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WHAT DO MATHEMATICIANS DO?

by George W. Mackey

There are a relatively small number of people in the world (perhaps a few thousand) who spend a large part of their time thinking about and trying to contribute to an esoteric subject called pure mathematics. The more active and successful number only in the hundreds and form a world community in which every one knows or knows of everyone else. The overwhelming majority make their living by teaching in universities, their investigations being subsidized by their being given less than full time teaching loads. For complicated historical and cultural reasons the great majority live in Europe, North America and Japan and are far from being uniformly distributed over these areas. Some European countries are almost completely unrepresented, and some, like France, are especially strong. Moreover, if the pure mathematicians of Paris, Moscow, greater Boston, Princeton and New York City were to be eliminated, the mathematical strength of the world would probably be reduced by at least two thirds.

If a non-mathematician listens to these people talk or attempts to read their journals, he confronts an incomprehensible jargon filled with words like differential equation, group, ring, manifold, homotopy,etc. If he asks for an explanation, he is overwhelmed by a concatenation of difficult to grasp abstract concepts held together by long chains of intricate argument. Whatever are these mathematicians doing? Why do they find it so interesting and what does it have to do with the rest of the world?

In the time at my disposal I can do little to answer these questions. Nevertheless, I am going to make an attempt. In a word, pure mathematicians are refining, developing, improving and (rather rarely) discovering the intellectual tools that have proved useful in analyzing and understanding the measurable aspects of the world in which we live. These measurable aspects are not so limited as they might seem. At the beginning there was just counting and later the measuring of distances, areas and volumes. However, the last three centuries or so have witnessed a steadily accelerating growth in the extent to which all natural phenomena can be understood in terms of relationships between measurable entities. In the 1920's, for example, the discovery of quantum mechanics went a very long way toward reducing chemistry to the solution of well-defined mathematical problems.

Indeed, only the extreme difficulty of many of these problems prevents the present day theoretical chemist from being able to predict the outcome of every laboratory experiment by making suitable calculations. More recently the molecular biologists have made startling progress in reducing the study of life back to the study of chemistry. The living cell is a miniature but extremely active and elaborate chemical factory and many, if not most, biologists today are confident that there is no mysterious "vital principle", but that life is just very complicated chemistry. With biology reduced to chemistry and chemistry to mathematics, the measurable aspects of the world become quite pervasive.

At this point I must make it emphatically clear that, in spite of what I have just said, pure mathematicians concern themselves very little with the external world — even in its measurable aspects. Their concern with the intellectual tools used in analyzing the external world is not so much in using these tools as in polishing them, improving them and very occasionally inventing brand new ones. Indeed it is their concern with the tools themselves, rather than with using the tools, that distinguishes them from applied mathematicians and the more mathematically minded scientists and engineers.

While it is natural to suppose that one cannot do anything very useful in tool making and tool improvement, without keeping a close eye on what the tool is to be used for, this supposition turns out to be largely wrong. Mathematics has sort of inevitable structure which unfolds as one studies it perceptively. It is as though it were already there and one had only to uncover it. Pure mathematicians are people who have a sensitivity to this structure and such a love for the beauties it presents that they will devote themselves voluntarily and with enthusiasm to uncovering more and more of it, whenever the opportunity presents itself.

Perhaps, because of the lack of arbitrariness in its structure, research in pure mathematics is a very cooperative activity in which everyone builds on the work of someone else and in turn has his own work built upon. On the other hand, mathematicians tend to work alone (and occasionally in pairs) and to be intensely individualistic. Thus, in a curious way, the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism. No one has enough of an overview to be at all effective in directing the development of mathematics. Indeed if anyone tried he would probably do more harm than good. Just as the social insects build marvelously designed intricate structures by apparently carrying materials around at random so have the mathematicians built a marvelously articulated body of abstract concepts by following their individual instincts with an eye to what their colleagues are doing. An interesting example occured during the first two decades of the twentieth century. While the physicists were struggling with contradictions and anomalies in the so-called "old quantum theory", two quite distinct branches of pure mathematics were being developed by two different sets of mathematicians with no thought for one another or for physics. Then the discoveries of Schroedinger and Heisenberg in 1924-25 provided the key to the mystery, and physics found its way to that subtle refinement of Newtonian mechanics known as quantum mechanics. Almost immediately it was found that these two separate new branches of pure mathematics were not only what quantum mechanics needed for its precise formulation and further development, but they could be regarded moreover as two facets of a bigger and better unified new branch which was even more adapted to the needs of quantum physics. Several decades later this unified new branch began to have important applications to some of the oldest problems in the theory of numbers.

The set of natural numbers $1, 2, 3, \ldots$ is perhaps the first mathematical tool discovered by man, but its study continues to provide pure mathematicians with an apparently inexhaustible supply of profound and challenging problems. Consider, for example, the problem of determining in how many different ways (if any) a given whole number can be written as a sum of two squared whole numbers. The answer to this question turns out to depend on the factorization of the number into primes. I remind you that a number is said to be a prime if it cannot be written as the product of two other positive numbers, neither of which is one. For example $2, 3, 5$ and 7 are primes while 4 and 15 are not since $4 = 2 \times 2$, and $15 = 3 \times 5$. One can find an answer for the problem expressed in terms of the answer when the given number is a prime. This much is fairly easy. Much more difficult to establish is the beautiful result that solutions exist for the prime 2 and for precisely those odd primes which leave a remainder of 1 when divided by 4. This theorem was announced without proof by Fermat in the middle of the seventeenth century. One hundred years later Euler, the great eighteenth century mathematician, worked for seven years before finding a proof. Nowadays quite simple proofs exist, but they use sophisticated new tools such as group theory and field theory. Similar but slightly more complicated problems remained unsolved until quite recently. Others are still beyond our reach but may become accessible when the new tool mentioned above and which arose in physics becomes further developed.

Such problems may seem trifling to the outsider, but a major lesson taught by the development of Science in the last three and a half centuries is that the way to progress lies in fine analysis — in looking very closely at the simplest aspects of things and then building from there. Galileo began modern mathematical physics by deciding that it would be worthwhile to time a falling body and discover just how much it accelerated as it fell.

Now let me return to my statement that the great majority of pure mathematicians make their livings by teaching in universities and have their work subsidized by reduced teaching loads. Nowadays many people criticize this arrangement on the grounds that it tempts faculty members to neglect their teaching. I think that this criticism is without serious foundation. In my oppinion a very high proportion enjoy the teaching they do and regard doing it well as a serious responsibility which is part of what they owe the University for supporting their research. It is, I think, a rather happy arrangement in that it makes it possible for at least some teaching to be done by genuine authorities in the field and at the same time supports an activity whose measurable economic benefits are so uncertain and so far into the future. On the other hand, there is a certain tension. One becomes extremely absorbed in one's research problems and longs for extra time in which to work on them. The summer vacation helps but is not enough. It is fortunate that other possibilites exist, such as sabbatical leaves and various institutes where one can go from time to time and concentrate exclusively on research. Actually there are all too few of the latter, and I would like to close by saying a few words about the two which I myself have visited — one of which is just a few miles outside of Paris in Bures sur Yvette.

The Institute for Advanced Study in Princeton, New Jersey is the older and became famous very quickly by having Einstein on its faculty. It was founded in 1933 and has played a very useful role in the mathematical world ever since. Its school of mathematics has an extremely distinguished permanent faculty of half a dozen or so and every year a group of 50 or 100 visitors. Most of the visitors are young — only a few years beyond the Ph.D. However, there is always of sprinkling of older mathematicians including a few distinguished foreigners. I have just come from a very pleasant and productive term there.

The institute at Bures sur Yvette (L'Institut des Hautes Etudes Scientifiques) is younger and has a smaller permanent faculty — but one which is probably no less distinguished. I spent an agreeable and profitable term there seven years ago. Like its older counterpart in Princeton, it plays a very important role in the mathematical world — not only by helping mathematicians find more time for their work, but by bringing those with similar interests together so they may exchange ideas.

On this visit to Paris I am not at Bures but am rather teaching a course at the University (Paris VI). However, my Harvard colleague, Professor Barry Mazur, is there and in fact is a frequent visitor. He is in the audience today and has agreed to try to answer questions any of you may have either about the nature of mathematical research or about the IHES.