Chapter 04.05

System of Equations

After reading this chapter, you should be able to:

1. *Setup simultaneous linear equations in matrix form and vice-versa,*
2. *Understand the concept of inverse of a matrix,*
3. *Know the difference between consistent and inconsistent system of linear equations,*
4. *Learn that system of linear equations can have a unique solution, no solution or infinite solutions.*

Matrix algebra is used for solving system of equations. Can you illustrate this concept?

Matrix algebra is used to solve a system of simultaneous linear equations. In fact, for many mathematical procedures such as solution of set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations. Let us illustrate with an example for interpolation.

Example 1

The upward velocity of a rocket is given at three different times on the following table

Time, t	Velocity, v
s	m/s
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = at^2 + bt + c, \quad 5 \le t \le 12.$$

Set up the equations in matrix form to find the coefficients a,b,c of the velocity profile.

Solution

The polynomial is going through three data points $(t_1,v_1),(t_2,v_2),and\ (t_3,v_3)$ where from the above table

$$t_1 = 5, v_1 = 106.8$$
$$t_2 = 8, v_2 = 177.2$$
$$t_3 = 12, v_3 = 279.2$$

Requiring that $v(t) = at^2 + bt + c$ passes through the three data points gives

$$v(t_1) = v_1 = at_1^2 + bt_1 + c$$
$$v(t_2) = v_2 = at_2^2 + bt_2 + c$$
$$v(t_3) = v_3 = at_3^2 + bt_3 + c$$

Substituting the data $(t_1,v_1),(t_2,v_2),(t_3,v_3)$ gives

$$a(5^2) + b(5) + c = 106.8$$
$$a(8^2) + b(8) + c = 177.2$$
$$a(12^2) + b(12) + c = 279.2$$

or

$$25a + 5b + c = 106.8$$
$$64a + 8b + c = 177.2$$
$$144a + 12b + c = 279.2$$

This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25a + & 5b + & c \\ 64a + & 8b + & c \\ 144a + & 12b + & c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above equation can be written as a linear combination as follows

$$a\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + b\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and further using matrix multiplications gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above is an illustration of why matrix algebra is needed. The complete solution to the set of equations is given later in this chapter.

For a general set of "*m*" linear equations and "*n*" unknowns,

$a_{11}x_1 + a_{22}x_2 + \cdots\cdots + a_{1n}x_n = c_1$

$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_m \end{bmatrix}$$

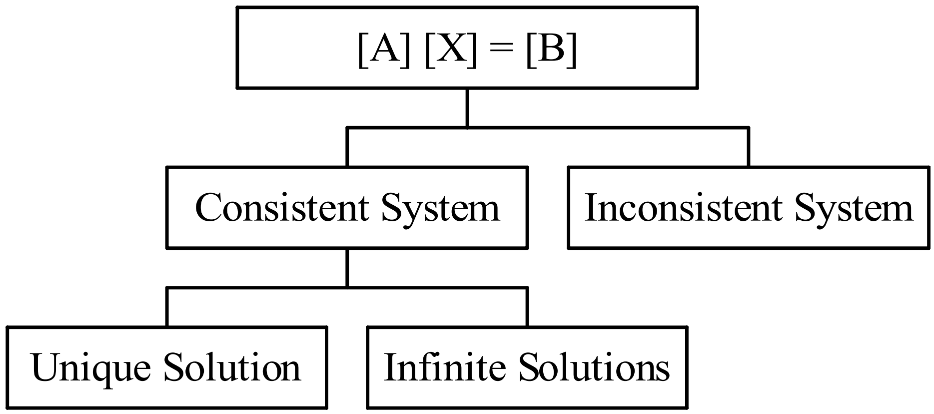
Denoting the matrices by $[A]$, $[X]$, and $[C]$, the system of equation is $[A]\ [X]=[C]$, where $[A]$ is called the coefficient matrix, $[C]$ is called the right hand side vector and $[X]$ is called the solution vector.

Sometimes $[A]\ [X]=[C]$ systems of equations is written in the augmented form. That is

$$[A \vdots C] = \begin{bmatrix} a_{11} & a_{12} & ...... & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & ...... & a_{2n} & \vdots c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & ...... & a_{mn} & \vdots c_n \end{bmatrix}$$

A system of equations can be consistent or inconsistent. What does that mean?

A system of equations $[A]\ [X]=[C]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, consistent system of equations does not mean a unique solution, that is, a consistent system of equation may have a unique solution or infinite solutions.



Example 2

Give examples of consistent and inconsistent system of equations.

Solution

a) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is a consistent system of equations as it has a unique solution, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

b) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

is also a consistent system of equations but it has infinite solutions as given as follows.

Expanding the above set of equations,

$$2x + 4y = 6$$

$$x + 2y = 3$$

you can see that they are the same equation. Hence any combination of (x, y) that satisfies

$$2x + 4y = 6$$

is a solution. For example $(x, y) = (1,1)$ is a solution and other solutions include $(x, y) = (0.5,1.25)$, $(x, y) = (0, 1.5)$ and so on.

c) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is inconsistent as no solution exists.

How can one distinguish between a consistent and inconsistent system of equations?

A system of equations [A] [X] = [C] is
consistent if the rank of A is equal to the rank of the augmented matrix $[A \vdots C]$,
inconsistent if the rank of A is less then the rank of the augmented matrix $[A \vdots C]$.
But, what do you mean by rank of a matrix?
The rank of a matrix is defined as the order of the largest square submatrix whose determinant is not zero.

Example 3

What is the rank of

$$[A] = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution

The largest square submatrix possible is of order 3 and that is [*A*] itself. Since det(*A*) = -25 $\neq$ 0, the rank of [*A*] = 3.

Example 4

What is the rank of

$$[A] = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 5 \\ 5 & 1 & 7 \end{bmatrix}$$

Solution

The largest square submatrix of [*A*] is of order 3, and is [*A*] itself. Since det(*A*) = 0, the rank of [*A*] is less than 3. The next largest square submatrix would be a 2×2 matrix. One of the square submatrices of [*A*] is

$$[B] = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$$

and det [*B*] = -2 ≠ 0. Hence the rank of [*A*] is 2.

Example 5

How do I now use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent or inconsistent system of equations?

Solution

The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side vector

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}.$$

The augmented matrix is

$$[B] = \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Since there are no square submatrices of order 4 as [*B*] is a 3x4 matrix, the rank of [*B*] is at most 3. So let us look at the square submatrices of [*B*] of order 3 and if any of these square submatrices have determinant not equal to zero, then the rank is 3. For example, a submatrix of the augmented matrix [*B*] is

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

has det (*D*) = -84 ≠ 0.

Hence the rank of the augmented matrix [B] is 3. Since [A]=[D], the rank of [A] =3. Since the rank of augmented matrix [B] = rank of coefficient matrix [A], the system of equations is consistent.

Example 6

Use the concept of rank of matrix to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is consistent or inconsistent?

Solution

The coefficient matrix is given by

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

and the right hand side

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

The augmented matrix is

$$[B] = \begin{bmatrix} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 89 & 13 & 2 & 284.0 \end{bmatrix}$$

Since there are no square submatrices of order 4 as [B] is a 4×3 matrix, the rank of the augmented [B] is at most 3. So let us look at square submatrices of the augmented matrix [B] of order 3 and see if any of these square submatrices have determinant not equal to zero, then the rank is 3. For example a square submatrix of the augmented matrix [B] is

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

has det(D) = 0. This means, we need to explore other square submatrices of order 3 of the augmented matrix [B].

That is,

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 284.0 \end{bmatrix}$$

$\det(E) = 0$,

$$[F] = \begin{bmatrix} 25 & 5 & 106.8 \\ 64 & 8 & 177.2 \\ 89 & 13 & 284.0 \end{bmatrix}$$

$\det(F) = 0$,

and

$$[G] = \begin{bmatrix} 25 & 1 & 106.8 \\ 64 & 1 & 177.2 \\ 89 & 2 & 284.0 \end{bmatrix}$$

$\det(G) = 0$.

All the square submatrices of order 3 of the augmented matrix [B] have a zero determinant. So the rank of the augmented matrix [B] is less than 3. Is the rank of [B] = 2? A 2×2 submatrix of the augmented matrix [B] is

$$[H] = \begin{bmatrix} 25 & 5 \\ 64 & 8 \end{bmatrix}$$

and

$\det(H) = -120 \neq 0$

So the rank of the augmented matrix [B] is 2.

Now we need to find the rank of the coefficient matrix [A].

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

and

$\det(A) = 0$.

So the rank of the coefficient matrix [A] is less than 3. A square submatrix of the coefficient matrix [A] is

$$[J] = \begin{bmatrix} 5 & 1 \\ 8 & 1 \end{bmatrix}$$

$\det(J) = -3 \neq 0$

So the rank of the coefficient matrix [A] is 2.

Hence, rank of the coefficient matrix [A] = rank of the augmented matrix [B].

So the system of equations [A] [X] = [B] is consistent.

Example 7

Use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 280.0 \end{bmatrix}$$

is consistent or inconsistent.

Solution

The augmented matrix is

$$[B] = \begin{bmatrix} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 4 & 177.2 \\ 89 & 13 & 2 & 280.0 \end{bmatrix}$$

Since there are no square submatrices of order 4 as the augmented matrix [B] is a 4×3 matrix, the rank of the augmented matrix [B] is at most 3. So let us look at square submatrices of the augmented matrix (B) of order 3 and see if any of the 3×3 submatrices have a determinant not equal to zero. For example a square submatrix of order 3×3 of [B]

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

$\det(D) = 0$

So it means, we need to explore other square submatrices of the augmented matrix [B],

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$$

$\det(E) \neq 12.0 \neq 0.$

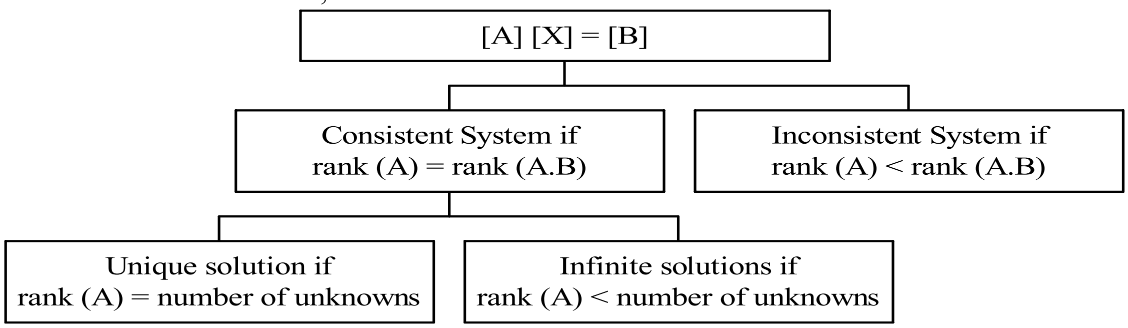
So rank of the augmented matrix [B] = 3.

The rank of the coefficient matrix [A] = 2 from the previous example.

Since rank of the coefficient matrix [A] < rank of the augmented matrix [B], the system of equations is inconsistent. Hence no solution exists for [A] [X] = [C].

If a solution exists, how do we know whether it is unique?

In a system of equations [A] [X] = [C] that is consistent, the rank of the coefficient matrix [A] is same as the augmented matrix [A|C]. If in addition, the rank of the coefficient matrix [A] is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix [A] is less than the number of unknowns, then infinite solutions exist.



Example 8

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent system of equations. Does the system of equations have a unique solution or does it have infinite solutions.

Solution

The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

While finding the whether the above equations were consistent in an earlier example, we found that rank of the coefficient matrix (A) = rank of augmented matrix $[A \vdots C] = 3$

The solution is unique as the number of unknowns = 3 = rank of (A).

Example 9

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is a consistent system of equations. Is the solution unique or does it have infinite solutions.

Solution

While finding the whether the above equations were consistent, we found that rank of coefficient matrix [A] = rank of augmented matrix $(A \vdots C) = 2$

Since rank of [A] = 2 < number of unknowns =3, infinite solutions exist.

If we have more equations than unknowns in [*A*] [*X*] = [*C*], does it mean the system is inconsistent?

No, it depends on the rank of the augmented matrix $[A \vdots C]$ and the rank of $[A]$.

a) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 284.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 3
rank of coefficient matrix = 3.

Now since rank of (A) = 3 = number of unknowns, the solution is not only consistent but also unique.

b) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 280.0 \end{bmatrix}$$

is inconsistent, since

rank of augmented matrix = 4
rank of coefficient matrix = 3.

c) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 50 & 10 & 2 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 213.6 \\ 280.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 2

rank of coefficient matrix = 2.

But since the rank of $[A]$ = 2 < the number of unknowns = 3, infinite solutions exist.

Consistent system of equations can only have a unique solution or infinite solutions. Can a system of equations have a finite (more than one but not infinite) number of solutions?

No, you can only have a unique solution or infinite solutions. Let us suppose $[A]\ [X]=[C]$ has two solutions $[Y]$ and $[Z]$ so that

$[A]\ [Y]=[C]$

$[A]\ [Z]=[C]$

If r is a constant, then from the two equations

$$r[A][\mathrm{Y}] = r[C]$$

$$(1-r)[A][\mathrm{Z}] = (1-r)[C]$$

Adding the above two equations gives

$$r[A][\mathrm{Y}] + (1-r)[A][\mathrm{Z}] = r[C] + (1-r)[C]$$

$$[A](r[Y] + (1-r)[Z]) = [C]$$

Hence

$$r[Y] + (1-r)[Z]$$

is a solution to

$$[A][\mathrm{X}] = [C].$$

Since r is any scalar, there are infinite solutions for $[A][X]=[C]$ of the form r[Y]+(1-r)[Z].

Can you divide two matrices?

If $[A]\ [B]=[C]$ is defined, it might seem intuitive that $[A]=\dfrac{[C]}{[B]}$, but matrix division is not defined. However an inverse of a matrix can be defined for certain types of square matrices. The inverse of a square matrix [A], if existing, is denoted by $[A]^{-1}$ such that $[A][A]^{-1} = [I] = [A]^{-1}[A]$.

In other words, let [A] be a square matrix. If [B] is another square matrix of same size such that [B][A] = [I], then [B] is the inverse of [A]. [A] is then called to be invertible or nonsingular. If $[A]^{-1}$ does not exist, [A] is called to be noninvertible or singular.

If [A] and [B] are two nxn matrices such that [B] [A] = [I], then these statements are also true

[B] is the inverse of [A]
[A] is the inverse of [B]
[A] and [B] are both invertible
[A] [B]=[I].
[A] and [B] are both nonsingular
all columns of [A] or [B]are linearly independent
all rows of [A] or [B] are linearly independent.

Example 10

Show if

$$[B] = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \textit{is the inverse of } [A] = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Solution

$$[B][A]$$
$$= \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= [I]$$

Since [B] [A] =[I], [B] is the inverse of [A] and [A] is the inverse of [B]. But we can also show that

$$[A][B]$$
$$= \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathrm{I}]$$

to show that [A] is the inverse of [B].

Can I use the concept of the inverse of a matrix to find the solution of a set of equations [A] [X] = [C]?

Yes, if the number of equations is same as the number of unknowns, the coefficient matrix $[A]$ is a square matrix.
Given

$$[A][X] = [C]$$

Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$.

$$[A]^{-1}[A][X] = [A]^{-1}[C]$$

$$[I][X] = [A]^{-1}[C]$$

$$[X] = [A]^{-1}[C]$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of $[A][X] = [C]$ is simply a multiplication of $[A]^{-1}$ and the right hand side vector, $[C]$.

How do I find the inverse of a matrix?

If $[A]$ is a $n \times n$ matrix, then $[A]^{-1}$ is a $n \times n$ matrix and according to the definition of inverse of a matrix

$[A][A]^{-1} = [I]$.

Denoting

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & a'_{nn} \end{bmatrix}$$

$$[I]=\begin{bmatrix}1 & 0 & \cdot & \cdot & \cdot & 0\\ 0 & 1 & & & & 0\\ 0 & & \cdot & & & \cdot\\ \cdot & & & 1 & & \cdot\\ \cdot & & & & \cdot & \cdot\\ 0 & \cdot & \cdot & \cdot & \cdot & 1\end{bmatrix}$$

Using the definition of matrix multiplication, the first column of the $[A]^{-1}$ matrix can then be found by solving

$$\begin{bmatrix}a_{11} & a_{12} & \cdot & \cdot & a_{1n}\\ a_{21} & a_{22} & \cdot & \cdot & a_{2n}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn}\end{bmatrix}\begin{bmatrix}a'_{11}\\ a'_{21}\\ \cdot\\ \cdot\\ a'_{n1}\end{bmatrix}=\begin{bmatrix}1\\ 0\\ \cdot\\ \cdot\\ 0\end{bmatrix}$$

Similarly, one can find the other columns of the $[A]^{-1}$ matrix by changing the right hand side accordingly.

Example 11

The upward velocity of the rocket is given by

Time, t	Velocity
s	m/s
5	106.8
8	177.2
12	279.2

In an earlier example, we wanted to approximate the velocity profile by $v(t)=at^2+bt+c,\quad 5\le 8\le 12$

We found that the coefficients a,b,c are given by

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 279.2\end{bmatrix}$$

First find the inverse of

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

and then use the definition of inverse to find the coefficients a, b, c.

Solution

If $[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$ is the inverse of [A],

Then

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives three sets of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the above three sets of equations separately gives

$$\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Now

$$[A][\mathrm{X}] = [C]$$

where

$$[X] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the definition of $[A]^{-1}$,

$$[A]^{-1}[\mathrm{A}][\mathrm{X}] = [A]^{-1}[\mathrm{C}]$$

$$[\mathrm{X}] = [A]^{-1}[\mathrm{C}]$$

$$= \begin{bmatrix} -0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix} \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

So

$$v(t) = 0.2900t^2 + 19.70t + 1.050, 5 \le t \le 12$$

Is there another way to find the inverse of a matrix?

For finding inverse of small matrices, inverse of an invertible matrix can be found by

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$

where

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & - & C_{nn} \end{bmatrix}^T$$

where C_{ij} are the cofactors of a_{ij}. The matrix $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{22} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & \cdots & \cdots & C_{nn} \end{bmatrix}$ itself is called the matrix of cofactors from [A]. Cofactors are defined in Chapter 4.

Example 12

Find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

From the example in Chapter 4, we found

$$\det(A) = -84 .$$

Next we need to find the adjoint of [A]. The cofactors of A are found as follows.

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 \\ 12 & 1 \end{vmatrix}$$

$$= -4$$

The cofactors of entry a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = -4$$

The minor of entry a_{12} is

$$M_{12} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix}$$

$$= -80$$

The cofactor of entry a_{12} is

$$C_{12} = (-1)^{1+2} M_{12}$$

$$= -M_{12}$$

$$= 80$$

Similarly

$$C_{13} = 384$$

$$C_{21} = 7$$

$$C_{22} = -119$$

$$C_{23} = 420$$

$$C_{31} = -3$$

$$C_{32} = 39$$

$$C_{33} = -120$$

Hence the matrix of cofactors of $[A]$ is

$$[C] = \begin{bmatrix} -4 & 80 & -384 \\ 7 & -119 & 420 \\ -3 & 39 & -120 \end{bmatrix}$$

The adjoint of matrix $[A]$ is $[C]^T$,

$$adj(A) = [C]^T = \begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

Hence

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{-84} \begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

If the inverse of a square matrix [A] exists, is it unique?

Yes, the inverse of a square matrix is unique, if it exists. The proof is as follows. Assume that the inverse of $[A]$ is $[B]$ and if this inverse is not unique, then let another inverse of $[A]$ exist called $[C]$.

$[B]$ is inverse of $[A]$, then

$$[B][A] = [I]$$

Multiply both sides by $[C]$,

$$[B][A][C] = [I][C]$$

$$[B][A][C] = [C]$$

Since $[C]$ is inverse of $[A]$, $[A][C] = [I]$

$$[B][I] = [C]$$

$$[B] = [C]$$

This shows that $[B]$ and $[C]$ are the same. So inverse of $[A]$ is unique.

Key Terms:

Consistent system	*Inconsistent system*
Infinite solutions	*Unique solution*
Rank	*Inverse*

SYSTEM OF EQUATIONS	
Topic	System of Equations
Summary	Textbook notes of System of Equations
Major	All Majors of Engineering
Authors	Autar Kaw
Last Revised	May 5, 2007
Web Site	http://numericalmethods.eng.usf.edu