THE MULTIDIMENSIONAL LUZIN THEOREM

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ABSTRACT. Each measurable map of an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is equal almost everywhere to the gradient of a continuous almost everywhere differentiable function defined on $\mathbb{R}^n$ that vanishes, together with its gradient, outside U.

A well-known theorem of Luzin [7, Théorème Fondamental, p. 90] asserts that for each real-valued measurable function f defined on the real line $\mathbb{R}$ there is an $F \in C(\mathbb{R})$ such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}$. The argument is based on a skilful utilization of singular functions (*devil's staircases*) defined on nondegenerate compact subintervals of $\mathbb{R}$. More recently, G. Alberti [1, Theorem 1] proved that for each Borel vector field v defined on an open set $U \subset \mathbb{R}^n$ of finite measure there are a compact set $K \subset U$ and an $F \in C_c^1(U)$ such that the measure of $U - K$ is as small as we wish and $\nabla F(x) = v(x)$ for each $x \in K$.

We show that each continuous real-valued function f defined on the boundary of a compact nondegenerate cube $Q \subset \mathbb{R}^n$ has a continuous almost everywhere differentiable extension $g : Q \to \mathbb{R}$ such that $\nabla g(x) = 0$ for almost all $x \in Q$. Using such extensions and Alberti's theorem, we prove the result stated in the abstract by following the main ideas of the original Luzin's argument. Note that as in Alberti's theorem, we do not require that $\operatorname{curl} v = 0$. An application to k-charges defined in Section 4 below generalizes the multidimensional version of Luzin's theorem obtained in [5], and a fortiori that mentioned in [10, p.218].

The authors wish to acknowledge helpful discussions with T. De Pauw and V. Sverak during the preparation of this note.

1. PRELIMINARIES

Throughout, the ambient space is $\mathbb{R}^n$ where $n \geq 2$ is a fixed integer. A *cube* is a compact nondegenerate cube in $\mathbb{R}^n$. The open ball in $\mathbb{R}^n$ of radius $r > 0$ centered at $x \in \mathbb{R}^n$ is denoted by $B(x, r)$. Given $E \subset \mathbb{R}^n$, we denote by ∂E, int E $d(E)$, and $|E|$ the *boundary, interior, diameter*, and *Lebesgue measure* of E, respectively. Unless specified otherwise, all concepts related to measure refer tacitly to Lebesgue measure in $\mathbb{R}^n$. All functions we consider are real-valued.

Let $E \subset \mathbb{R}^n$ and $x \in \operatorname{int} E$. For a map $\phi : E \to \mathbb{R}^m$, $m = 1, 2, \ldots$, we denote by $D\phi(x)$ the differential of ϕ at x; if $m = 1$, the i-th partial derivative of ϕ at x is denoted by $D_i\phi(x)$. The symbols $C(E)$, $C(E; \mathbb{R}^m)$, $C^1(\operatorname{int} E; \mathbb{R}^m)$, and $C^\infty(\operatorname{int} E; \mathbb{R}^m)$ have the usual meaning; as customary, a subscript c will indicate

compact supports. If E is measurable, we denote by $L^0(E; \mathbb{R}^m)$ the space of all measurable maps $v : E \to \mathbb{R}^m$.

In each $\mathbb{R}^m$, we employ the *Euclidean norm* $|\cdot|$ induced by the usual inner product $x \cdot y$. Given $E \subset \mathbb{R}^n$ and $\phi : E \to \mathbb{R}^m$, the extended real number

$$\operatorname{osc}(\phi) := \sup\Big\{ |\phi(x) - \phi(y)| : x, y \in E \Big\}$$

is called the *oscillation* of ϕ.

Observation 1.1. *Let $E \subset \mathbb{R}^n$, and let $\phi_i : E \to \mathbb{R}^m$ be such that $\sum_{i=1}^{\infty} \operatorname{osc}(\phi_i)$ converges. If there is an $x_0 \in E$ such that $\sum_{i=1}^{\infty} |\phi_i(x_0)|$ converges, then $\sum_{i=1}^{\infty} \phi_i$ converges uniformly to a $\phi : E \to \mathbb{R}^m$, and $\operatorname{osc}(\phi) \leq \sum_{i=1}^{\infty} \operatorname{osc}(\phi_i)$.*

Proof. The sum $\sum_{i=1}^{\infty} |\phi_i|$ converges uniformly, since

$$|\phi_i(x)| \leq |\phi_i(x) - \phi_i(x_0)| + |\phi_i(x_0)| \leq \operatorname{osc}(\phi_i) + |\phi_i(x_0)|$$

for each $x \in E$. As

$$|\phi(x) - \phi(y)| \leq \sum_{i=1}^{\infty} |\phi_i(x) - \phi_i(y)| \leq \sum_{i=1}^{\infty} \operatorname{osc}(\phi_i)$$

for each $x, y \in E$, the observation follows. $\square$

2. A singular extension

Proposition 2.1. *Let Q be a cube, and let $f \in C(\partial Q)$. There is an almost everywhere differentiable $g \in C(Q)$ such that $g \restriction \partial Q = f$, $\operatorname{osc}(g) = \operatorname{osc}(f)$, and $Dg(x) = 0$ for almost all $x \in Q$.*

Proof. For closed sets $A_i \subset Q$ and functions $h_i \in C(A_i)$ satisfying $h_i(x) = h_j(x)$ for each $x \in A_i \cap A_j$ and each $i, j \in I$, we denote by $\bigvee_{i \in I} h_i$ the unique function $h : \bigcup_{i \in I} A_i \to \mathbb{R}$ such that $h(x) = h_i(x)$ for every $x \in A_i$. Note that $\bigvee_{i \in I} h_i$ is continuous whenever I is finite.

For $i = 0, 1, \ldots$, let $\mathcal{C}_i$ be the collection of all nonoverlapping congruent cubes of diameter $d(Q)/3^i$ whose union is Q, and let $\mathcal{C} := \bigcup_{i=0}^{\infty} \mathcal{C}_i$. Given $C \in \mathcal{C}_i$, denote by $C^\frown$ the unique cube in $\mathcal{C}_{i+1}$ contained in the interior of C, and let

$$\mathcal{C}(C) := \{K \in \mathcal{C}_{i+1} : K \subset C \text{ and } K \neq C^\frown\};$$

if $i \geq 1$, the unique cube in $\mathcal{C}_{i-1}$ containing C is denoted by $C^\vee$.

Choose a $C \in \mathcal{C}_i$. For $k = 0, \ldots, n-1$, a *k-face* of C is a closed k-dimensional face of C. As usual, 0-faces and 1-faces of C are called, respectively, *vertices* and *edges* of C. The *link* of a k-face A of C is the union B of all $(n-1)$-faces of C that do not meet A. Given $x \in C - (A \cup B)$, let ℓ_x be the line passing through x and A if A is a vertex of C, and the line passing through x and perpendicular to A otherwise. Denoting by x_A and x_B the intersection of ℓ_x with A and B, respectively, we have $x = t x_A + (1-t) x_B$ for a unique $t \in (0, 1)$. Now for a $\varphi \in C(A \cup B)$, let

$$\varphi^\sim(x) := \begin{cases} \varphi(x) & \text{if } x \in A \cup B, \\ t\varphi(x_A) + (1-t)\varphi(x_B) & \text{if } x \in C - (A \cup B), \end{cases}$$

and observe that $\varphi^\sim \in C(C)$ and $\mathrm{osc}(\varphi^\sim) = \mathrm{osc}(\varphi)$. Next we describe a specific extension of $\varphi \in C(\partial C)$ to a φ^* defined on $C^* := C^\frown \cup \bigcup_{K \in \mathcal{C}(C)} \partial K$. Denote by $\mathcal{H}$ the $(n-1)$-dimensional *Hausdorff measure* in $\mathbb{R}^n$, and let

$$\varphi_C(x) := \frac{1}{\mathcal{H}(\partial C)} \int_{\partial C} \varphi \, d\mathcal{H}$$

for each $x \in C$. If $K \in \mathcal{C}(C)$, then the intersection $A_K := C^\frown \cap K$ is a k-face of K whose link B_K is contained in ∂C, and we let

$$\varphi_K := \big[(\varphi_C \upharpoonright A_K) \vee (\varphi \upharpoonright B_K)\big]^\sim \quad \text{and} \quad \varphi^* := (\varphi_C \upharpoonright C^\frown) \vee \bigvee_{K \in \mathcal{C}(C)} (\varphi_K \upharpoonright \partial K).$$

Clearly, $\varphi^* \in C(C^*)$ extends φ and $\mathrm{osc}(\varphi^*) = \mathrm{osc}(\varphi)$.

The function $f_0 := f^*$ is defined and continuous on $Q_0 := Q^*$. Assuming that Q_i and $f_i \in C(Q_i)$ have been defined for an $i \geq 0$, let

$$Q_{i+1} := Q_i \cup \bigcup_{C \in \mathcal{C}_i} \bigcup_{K \in \mathcal{C}(C)} K^* \quad \text{and} \quad f_{i+1} := f_i \vee \bigvee_{C \in \mathcal{C}_i} \bigvee_{K \in \mathcal{C}(C)} (f_i \upharpoonright \partial K)^*,$$

and observe that $f_{i+1} \in C(Q_{i+1})$ extends f_i. The function $g := \bigvee_{i=0}^\infty f_i$ is defined on $D := \bigcup_{i=0}^\infty Q_i$ and extends f. If $C \in \mathcal{C}$, then

$$\mathrm{osc}\big[g \upharpoonright (C \cap D)\big] = \mathrm{osc}(g \upharpoonright \partial C), \tag{2.1}$$

$C^\frown \subset D$, and g is constant on $C^\frown$. Since $\sum_{C \in \mathcal{C}} |C^\frown| = |Q|$, we only need to show that g is continuous and has a continuous extension to Q. To this end, select a sequence $\{x_i\}$ in D converging to an $x \in Q$, and let $N := 2^n$.

Assume $x \in \mathrm{int}\, Q$. For sufficiently large i, the function g is piecewise linear on ∂C for each $C \in \mathcal{C}_i$ containing x. Select such a C, and denote by $v_1, \ldots, v_N$ its vertices. Then $g(x) = (1/N) \sum_{s=1}^N g(v_s)$ for each $x \in C^\frown$, and

$$\mathrm{osc}(g \upharpoonright \partial C) = \max\Big\{ \big|g(v_s) - g(v_t)\big| : s, t = 1, \ldots, N \Big\}.$$

Let $x \in C^\frown$ and $y \in \partial C$. As $y = \sum_{t=1}^N \alpha_t v_t$ where $\alpha_t \geq 0$ for $t = 1, \ldots, N$ and $\sum_{t=1}^N \alpha_t = 1$, we obtain

$$\begin{aligned} \big|g(y) - g(x)\big| &\leq \frac{1}{N} \sum_{s=1}^N \big|g(y) - g(v_s)\big| \leq \frac{1}{N} \sum_{s=1}^N \sum_{t=1}^N \alpha_t \big|g(v_t) - g(v_s)\big| \\ &= \frac{1}{N} \sum_{t=1}^N \alpha_t \sum_{s=1}^N \big|g(v_t) - g(v_s)\big| \leq \frac{N-1}{N} \mathrm{osc}(g \upharpoonright \partial C). \end{aligned} \tag{2.2}$$

If v and w are vertices of $K \in \mathcal{C}(C)$ contained in ∂C, it is easy to see that

$$\big|g(v) - g(w)\big| \leq \frac{1}{3} \mathrm{osc}(g \upharpoonright \partial C) \leq \frac{N-1}{N} \mathrm{osc}(g \upharpoonright \partial C). \tag{2.3}$$

Since each vertex of K belongs either to ∂C or to $C^\frown$, from (2.1)–(2.3) we obtain

$$\mathrm{osc}\big[g \upharpoonright (K \cap D)\big] \leq \frac{N-1}{N} \mathrm{osc}\big[g \upharpoonright (C \cap D)\big].$$

It follows that $\mathrm{osc}\big[g \upharpoonright (C \cap D)\big] \to 0$ uniformly on $\{C \in \mathcal{C}_i : x \in C\}$ as $i \to \infty$, and the sequence $\{g(x_i)\}$ is Cauchy. In particular, $\{g(x_i)\}$ converges to $g(x)$ whenever $x \in D$; indeed, $\{x_1, x, x_2, x, \ldots\}$ is a sequence in D converging to x.

Assume $x \in \partial Q$. Choose an $\varepsilon > 0$, end find a $\delta > 0$ so that $\big|f(y) - f(x)\big| < \varepsilon$ for each $y \in \partial Q$ such that $|y - x| < \delta$. Next find an integer $k \geq 1$ so that $C \subset B(x, \delta)$ for each $C \in \mathcal{C}_k$ with $x \in C$. For such a C, the function g is equal to a constant c on $(C^\vee)^\wedge \cap \partial C$, and it is linear on each one-dimensional segment in ∂C that is parallel to an edge of C not contained in ∂Q. Consequently,

$$\operatorname{osc}(g \restriction \partial C) \leq \operatorname{osc}\big[f \restriction (C \cap \partial Q)\big] + \sup\Big\{\big|f(y) - c\big| : y \in C \cap \partial Q\Big\}$$

$$< 2\varepsilon + \sup\Big\{\big|f(y) - f(x)\big| : y \in C \cap \partial Q\Big\} + \big|f(x) - c\big| \qquad (2.4)$$

$$< 3\varepsilon + \big|f(x) - c\big|.$$

If p is the number of vertices of C contained in ∂Q, then $N/2 \leq p \leq N - 1$. Let $E_b := \big[pb + (N - p)c\big]/N$, and observe that $\big|f(x) - E_{f(x)}\big| \leq \big|f(x) - c\big|/2$ and

$$E_{f(x)} - \varepsilon < E_{f(x)-\varepsilon} < g(z) < E_{f(x)+\varepsilon} < E_{f(x)} + \varepsilon$$

for each $z \in C^\wedge$. Thus given $K \in \mathcal{C}(C)$ containing x, and any $z \in K^\wedge$,

$$\big|f(x) - g(z)\big| \leq \big|f(x) - E_{f(x)}\big| + \big|E_{f(x)} - g(z)\big| < \frac{1}{2}\big|f(x) - c\big| + \varepsilon.$$

An induction on $i = 1, 2, \ldots$ implies that if $L \in \mathcal{C}_{k+i}$ contains x, then

$$\big|f(x) - g(z)\big| < 2^{-i}\big|f(x) - c\big| + \varepsilon \sum_{j=0}^{i-1} 2^{-j} \qquad (2.5)$$

for each $z \in L^\wedge$. Now (2.4), (2.5), and (2.1) yield $\operatorname{osc}\big[g \restriction (L \cap D)\big] < 6\varepsilon$ for all sufficiently large i. We conclude the sequence $\big\{g(x_i)\big\}$ converges to $f(x)$. $\qquad \square$

A function g that satisfies the conditions of Proposition 2.1 is called a *singular extension* of f.

Corollary 2.2. *Let $U \subset \mathbb{R}^n$ be open, and let $f \in C(U)$ be differentiable almost everywhere. Given $\eta > 0$, there is a $g \in C(\mathbb{R}^n)$ such that $\{g \neq 0\} \subset U$, $\operatorname{osc}(g) < \eta$, and $Dg(x) = Df(x)$ for almost all $x \in U$.*

Proof. Let $C_1, C_2, \ldots$ be nonoverlapping cubes whose union is U. As f is uniformly continuous in each C_i, there are nonoverlapping cubes $Q_{i,1}, \ldots, Q_{i,k_i}$ such that $C_i = \bigcup_{j=1}^{k_i} Q_{i,j}$ and $\operatorname{osc}(f \restriction Q_{i,j}) < \eta/2^{i+2}$ for $j = 1, \ldots, k_i$, $i = 1, 2, \ldots$. Find a singular extension $g_{i,j}$ of $f \restriction \partial Q_{i,j}$, and let

$$f_{i,j}(x) := \begin{cases} f(x) - g_{i,j}(x) & \text{if } x \in Q_{i,j}, \\ 0 & \text{if } x \in \mathbb{R}^n - Q_{i,j}. \end{cases}$$

Observation 1.1 implies that $g := \sum_{i=1}^{\infty} \sum_{j=1}^{k_i} f_{i,j}$ is the desired function. $\qquad \square$

3. Luzin's theorem

The following proposition is a mere reformulation, convenient for our purpose, of the aforementioned result due to G. Alberti.

Proposition 3.1 (Alberti). *Let $U \subset \mathbb{R}^n$ be an open set, and let $v : U \to \mathbb{R}^n$ be a Borel vector field. Given $\varepsilon > 0$, there are a closed set $C \subset U$ and an $f \in C^1(\mathbb{R}^n)$ such that $|U - C| < \varepsilon$, $\{f \neq 0\} \subset U$, and $Df(x) = v(x)$ for each $x \in C$.*

Proof. Let $U_i := \big\{ x \in U : i - 1 < |x| < i \big\}$ for $i = 1, 2, \ldots$. According to [1, Theorem 1], there are compact sets $K_i \subset U_i$ and functions $f_i \in C^1(\mathbb{R}^n)$ such that $|U_i - K_i| < \varepsilon/2^i$, $\{f_i \neq 0\} \subset U_i$, and $Df_i(x) = v(x)$ for each $x \in K_i$. Clearly, it suffices to let $C := \bigcup_{i=1}^{\infty} K_i$ and $f := \sum_{i=1}^{\infty} f_i$. $\square$

Lemma 3.2. *Let $U \subset \mathbb{R}^n$ be an open set, and let $v \in L^0(U; \mathbb{R}^n)$. Given $\varepsilon > 0$ and a closed set $A \subset U$, there are a $g \in C(\mathbb{R}^n)$ and a closed set $C \subset U$ satisfying the following conditions.*

(1) $|U - C| < \varepsilon$, $\operatorname{osc}(g) < \varepsilon$, *and* $\{g \neq 0\} \subset U - A$.

(2) g *is differentiable almost everywhere in* $U - A$.

(3) $Dg(x) = v(x)$ *for each* $x \in C - A$.

(4) *If* $x \in A \cup (\mathbb{R}^n - U)$ *and* $h \in \mathbb{R}^n$, *then* $\big|g(x + h)\big| \leq \varepsilon|h|$.

(5) *If* $x \in A \cup (\mathbb{R}^n - U)$, *then* g *is differentiable at* x *and* $Dg(x) = 0$.

Proof. Since v equals almost everywhere to a Borel vector field $w : U \to \mathbb{R}^n$, we may assume that v is Borel. By Proposition 3.1, there are an $f \in C^1(\mathbb{R}^n)$ and a closed set $C \subset U$ such that $|U - C| < \varepsilon$, $\{f \neq 0\} \subset U$, and $Df(x) = v(x)$ for each $x \in C$. Find disjoint nonoverlapping cubes Q_i with $\bigcup_{i=1}^{\infty} Q_i = U - A$. Select a $\Delta_i \in (0, 1]$ that is smaller than the distance between Q_i and $A \cup (\mathbb{R}^n - U)$, and let $\eta_i := \varepsilon 2^{-i} \Delta_i$. If $g_i \in C(\mathbb{R}^n)$ is associated with $f \upharpoonright \operatorname{int} Q_i$ and η_i according to Corollary 2.2, then $\{g_i \neq 0\} \subset \operatorname{int} Q_i$, $\operatorname{osc}(g_i) < \eta_i$, and $Dg_i(x) = v(x)$ for almost all $x \in C \cap Q_i$. Observation 1.1 implies that $g := \sum_{i=1}^{\infty} g_i$ belongs to $C(\mathbb{R}^n)$ and $\operatorname{osc}(g) < \varepsilon$. Clearly, $\{g \neq 0\} \subset U - A$ and g is differentiable at almost all $x \in U - A$. Moreover, $Dg(x) = v(x)$ for almost all $x \in C - A$. Making C smaller, we may assume that $Dg(x) = v(x)$ holds for all $x \in C - A$.

Choose an $x \in A \cup (\mathbb{R}^n - U)$ and $h \in \mathbb{R}^n$. If $x + h$ does not belong to Q_i, then $g_i(x + h) = 0$. On the other hand, if $x + h$ is in Q_i, then $\Delta_i < |h|$ and hence

$$\big|g_i(x + h)\big| = \big|g_i(x + h) - g_i(x)\big| \leq \operatorname{osc}(g_i) < \eta_i < \varepsilon 2^{-i}|h|.$$

Consequently $\big|g(x + h)\big| \leq \varepsilon|h|$. More precisely, if $i(h)$ is the least positive integer with $x + h \in Q_i$, then

$$\big|g(x + h) - g(x)\big| = \big|g(x + h)\big| \leq \sum_{i = i(h)}^{\infty} \big|g_i(x + h)\big| < \varepsilon|h| \sum_{i = i(h)}^{\infty} 2^{-i}.$$

Since $i(h) \to \infty$ as $|h| \to 0$, we conclude g is differentiable at x and $Dg(x) = 0$. $\square$

Lemma 3.3. *Let $U \subset \mathbb{R}^n$ be an open set, let $v \in L^0(U; \mathbb{R}^n)$, and let $C_0 := \emptyset$. Given $\varepsilon > 0$, there are $g_i \in C(\mathbb{R}^n)$ and closed sets $C_i \subset U$ such that the following conditions are met for $i = 1, 2, \ldots$.*

(1) $C_{i-1} \subset C_i$ *and* $|U - C_i| < \varepsilon/2^i$.

(2) $\operatorname{osc}(g_i) < \varepsilon/2^i$ *and* $\{g_i \neq 0\} \subset U - C_{i-1}$.

(3) g_i *is differentiable almost everywhere in* $U - C_{i-1}$.

(4) *If* $f_i := \sum_{j=1}^{i} g_j$, *then* $Df_i(x) = v(x)$ *for each* $x \in C_i$.

(5) *If* $x \in C_{i-1} \cup (\mathbb{R}^n - U)$ *and* $h \in \mathbb{R}^n$, *then* $\big|g_i(x + h)\big| \leq 2^{-i} \varepsilon|h|$.

(6) *If* $x \in C_{i-1} \cup (\mathbb{R}^n - U)$, *then* g_i *is differentiable at* x *and* $Dg_i(x) = 0$.

Proof. Using Lemma 3.2, find an almost everywhere differentiable $g_1 \in C(\mathbb{R}^n)$ and a closed set $C_1 \subset U$ such that $|U - C_1| < \varepsilon/2$, $\mathrm{osc}(g_1) < \varepsilon/2$, $\{g_1 \neq 0\} \subset U$, and $Dg_1(x) = v(x)$ for each $x \in C_1$. Since $f_1 = g_1$ and $C_0 = \emptyset$, the function g_1 satisfies conditions (1)–(6).

Proceeding by induction, suppose that $g_i \in C(\mathbb{R}^n)$ and closed sets $C_i \subset U$ have been defined for each positive integer $i \leq n-1$ so that conditions (1)–(6) are satisfied with $f_i := \sum_{j=1}^{i} g_j$. Using condition (3), find a $w \in L^0(U; \mathbb{R}^n)$ so that $w(x) = v(x) - Df_{n-1}(x)$ for almost all $x \in U$. Let $g_n \in C(\mathbb{R}^n)$ and a closed set $C \subset U$ be associated with w, $2^{-n}\varepsilon$, and C_{n-1} according to Lemma 3.2. Clearly g_n satisfies conditions (2), (3), (5), and (6), and making C smaller, we may assume that $Dg_n(x) = v(x) - Df_{n-1}(x)$ for all $x \in C$. The closed set $C_n := C_{n-1} \cup C$ satisfies condition (1), and for each $x \in C - C_{n-1}$,

$$Df_n(x) = Df_{n-1}(x) + Dg_n(x) = v(x).$$

However, if $x \in C_{n-1}$, then $Dg_n(x) = 0$ by condition (5) of Lemma 3.2. In view of this and the induction hypothesis, $Df_n(x) = Df_{n-1}(x) = v(x)$ for each $x \in C_{n-1}$, which establishes condition (4). $\square$

Theorem 3.4. *Let $U \subset \mathbb{R}^n$ be an open set, and let $v \in L^0(U; \mathbb{R}^n)$. Given $\varepsilon > 0$, there is an almost everywhere differentiable $f \in C(\mathbb{R}^n)$ satisfying the following conditions.*

(1) $\mathrm{osc}(f) < \varepsilon$ *and* $\{f \neq 0\} \subset U$.
(2) $Df(x) = v(x)$ *for almost all* $x \in U$.
(3) *If* $x \in \mathbb{R}^n - U$, *then* f *is differentiable at* x *and* $Df(x) = 0$.

Proof. For $i = 1, 2, \ldots$, let $C_i \subset U$ and $g_i \in C(\mathbb{R}^n)$ be associated with v and a positive $\varepsilon \leq 1$ according to Lemma 3.3, and let $f_i := \sum_{j=1}^{i} g_j$. Observation 1.1 shows that $f := \sum_{i=1}^{\infty} g_i$ belongs to $C(\mathbb{R}^n)$ and $\mathrm{osc}(f) < \varepsilon$. Since $\{f \neq 0\} \subset U$, we only need to establish the differentiability conditions. Condition (3) holds, since

$$\left|f(x+h) - f(x)\right| \leq \sum_{i=1}^{\infty} \left|g_i(x+h)\right| \leq \varepsilon|h| \sum_{i=1}^{\infty} 2^{-i} = \varepsilon|h|$$

for each $x \in \mathbb{R}^n - U$ and each $h \in \mathbb{R}^n$. Note $\left|U - \bigcup_{i=1}^{\infty} C_i\right| = 0$, and choose an $x \in \bigcup_{i=1}^{\infty} C_i$. If $x \in C_p$, then $g_i(x) = Dg_i(x) = 0$ for $i > p$; since $\{C_i\}$ is an increasing sequence of sets. Let $D_i := Dg_i(x)$ and $D := \sum_{i=1}^{p} D_i$. Choose a $\Delta > 0$, and find an integer $q \geq p$ with $2^{-q} < \Delta/2$. There is a $\delta > 0$ such that

$$\left|g_i(x+h) - g_i(x) - D_i(h)\right| \leq \frac{\Delta}{2q}|h|$$

for each $h \in \mathbb{R}^n$ with $|h| < \delta$, and $i = 1, \ldots, q$. Consequently

$$\left|f(x+h) - f(x) - D(h)\right| \leq \sum_{i=1}^{q} \left|g_i(x+h) - g_i(x) - D_i(h)\right| + \sum_{i=q+1}^{\infty} \left|g_i(x+h)\right|$$

$$\leq \frac{\Delta}{2}|h| + |h| \sum_{i=q+1}^{\infty} 2^{-i} \leq \Delta|h|$$

for each $h \in \mathbb{R}^n$ with $|h| < \delta$, and we conclude

$$Df(x) = D = \sum_{i=1}^{p} Dg_i(x) = Df_p(x) = v(x). \qquad \square$$

4. Charges

For $m = 0, \ldots, n$, the linear spaces of all *m-forms* and all *m-vectors* in $\mathbb{R}^n$ are denoted by $\wedge^m \mathbb{R}^n$ and $\wedge_m \mathbb{R}^n$, respectively. Identifying $\wedge^m \mathbb{R}^n$ and $\wedge_m \mathbb{R}^n$ with $\mathbb{R}^N$ where $N := \binom{n}{m}$, we denote by $|\omega|$ and $|\xi|$ the Euclidean norms of $\omega \in \wedge^m \mathbb{R}^n$ and $\xi \in \wedge_m \mathbb{R}^n$. The collection of all *simple m-vectors* $\xi \in \wedge_m \mathbb{R}^n$ with $|\xi| = 1$ is the *Grassmanian* $\mathbf{G}_0(n, m)$.

Throughout this section, $U \subset \mathbb{R}^n$ is a fixed open set. If $0 \le m < n$ is an integer and $\omega : U \to \wedge^m \mathbb{R}^n$ is differentiable at $x \in U$, then $d\omega(x)$ denotes the *exterior derivative* of ω at x; the differential $D\omega(x)$ is defined in the obvious way.

Proposition 4.1. *Let* $\omega \in L^0(U; \wedge^{m+1} \mathbb{R}^n)$ *where* $0 \le m \le n - 1$. *For each* $\varepsilon > 0$, *there is an almost everywhere differentiable* $\phi \in C(\mathbb{R}^n; \wedge^m \mathbb{R}^n)$ *satisfying the following conditions.*

(1) $\operatorname{osc}(\phi) < \varepsilon$ *and* $\{\phi \ne 0\} \subset U$.

(2) $d\phi(x) = \omega(x)$ *for almost all* $x \in U$.

(3) *If* $x \in \mathbb{R}^n - U$, *then* ϕ *is differentiable at* x *and* $D\phi(x) = 0$.

Proof. Let $\omega := \sum_{i_1 < \cdots < i_{m+1}} a_{i_1 \cdots i_{m+1}} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_{m+1}}$ where $a_{i_1 \cdots i_{m+1}} \in L^0(U)$. Using Theorem 3.4, find almost everywhere differentiable $A_{i_1 \cdots i_m} \in C(\mathbb{R}^n)$ satisfyig the following conditions.

(1) $\operatorname{osc}(A_{i_1 \cdots i_m}) < \varepsilon/N$ where $N := \binom{n}{m}$, and $\{A_{i_1 \cdots i_m} \ne 0\} \subset U$.

(2) For almost all $x \in U$ and $i = 1, \ldots, n$,

$$D_i A_{i_1 \cdots i_m}(x) = \begin{cases} a_{i\, i_1 \cdots i_m}(x) & \text{if } i < i_1, \\ 0 & \text{if } i \ge i_1. \end{cases}$$

(3) if $x \in \mathbb{R}^n - U$, then $A_{i_1 \cdots i_m}$ is differentiable at x and $DA_{i_1 \cdots i_m}(x) = 0$.

Now $\phi := \sum_{i_1 < \cdots < i_m} A_{i_1 \cdots i_m} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_m}$ is the desired m-form, since

$$d\phi(x) = \sum_{i_1 < \cdots < i_m} \sum_{i=1}^{n} D_i A_{i_1 \cdots i_m}(x)\, d\xi_i \wedge d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_m}$$

$$= \sum_{i < i_1 < \cdots < i_m} a_{i\, i_1 \cdots i_m}(x) d\xi_i \wedge d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_m} = \omega(x)$$

for almost all $x \in U$. $\qquad \square$

For each nonnegative integer $m \le n$, we denote by $\mathbf{N}_{m,K}(U)$ the linear space of all m-dimensional *normal currents* in U supported in a compact set $K \subset U$, and let $\mathbf{N}_m(U) := \bigcup_K \mathbf{N}_{m,K}(U)$ where the union is taken over all compact sets $K \subset U$. All properties of currents we will use can be found in [4, Section 4.1].

Let $T \in \mathbf{N}_m(U)$. There are a compactly supported Radon measure $\|T\|$ in U and a Borel map $\overrightarrow{T} : U \to \wedge_m \mathbb{R}^n$ such that $\left|\overrightarrow{T}(x)\right| = 1$ for $\|T\|$ almost all $x \in U$ and

$$\langle T, \phi \rangle = \int_U \langle \phi, \overrightarrow{T} \rangle \, d\|T\|$$

for every $\phi \in C(U, \wedge^m \mathbb{R}^n)$. Denote by ∂T and $\operatorname{spt} T$ the *boundary* and *support* of T, respectively, and let

$$\mathbf{M}(T) := \|T\|(U) \quad \text{and} \quad \mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T).$$

If $K \subset U$ is compact, we define the *flat seminorm* of T in K as

$$\mathbf{F}_K(T) := \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathbf{N}_{m+1,K}(U)\}.$$

Definition 4.2. A linear functional $F : \mathbf{N}_m(U) \to \mathbb{R}$ is called an *m-charge* if given $\varepsilon > 0$ and a compact set $K \subset U$, there is a $\delta > 0$ such that $F(T) < \varepsilon$ for each $T \in \mathbf{N}_{m,K}(U)$ with $\mathbf{F}_K(T) < \delta$ and $\mathbf{N}(T) < 1/\varepsilon$.

Note. Definition 4.2 was introduced jointly by T. De Pauw and W.F. Pfeffer in 2004 — a private communication on middle dimensional integration. It generalizes the concepts defined [9, Section 2.1]. For another connection see Remark 4.5 below.

Observation 4.3. *A linear functional $F : \mathbf{N}_m(U) \to \mathbb{R}$ is an m-charge if and only if given $\varepsilon > 0$ and a compact set $K \subset U$, there is a $\theta > 0$ such that*

$$F(T) \leq \theta \mathbf{F}_K(T) + \varepsilon \mathbf{N}(T)$$

for each $T \in \mathbf{N}_{m,K}(U)$.

Proof. As the converse is obvious, assume F is an m-charge. Choose a positive $\varepsilon \leq 1$ and a compact set $K \subset U$. There is a $\delta > 0$ such that $F(S) \leq \varepsilon$ for each $S \in \mathbf{N}_{m,K}(U)$ with $\mathbf{F}_K(S) \leq \delta$ and $\mathbf{N}(S) \leq 1/\varepsilon$. Let $\theta := \varepsilon/\delta$, and select a $T \in \mathbf{N}_{m,K}(U)$ with $\mathbf{N}(T) = 1$. As $\mathbf{N}(T) \leq 1/\varepsilon$, we have $F(T) \leq \varepsilon \mathbf{N}(T)$ whenever $\mathbf{F}_K(T) \leq \delta$. If $\mathbf{F}_K(T) > \delta$, let $S := \left[\delta/\mathbf{F}_K(T)\right]T$ and observe that $F(S) \leq \varepsilon$; since $\mathbf{F}_K(S) = \delta$ and $\mathbf{N}(S) = \delta/\mathbf{F}_K(T) \leq 1/\varepsilon$. Thus $F(T) \leq (\varepsilon/\delta)\mathbf{F}_K(T)$, and

$$F(T) \leq \theta \mathbf{F}_K(T) + \varepsilon \mathbf{N}(T)$$

in either case. As the last inequality is homogenous, the observation follows. $\square$

The *oscillation* of an m-charge F, denoted by $\operatorname{osc}(F)$, is the infimum of all $\varepsilon > 0$ such that $F(T) < \varepsilon$ for each $T \in \mathbf{N}_m(U)$ with $\mathbf{N}(T) < 1/\varepsilon$.

Proposition 4.4. *If $\omega \in C(U; \wedge^{m-1} \mathbb{R}^n)$, then*

$$F_\omega : T \mapsto \langle \partial T, \omega \rangle : \mathbf{N}_m(U) \to \mathbb{R}$$

is an m-charge, and $\operatorname{osc}(F_\omega) \leq \sqrt{\operatorname{osc}(\omega)}$.

Proof. Choose an $\varepsilon > 0$ and a compact set $K \subset U$. Find a $\phi \in C_c^\infty(U; \wedge^{m-1} \mathbb{R}^n)$ such that $\left|\omega(x) - \phi(x)\right| < \varepsilon^2/2$ for each $x \in K$, and let $\delta := \varepsilon/(2c)$ where c is larger than $\left|d\phi(x)\right|$ for each $x \in U$. Now let $T \in \mathbf{N}_{m,K}(U)$ be such that $\mathbf{N}(T) < 1/\varepsilon$ and $\mathbf{F}_K(T) < \delta$, and select an $S \in \mathbf{N}_{m+1,K}(U)$ with

$$\mathbf{M}(T - \partial S) \leq \mathbf{M}(T - \partial S) + \mathbf{M}(S) < \delta.$$

Since spt $\partial T \subset K$, we obtain

$$F_\omega(T) = \langle \partial T, \omega - \phi \rangle + \langle \partial(T - \partial S), \phi \rangle = \int_K \langle \omega - \phi, \overrightarrow{\partial T} \rangle \, d\|\partial T\| + \langle T - \partial S, d\phi \rangle$$

$$\leq \frac{\varepsilon^2}{2} \mathbf{N}(T) + \sup\Big\{ \big|d\phi(x)\big| : x \in U \Big\} \mathbf{M}(T - \partial S) < \frac{\varepsilon}{2} + c\delta = \varepsilon.$$

As the linearity of F_ω is obvious, F_ω is a m-charge. Assume $\operatorname{osc}(\omega) > 0$, and choose an $x \in U$ and a $T \in \mathbf{N}_m(U)$ with $\mathbf{N}(T) < 1/\sqrt{\operatorname{osc}(\omega)}$. Then

$$F_\omega(T) = \langle \partial T, \omega - \omega(x) \rangle = \int_U \langle \omega(y) - \omega(x), \overrightarrow{\partial T} \rangle \, d\|\partial T\|(y)$$

$$\leq \int_U \big|\omega(y) - \omega(x)\big| d\|\partial T\|(y) \leq \operatorname{osc}(\omega) \mathbf{N}(T) < \sqrt{\operatorname{osc}(\omega)}$$

and we conclude that $\operatorname{osc}(F_\omega) \leq \sqrt{\operatorname{osc}(\omega)}$. $\qquad\square$

The charge F_ω defined in Proposition 4.4 is called the *flux* of ω.

Remark 4.5. Note that $\omega \in C(U; \wedge^{m-1}\mathbb{R}^n)$ can be thought of as a *weak solution* of the equation $d\omega = F_\omega$. This assertion is more transparent when $m = n$. Identifying $C(U; \wedge^n \mathbb{R}^n)$ and $C(U; \wedge^{n-1}\mathbb{R}^n)$ with $C(U)$ and $C(U; \mathbb{R}^n)$, respectively, n-charges are distributions, called *strong charges* in [2]. Proposition 4.6 and Theorem 4.7 below, proved in [2, Proposition 2.9 and Theorem 4.7], indicate their usefulness.

Proposition 4.6. *Each $f \in L^n_{\mathrm{loc}}(U)$ defines an n-charge by the formula*

$$F(\varphi) = \int_U f(x)\varphi(x) dx$$

for each test function $\varphi \in C^\infty_c(U)$.

Theorem 4.7. *Let $F : C^\infty_c(U) \to \mathbb{R}$ be a distribution. The equation $\operatorname{div} v = F$ has a distributional solution $v \in C(U; \mathbb{R}^n)$ if and only if F is an n-charge.*

If $(T, x) \in \mathbf{N}_m(U) \times U$, we let $\operatorname{diam}(T, x) := d(\{x\} \cup \operatorname{spt} T)$ and

$$\operatorname{reg}(T, x) := \begin{cases} \dfrac{\mathbf{M}(T)}{\mathbf{M}(\partial T) \operatorname{diam}(T, x)} & \text{if } T \neq 0 \text{ and } \partial T \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a sequence $\{T_i\}$ in $\mathbf{N}_m(U)$ *tends to* $(x, \xi) \in U \times \mathbf{G}_0(n, m)$ if the following conditions are satisfied

(1) $\lim \operatorname{diam}(T_i, x) = 0$ and $\inf \operatorname{reg}(T_i, x) > 0$,

(2) $\lim \frac{1}{\mathbf{M}(T_i)} \int_U \big|\overrightarrow{T_i}(y) - \xi\big| d\|T_i\|(y) = 0$.

An m-charge F is *derivable* at $(x, \xi) \in U \times \mathbf{G}_0(n, m)$ if a limit

$$\lim \frac{F(T_i)}{\mathbf{M}(T_i)} \neq \pm\infty$$

exists for each sequence $\{T_i\}$ in $\mathbf{N}_m(U)$ that tends to (x, ξ); in which case all these limits have a common value called the *derivative* of F at (x, ξ), denoted by $\mathfrak{D}F(x, \xi)$. Since the set of sequences $\{T_i\}$ which tend to (x, ξ) is nonempty, the meaning of $\mathfrak{D}F(x, \xi)$ is unambiguous.

Note. For $m = n$, the derivation base defined above was used in [6, 8]. A very general Gauss-Green theorem has been obtained in [9] by employing a derivation base consisting of bounded BV sets, i.e, a very specialized n-dimensional normal currents, satisfying the above regularity condition. More details about this top-dimensional derivation base are given in the last paragraph of this section.

Proposition 4.8. *If $\omega \in C(U; \wedge^{m-1}\mathbb{R}^n)$ is differentiable at $x \in U$, then the flux F_ω of ω is derivable at (x, ξ) for each $\xi \in \mathbf{G}_0(n, m)$ and*
$$\mathfrak{D}F_\omega(x, \xi) = \langle d\omega(x), \xi \rangle.$$

Proof. Let $\xi \in \mathbf{G}_0(n, m)$, and let $\{T_i\}$ be a sequence in $\mathbf{N}_m(U)$ which tends to (x, ξ). Choose $\varepsilon > 0$, and let $\eta := \inf \operatorname{reg}(T_i, x)$. There are a linear map $\lambda : \mathbb{R}^n \to \wedge^{m-1}\mathbb{R}^n$ and a $\delta > 0$ such that
$$\big|\omega(y) - \omega(x) - \lambda(y - x)\big| \le \varepsilon\eta|y - x|$$
for $y \in U \cap B(x, \delta)$. If $\phi(y) := \omega(x) + \lambda(y - x)$ for each $y \in \mathbb{R}^n$, then it is easy to see that $d\phi$ is a constant m-form equal to $d\omega(x)$. Passing to a subsequence, we may assume that for $i = 1, 2, \ldots,$
$$\operatorname{spt} T_i \subset B(x, \delta) \quad \text{and} \quad \int_U \big|\overrightarrow{T_i}(y) - \xi\big|\, d\|T_i\|(y) < \varepsilon \mathbf{M}(T_i).$$
In the inequality
$$\Big|F_\omega(T_i) - \langle d\omega(x), \xi \rangle \mathbf{M}(T_i)\Big| \le$$
$$\big|\langle \partial T_i, \omega - \phi \rangle\big| + \Big|\langle \partial T_i, \phi \rangle - \langle d\omega(x), \xi \rangle \mathbf{M}(T_i)\Big| = A_1 + A_2$$
we estimate separately the terms A_1 and A_2:
$$A_1 \le \int_U \big|\omega(y) - \phi(y)\big|\, d\|\partial T_i\|(y) \le \varepsilon\eta \int_{\operatorname{spt} T_i} |y - x|\, d\|\partial T_i\|(y)$$
$$\le \varepsilon\,\eta \operatorname{diam}(T_i, x)\mathbf{M}(\partial T_i) < \varepsilon \mathbf{M}(T_i),$$
$$A_2 = \left|\langle T_i, d\phi \rangle - \int_U \langle d\omega(x), \xi \rangle\, d\|T_i\|(y)\right| \le \int_U \big|\langle d\omega(x), \overrightarrow{T_i}(y) - \xi \rangle\big|\, d\|T_i\|(y)$$
$$\le \big|d\omega(x)\big| \int_U \big|\overrightarrow{T_i}(y) - \xi\big|\, d\|T_i\|(y) < \varepsilon\big|d\omega(x)\big|\mathbf{M}(T_i).$$
The switch $\langle \partial T_i, \phi \rangle = \langle T_i, d\phi \rangle$ in the estimate of A_2 is possible, since there is a $\psi \in C_c^\infty(U; \wedge^{m-1}\mathbb{R}^n)$ with $\psi(y) = \phi(y)$ for each $y \in B(x, \delta)$. Combining the last three inequalities, we obtain
$$\Big|\langle d\omega(x), \xi \rangle \mathbf{M}(T_i) - F_\omega(T_i)\Big| < \varepsilon\Big(1 + \big|d\omega(x)\big|\Big)\mathbf{M}(T_i)$$
and the proposition follows. $\square$

Theorem 4.9. *Let $\omega \in L^0(U, \wedge^{m+1}\mathbb{R}^m)$ where $0 \le m \le n - 1$. Given $\varepsilon > 0$, there is an almost everywhere derivable $(m+1)$-charge $F : N_{m+1}(\mathbb{R}^n) \to \mathbb{R}$ satisfying the following conditions.*

(1) $\operatorname{osc}(F) < \varepsilon$ *and* $F(T) = 0$ *whenever* $U \cap \operatorname{spt} T = \emptyset$.
(2) $\mathfrak{D}F(x, \xi) = \langle \omega(x), \xi \rangle$ *for almost all $x \in U$ and each $\xi \in \mathbf{G}_0(n, m + 1)$.*
(3) *If $x \in \mathbb{R}^n - U$, then $\mathfrak{D}F(x, \xi) = 0$ for each $\xi \in \mathbf{G}_0(n, m + 1)$.*

Proof. Let $\phi \in C(\mathbb{R}^n; \wedge^m \mathbb{R}^n)$ be associated with ω and ε^2 according to Proposition 4.1. In view of Propositions 4.4 and 4.8, it suffices to let $F := F_\phi$ be the flux of ϕ. □

Let $\mathcal{BV}_c(U)$ be the collection of all BV sets [3, Chapter 5] whose closures are compact subsets of U, and denote by $\|B\|$ the *perimeter* of $B \in \mathcal{BV}_c$. In accordance with [4, Section 4.5], view $\mathcal{BV}_c(U)$ as a subspace of $\mathbf{N}_n(U)$, and observe that

$$\mathrm{reg}\,(B, x) = \begin{cases} \frac{|B|}{d(B \cup \{x\}) \|B\|} & \text{if } |B| > 0, \\ 0 & \text{if } |B| = 0, \end{cases}$$

for each $(B, x) \in \mathcal{BV}_c \times U$. For every $B \in \mathcal{BV}_c$, the n-vector $\overrightarrow{B}$ equals to a fixed $\xi_0 \in \mathbf{G}_0(n, n)$, which orients $\mathbb{R}^n$. It follows that a sequence $\{B_i\}$ in $\mathcal{BV}_c$ tends to $(x, \xi) \in U \times \mathbf{G}_0(n, n)$ whenever $\xi = \xi_0$, and

$$\lim d\big(B_i \cup \{x\}\big) = 0 \quad \text{and} \quad \inf \mathrm{reg}\,(B_i, x) > 0. \tag{4.1}$$

Thus if F is an n-charge derivable at (x, ξ_0), then

$$\lim \frac{F(B_i)}{|B_i|} = \mathfrak{D}F(x, \xi_0)$$

for each sequence $\{B_i\}$ in $\mathcal{BV}_c$ satisfying conditions (4.1). From this it follows that Theorem 4.9 generalizes the main result of [5].

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