i,j} \alpha_i \beta_j s_i t_j) \\
&= \sum_{i,j} \alpha_i \beta_j \partial(s_i t_j) \\
&= \sum_{i,j} \alpha_i \beta_j (\partial(s_i) t_j + (-1)^p s_i \partial(t_j)) \\
&= \sum_i \alpha_i \partial(s_i) \sum_j \beta_j t_j + (-1)^p \sum_i \alpha_i s_i \sum_j \beta_j \partial(t_j) \\
&= (\partial\phi)\psi + (-1)^p \phi\partial\psi.
\end{aligned}$$

□

Theorem 17. *For every homogeneous subset F of $\mathcal{K}[n,m]$, $(F)_\partial = (F \cup \partial F)$.*

Proof. Since $F \cup \partial F \subseteq (F)_\partial$, we have $(F \cup \partial F) \subseteq (F)_\partial$.

Conversely, let $\theta \in (F \cup \partial F) = (F \cup \partial F)_L$ by Theorem 15. Then by using distributivity, it is possible to express θ in the form $\theta = \sum p_i f_i + \sum q_j \partial g_j$, where the f_i and g_j are not necessarily distinct elements of F, and the p_i and q_j are homogeneous elements of $\mathcal{K}[n,m]$. Then by Theorem 16,

$$\begin{aligned}
\partial\theta &= \sum \partial(p_i f_i) + \sum \partial(q_j \partial g_j) \\
&= \sum((\partial p_i) f_i + p_i' \partial f_i) + \sum(\partial q_j \partial g_j + q_j' \partial^2 g_j) \\
&= \sum(\partial p_i) f_i + \sum p_i' \partial f_i) + \sum \partial q_j \partial g_j \\
&\in (F \cup \partial F),
\end{aligned}$$

where $p_i' = \pm p_i$ and $q_j' = \pm q_j$. Hence $(F \cup \partial F)$ is a ∂-ideal. It contains F, so $(F)_\partial \subseteq (F \cup \partial F)$, and the proof is complete. □

Because no non-trivial linear combination of m-points can equal a sum of non-trivial linear combinations of n-points for various $n \neq m$, we have the following

Theorem 18. *For $F \subseteq \mathcal{K}[n, m]$, $\mathcal{C}(F)$ is homogeneous.*

There are many homogeneous ideals not of the form $\mathcal{C}(F)$ however. For instance, (P) is a homogeneous ideal in $\mathcal{K}[1, 0]$, yet $\partial P = 1 \notin (P)$. So not every homogeneous ideal is a ∂-ideal. Similarly, for

$$f = P_1 - P_2 + (P_1 - P_3)(P_1 - P_4)(P_1 - P_5),$$

it is easily shown that $(f)_\partial = (f)_L$ and hence not every ∂-ideal is homogeneous. We shall later see an example of a homogeneous ∂-ideal not of the form $\mathcal{C}(F)$.

In the Grassmann algebraic formulation of a geometry theorem, it is homogeneous polynomials which feature, as they are the important polynomials in geometry.

Corollary 19. *If F is a set of homogeneous polynomials, then $(F)_\partial = (F \cup \partial F)_L$.*

Proof. If F is homogeneous, so is $F \cup \partial F$ so $(F \cup \partial F)_L$ is a homogeneous left ideal, as follows from a basic fact concerning graded rings. Thus $(F \cup \partial F)_L$ is an ideal by Theorem 15, and so $(F \cup \partial F)_L = (F \cup \partial F) = (F)_\partial$ by Theorem 17. □

10 Gröbner Bases and Theorem Proving

We next introduce the relevant variant of Gröbner bases and the Buchberger algorithm. There are some basic notions we must define first, such as admissible orders, reduction and so on; they are mostly straight-forward variants of the usual commutative notions, so we will be fairly brief. In any case, many generalisations of Buchberger's algorithm and the Gröbner basis idea to various kinds of generalised polynomial have already been considered in the literature. Indeed the current one may be viewed as being a special case of one considered by [1]. However, some of the special features of the Grassmann case, such as homogeneity and the boundary map, have not been considered elsewhere apart from in [26].

Let $\leq$ be a total ordering of $T_{n,m}$. For $p \in \mathcal{K}[n, m]$, we denote the highest term in $T_{n,m}$ occurring in f with respect to $\leq$ by $Hterm_{\leq}(f)$, or, if there is no ambiguity, by $Hterm(f)$.

The ordering $\leq$ is *admissible* if, for all s, t, u

1. $1 \leq t$,
2. if $s < t$ and $P(u) \cap P(s) = P(u) \cap P(t) = \emptyset$, then $T(us)^* < T(ut)^*$.

So 1 is the smallest term, and pre-multiplying by a term preserves the ordering providing neither term becomes zero. In fact, because of the commutativity and anti-commutativity relations in the Grassmann polynomial algebra, any such order also satisfies the condition that if $s < t$ and $P(u) \cap P(s) = P(u) \cap P(t) = \emptyset$, then $T(su)^* < T(tu)^*$, so post-multiplication is order preserving providing neither term becomes zero.

We note that if $P(u) \backslash P(t) = \emptyset$, then ut and tu are multiples of $T(ut)^* = T(tu)^* = Hterm(ut)^* = Hterm(tu)^*$. Examples are variants of the commutative case: for instance, we have the *total degree order* on $T_{n,m}$. With $n = 2$ and $m = 1$, this is as follows:

$$\begin{aligned} &1 < P_1 < P_2 < x_1 < P_1P_2 < P_1x_1 < P_2x_1 < x_1^2 \\ &< P_1P_2x_1 < P_1x_1 < P_2x_1 < x_1^3 < \cdots, \end{aligned}$$

Here is the *lexicographic order* for the same n, m:

$$1 < P_1 < P_1P_2 < P_2 < x_1 < P_1x_1 < P_2x_1 < P_1P_2x_1 < x_1^2 < P_1x_1^2 < \cdots.$$

It should be clear how these generalise.

Let $\leq$ be a fixed admissible order in what follows (the total degree order in all computed examples). We use the following abbreviations throughout:

$coef(f, t)$ is the coefficient of the term t in f, where $f \in \mathcal{K}[n, m]$.
$Hcoef(f)$ is $coef(f, Hterm(f))$.

For example, with the total degree order on the terms of $\mathcal{K}[4, 0]$,

$coef(P_1 + 2P_3P_1, P_1P_3) = -2$,
$Hterm(P_1P_2 + P_2P_3 + P_4P_3) = P_3P_4$, and
$Hcoef(P_1P_2 + P_2P_3 + P_4P_3) = coef(P_1P_2 + P_2P_3 + P_4P_3, P_3P_4) = -1$.

Every admissible order on $T_{n,m}$ is a well order (the proof of which is very similar to the commutative, $n = 0$ case), so the terms can be listed in order from the smallest element 1 up as far as one wishes. If $m = 0$, the list is finite.

We say $g \rightarrow_F h$ (*g left reduces to h modulo F*) if there are b, u, and $f \in F$ with $P(u) \cap P(Hterm(f)) = \emptyset$ and such that

$h = g - buf$,
$coef(g, T(u \cdot Hterm(f))) \neq 0$, and
$b = coef(g, T(u \cdot Hterm(f)))/Hcoef(uf) \neq 0$.

In such circumstances we say that $g \rightarrow_{f,b,u}$ and that $g \rightarrow_{t,f} h$ where $t = T(u \cdot Hterm(f))$. $\rightarrow_F$ is the *reduction relation for F*.

These are just the analogs of the commutative polynomial notions. To say that $g \rightarrow_F h$ is to say that there is a polynomial f in F whose largest term can be used to replace a term in g by a linear combination of smaller terms to give a polynomial h. Thus reduction is just a formalisation of the process of substitution and simplification which is the basis of our equational approach to geometric reasoning. The reduction relation $\rightarrow_F$ is a Noetherian relation (so that reductions cannot go on forever), a straightforward generalisation of the corresponding result for standard polynomials appearing in [2].

For example, in $\mathcal{K}[3,0]$, let $F = \{f\}$, $f = P_1P_3 - 2P_1P_2$, $g = P_1P_2P_3$. Then $g \rightarrow_F P_1P_2P_3 - (-1)P_2(P_1P_3 - 2P_1P_2) = 0$. Thus $g \rightarrow_{t,f} 0$ where $t = P_1P_2P_3$, and $g \rightarrow_{f,-1,P_2}$.

Let $\mathcal{K}$ be a computable field (meaning one which can be implemented on a computer, such as the rationals). The following algorithm yields a normal form $N(g)$ of a given polynomial g, modulo F.

begin
  $N(g) := g$
  while exist $f \in F, b, u$ such that $N(g) \rightarrow f, b, u$, do
    choose $f \in F$, $u \in T_{n,m}$, $b \in \mathcal{K}$ such that $N(g) \rightarrow_{f,b,u}$
  $N(g) := N(g) - b \cdot u \cdot f$.
end

Correctness is clear. Termination follows from the Noetherian property of $\rightarrow_F$.

For $u, v \in T_{n,m}$, with $P(u) = P(v) = \emptyset$, let $lcm(u, v)$ denote the usual least common multiple of u and v as polynomial terms. Then for arbitrary $s = fp, t = gq \in T_{n,m}$, with f, g containing only number variables and p, q only point variables, $lcm(s,t)$ is defined to be $lcm(f,g)T(P(s) \cup P(t))$. So $lcm(s,t)$ is the term of lowest degree (both number and point degree) divisible by both s and t.

The *S-polynomial*, $SP(f_1, f_2)$, corresponding to f_1 and f_2 is defined to be

$$SP(f_1, f_2) = a_1 \cdot Hcoef(u_2f_2) \cdot u_1f_1 - a_2 \cdot Hcoef(u_1f_1) \cdot u_2f_2,$$

with a_1, a_2, u_1, u_2 such that

$$P(u_1) \cap P(Hterm(f_1)) = P(u_2) \cap P(Hterm(f_2)) = \emptyset$$

and

$$\begin{aligned} a_1u_1 \cdot Hterm(f_1) &= T(u_1 \cdot Hterm(f_1))^* \\ &= Hterm(u_1f_1) \\ &= lcm(Hterm(f_1), Hterm(f_2)) \\ &= a_2u_2 \cdot Hterm(f_2) \\ &= T(u_2Hterm(f_2))^* \\ &= Hterm(u_2f_2). \end{aligned}$$

The existence and uniqueness of $SP(f_1, f_2)$ for given $f_1, f_2 \in P$ are easily shown - let $u_1 = T((P(Hterm(f_1)) \cup P(Hterm(f_2)) \backslash P(Hterm(f_1)))$, and likewise with u_2. The S-polynomial is just the difference between the results of reducing the lcm of the highest terms of two polynomials using each of them in turn.

To illustrate these definitions, letting $f = P_3P_5x_2^2 - 2P_2P_4$ and $g = P_3P_4x_2 + 3P_1P_2$, we have that $lcm(f,g) = lcm(P_3P_5x_2^2, P_3P_4x_2) = x_2^2P_3P_4P_5$, and so

$$\begin{aligned} SP(f,g) &= -P_4(P_3P_5x_2^2 - 2P_2P_4) - P_5x_2(P_3P_4x_2 + 3P_1P_2) \\ &= P_3P_4P_5x_2^2 + 0 - P_3P_4P_5x_2^2 - 3P_1P_2P_5x_2 \\ &= -3P_1P_2P_5x_2. \end{aligned}$$

F is a *Gröbner basis* if every $f \in (F)_L$ can be reduced to zero by a sequence of reductions involving $\rightarrow_F$.

Theorem 20. *The following statements are equivalent:*

1. *F is a Gröbner basis.*
2. *If $f_1, f_2 \in F$, then for any $P_i \in P(Hterm(f_1))$, $P_i \cdot f_1 \rightarrow_F^* 0$ and $SP(f_1, f_2)$ $\rightarrow_F^* 0$.*

The proof follows from a more general result of Apel [1].

Theorem 21. *Given a finite subset F of $\mathcal{K}[n, m]$, the following algorithm constructs a Gröbner basis G such that $(F)_L = (G)_L$.*

```
begin
  V := {point variables in P}
  G := F
  H := F
  B := {{f1, f2}|f1, f2 ∈ F, f1 ≠ f2}
  comment: H plays two roles in what follows
  while H ≠ ∅ do
  comment: in the next procedure H is the subset of G which supplies
  polynomials which, when multiplied by appropriate point variables, are to
  be included in G if they are not of normal form zero modulo G
  begin
    while H ≠ ∅ do
    begin
      f := an element of H
      H := H\{f}
      W := P(Hterm(f))
      comment: W is the set of all point variables that do not occur in
      Hterm(f)
      while W ≠ ∅ do
      begin
        P := an element of W
        k := P · f
        k' := N(G, k)
        if k' ≠ 0 then
          H := H ∪ {k'}
          G := G ∪ {k'}
          B := B ∪ {{g, k'}|g ∈ G}
          comment: if the normal form of P · f is not zero, then it is added to
          both G and H, thereby enlarging the basis and providing more
          polynomials for multiplication by appropriate terms as above
      end
    end
  end
```

```
  comment: H is now empty; in the next procedure, H will comprise the
  additions to G arising in the course of enlarging G by means of
  S-polynomials
  while B ≠ ∅ do
  begin
    {f1, f2} := an element of B
    B := B\{{f1, f2}}
    h := N(G, Spoly(f1, f2))
    if h ≠ 0 then
      B := B ∪ {{g, h'}|g ∈ G}
      G := G ∪ {h'}
      H := H ∪ {h'}
  end
end
```

Proof. Termination occurs since the only polynomials that are used to extend G or H are in normal form modulo G.

The algorithm will not terminate unless G has the two properties as in (3) of Theorem 20, so that G is a Gröbner basis. Furthermore, every polynomial used to extend the original set F in the initial run through the routine is either a left multiple of an element of F or a linear combination of such left multiples (a left multiple of an S-polynomial). Hence it is in $(F)_L$, so by induction $G \subset (F)_L$, and hence $(G)_L = (F)_L$, since $F \subseteq G$. □

Thus the left ideal membership problem has an algorithmic solution. In fact, if F is a set of homogeneous polynomials, $(F) = (F)_L$ by 15, and we have the obvious

Corollary 22. *Given a finite subset F of $\mathcal{K}[n, m]$ consisting of homogeneous polynomials, the above algorithm constructs a Gröbner basis G such that $(F) = (G)$.*

From Corollary 19, we also have the following

Corollary 23. *Given a finite subset F of $\mathcal{K}[n, m]$ consisting of homogeneous Grassmann polynomials, the above algorithm constructs a Gröbner basis G from $F' = F \cup \partial F$ such that $(G) = (F)_\partial$.*

Hence the ∂-ideal membership problem for homogeneous polynomials has an algorithmic solution; note that the boundary map is only needed initially in order to compute the boundaries of all elements of F. Thus any equational consequence of a set of hypotheses in which the boundary map is freely used will be reduced to zero by the Gröbner basis of the hypothesis polynomials together with their boundaries, and conversely.

In geometry, it is more natural to encode collinearity information (and the higher-dimensional analogs) using a single product: thus collinear(A, B, C) becomes $ABC = 0$; similarly $M =$ midpoint(A, C) can be rendered as $AM = MC$.

Then $ABCD = 0$ says that the four points A, B, C, D are coplanar, from which one can deduce via the boundary map that $ABC = ABD + BCD + CAD$, which has an interpretation in terms of oriented areas.

Consider the problem of proving Proposition 3. Encoding the hypotheses as $ADP = BCP = 0$, $AM = MC$ and $BN = ND$, it is impossible to prove the conclusion that $AB + BC + CD + DA = 4(MN + NP + PM)$ using only substitutions and ideal-theoretic manipulations generally: the conclusion is not in the ideal of the hypotheses, nor even is the stronger conclusion polynomial $ABC + ACD - 4MNP$ from which the original conclusion may be derived by applying the boundary map. Instead one must first apply the boundary map to the hypotheses, and then reduction of both conclusions to zero is possible.

A more striking example arises by considering the one-dimensional analog of oriented area. Thus if $AB = CD$, it follows that $AC + CD = AD$, although clearly $AC + CD - AD$ is not a multiple of $AB - CD$. However, taking the boundary of $AB - CD$ gives $A - B - C + D$, and then one can easily check that $AC + CD - AD$ is in the ideal $(AB - CD, A - B - C + D)$.

A final observation. For vectors U, V, W of a Grassmann algebra, it follows from the Exchange Theorem that if $UV = 0$ and $UW = 0$ then either $U = 0$ or $VW = 0$. Hence in $\mathcal{K}[5, 0]$, letting $F = \{(P_1 - P_2)(P_1 - P_3), (P_1 - P_2)(P_1 - P_4), (P_1 - P_2)(P_1 - P_5) - (P_1 - P_3)(P_1 - P_4)\}$, we have that $(P_1 - P_3)(P_1 - P_4) \in \mathcal{C}(F)$. Note that $\partial F = \{0\}$, so by Theorem 17, $(F) = (F)_\partial$, a homogeneous ∂-ideal. Now one can apply the above algorithm to show that $(P_1 - P_3)(P_1 - P_4) \notin (F) = (F)_\partial$. This shows that a theorem prover based solely upon the computation of Gröbner bases of ∂-ideals is not complete. We conjecture that there is an algorithm which incorporates use of the Exchange Theorem to produce two or more new subcases of the hypotheses every time it is used and which is complete; such an algorithm should also be able to deal with genericity issues.

11 Conclusion

The origins of the ideas in this paper lie in the great works *Die Lineale Ausdehnungslehre,* of 1844, and *Die Ausdehnungslehre,* of 1862, of Hermann Grassmann, in which he founded a core discipline of modern mathematics: linear algebra, including exterior algebra. Although the second of these books was published as a new edition of the first, it really offered a very different approach to the same material and both are included in full in the *Collected Works* [15]. An historical analysis of Grassmann's work appears in [10]; also, in [12] it is argued that Grassmann anticipated some aspects of modern universal algebra. The approach to area advocated in the present paper has been used by one of us in geometry courses for twenty years or so.

For an overview of automated geometry theorem proving and of algebraic methods, [29] and [18] are recommended. The algorithmic method for proving Euclidean geometry theorems, beautifully presented by Shang-Ching Chou in [4], has its origin in the pioneering work of Wu Wen-tsün; see also [3], [30] and [31].

A somewhat different approach is to use Buchberger's Gröbner base algorithm; see, for example, [20], [21], [22] and [24]. The algorithms presented in this paper (and in [26]) extend Buchberger's algorithm to take in exterior products.

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