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Hodge Number Polynomials for Nearby and Vanishing Cohomology

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Introduction

The behaviour of the cohomology of a degenerating family of complex projective manifolds has been intensively studied in the nineteen-seventies by Clemens, Griffiths, Schmid and others. See [Gr] for a nice overview. Recently, the theory of motivic integration, initiated by Kontsevich and developed by Denef and Loeser, has given a new impetus to this topic. In particular, in the case of a one-parameter degeneration it has produced an object ψ_f in the Grothendieck group of complex algebraic varieties, called the *motivic nearby fibre* [B05], which reflects the limit mixed Hodge structure of the family in a certain sense. The purpose of this paper is twofold. First, we prove that the motivic nearby fibre is well-defined without using the theory of motivic integration. Instead we use the Weak Factorization Theorem [AKMW]. Second, we give a survey of formulas containing numerical invariants of the limit mixed Hodge structure, and in particular of the vanishing cohomology of an isolated hypersurface singularity, without using the theory of mixed Hodge structures or of variations of Hodge structure.

We hope that in this way this interesting topic becomes accessible to a wider audience.

19.1 Real Hodge structures

A real Hodge structure on a finite dimensional real vector space V consists of a direct sum decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}, \text{with } V^{p,q} = \overline{V^{q,p}}$$

on its complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$. The corresponding *Hodge filtration* is given by

$$F^p(V) = \bigoplus_{r \geq p} V^{r,s}.$$

The numbers

$$h^{p,q}(V) := \dim V^{p,q}$$

are the Hodge numbers of the Hodge structure. If for some integer k we have $h^{p,q} = 0$ for all (p,q) with $p+q \neq k$ the Hodge structure is pure of weight k. Any real Hodge structure is the direct sum of pure Hodge structures. The polynomial

$$\begin{aligned} P_{\mathrm{hn}}(V) &= \sum_{p,q \in \mathbb{Z}} h^{p,q}(V) u^p v^q \\ &= \sum h^{p,k-p}(V) u^p v^{k-p} \in \mathbb{Z}[u,v,u^{-1},v^{-1}] \end{aligned} \tag{19.1}$$

is its associated *Hodge number polynomial.* † A classical example of a weight k Hodge real structure is furnished by the rank k (singular) cohomology group $H^k(X)$ (with $\mathbb{R}$-coefficients) of a compact Kähler manifold X.

Various multilinear algebra operations can be applied to Hodge structures as we now explain. Suppose that V and W are two real vector spaces with a Hodge structure of weight k and ℓ respectively. Then:

(i) $V \otimes W$ has a Hodge structure of weight $k + \ell$ given by

$$F^p(V \otimes W)_{\mathbb{C}} = \sum_m F^m(V_{\mathbb{C}}) \otimes F^{p-m}(W_{\mathbb{C}}) \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$$

and with Hodge number polynomial given by

$$P_{\mathrm{hn}}(V \otimes W) = P_{\mathrm{hn}}(V) P_{\mathrm{hn}}(W). \tag{19.2}$$

(ii) On $\mathrm{Hom}(V,W)$ we have a Hodge structure of weight $\ell - k$:

$$F^p \,\mathrm{Hom}(V,W)_{\mathbb{C}} = \{f : V_{\mathbb{C}} \to W_{\mathbb{C}} \mid f F^n(V_{\mathbb{C}}) \subset F^{n+p}(W_{\mathbb{C}}) \quad \forall n\}$$

with Hodge number polynomial

$$P_{\mathrm{hn}}(\mathrm{Hom}(V,W)(u,v) = P_{\mathrm{hn}}(V)(u^{-1},v^{-1}) P_{\mathrm{hn}}(W)(u,v). \tag{19.3}$$

In particular, taking $W = \mathbb{R}$ with $W_{\mathbb{C}} = W^{0,0}$ we get a Hodge structure of weight $-k$ on the dual $V^\vee$ of V with Hodge number polynomial

$$P_{\mathrm{hn}}(V^\vee)(u,v) = P_{\mathrm{hn}}(V)(u^{-1},v^{-1}). \tag{19.4}$$

† There are other conventions in the litterature, for instance, some authors put a sign $(-1)^{p+q}$ in front of the coefficient $h^{pq,}(V)$ of $u^p v^q$.

The category $\mathfrak{hs}$ of real Hodge structures leads to a ring, the *Grothendieck ring* $K_0(\mathfrak{hs})$ which is is the free group on the isomorphism classes $[V]$ of real Hodge structures V modulo the subgroup generated by $[V]-[V']-[V'']$ where

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence of pure Hodge structures and where the complexified maps preserve the Hodge decompositions. Because the Hodge number polynomial(19.1) is clearly additive and by (19.2) behaves well on products the Hodge number polynomial defines a ring homomorphism

$$P_{\mathrm{hn}} : \mathrm{K}_0(\mathfrak{hs}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

As remarked before, pure Hodge structures of weight k in algebraic geometry arise as the (real) cohomology groups $H^k(X)$ of smooth complex projective varieties. We combine these as follows:

$$\chi_{\mathrm{Hdg}}(X) \ := \ \sum(-1)^k[H^k(X)] \in \mathrm{K}_0(\mathfrak{hs}); \tag{19.5}$$

$$e_{\mathrm{Hdg}}(X) \ := \ \sum(-1)^k P_{\mathrm{hn}}(H^k(X)) \in \mathbb{Z}[u, v, u^{-1}, v^{-1}] \tag{19.6}$$

which we call the Hodge-Grothendieck character and the Hodge-Euler polynomial of X respectively.

Let us now recall the definition of the naive Grothendieck group $\mathrm{K}_0(\mathrm{Var})$ of (complex) algebraic varieties. It is the quotient of the free abelian group on isomorphism classes $[X]$ of algebraic varieties over $\mathbb{C}$ with the so-called *scissor relations* $[X] = [X-Y] + [Y]$ for $Y \subset X$ a closed subvariety. The cartesian product is compatible with the scissor relations and induces a product structure on $\mathrm{K}_0(\mathrm{Var})$, making it into a ring. There is a nice set of generators and relations for $\mathrm{K}_0(\mathrm{Var})$. To explain this we first recall:

Lemma 19.1.1. *Suppose that X is a smooth projective variety and $Y \subset X$ is a smooth closed subvariety. Let $\pi : Z \to X$ be the blowing-up with centre Y and let $E = \pi^{-1}(Y)$ be the exceptional divisor. Then*

$$\begin{aligned} \chi_{\mathrm{Hdg}}(X) - \chi_{\mathrm{Hdg}}(Y) &= \chi_{\mathrm{Hdg}}(Z) - \chi_{\mathrm{Hdg}}(E); \\ e_{\mathrm{Hdg}}(X) - e_{\mathrm{Hdg}}(Y) &= e_{\mathrm{Hdg}}(Z) - e_{\mathrm{Hdg}}(E). \end{aligned}$$

Proof By [GH, p. 605]

$$0 \to H^k(X) \to H^k(Z) \oplus H^k(Y) \to H^k(E) \to 0$$

is exact. □

Theorem 19.1.2 ([B04, Theorem 3.1]). *The group* $\mathrm{K}_0(\mathrm{Var})$ *is isomorphic to the free abelian group generated by the isomorphism classes of smooth complex projective varieties subject to the relations* $[\varnothing] = 0$ *and* $[Z] - [E] = [X] - [Y]$ *where* X, Y, Z, E *are as in Lemma 19.1.1.*

It follows that for every complex algebraic variety X there exist projective smooth varieties $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ such that

$$[X] = \sum_i [X_i] - \sum_j [Y_j] \text{ in } \mathrm{K}_0(\mathrm{Var})$$

and so, using Lemma 19.1.1 we have:

Corollary 19.1.3. *The Hodge Euler character extends to a ring homomorphism*

$$\chi_{\mathrm{Hdg}} : K_0(\mathrm{Var}) \to K_0(\mathfrak{hs})$$

and the Hodge number polynomial extends to a ring homomorphism

$$e_{\mathrm{Hdg}} : \mathrm{K}_0(\mathrm{Var}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$$

Remark 19.1.4. By Deligne's theory [Del71], [Del74] there is a mixed Hodge structure on the real vector spaces $H^k(X)$. For our purposes, since we are working with real coefficients, a mixed Hodge structure is just a real Hodge structure, i.e. a direct sum of real Hodge structures of various weights, and so the Hodge character and Hodge number polynomial are defined for any real mixed Hodge structure. However, ordinary cohomology does not behave well with respect to the scissor relation; we need compactly supported cohomology $H^k_c(X; \mathbb{R})$. But these also carry a Hodge structure and we have the following explicit expression for the above characters.

$$\chi_{\mathrm{Hdg}}(X) = \sum (-1)^k [H^k_c(X))];$$
$$e_{\mathrm{Hdg}}(X) = \sum (-1)^k P_{\mathrm{hn}}(H^k_c(X)).$$

Example 19.1.5.
1) Let U be a smooth, but not necessarily compact complex algebraic manifold. Such a manifold has a *good compactification* X, i.e. X is a compact complex algebraic manifold and $D = X - U$ is a normal crossing divisor, say $D = D_1 \cup \cdots D_N$ with D_j smooth and irreducible. We introduce

$$\begin{aligned} D_I &= D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_m}, \quad I = \{i_1, \ldots, i_m\}; \\ a_I &: D_I \hookrightarrow X \end{aligned}$$

and we set

$$\begin{aligned} D(0) &= X; \\ D(m) &= \textstyle\coprod_{|I|=m} D_I, \quad m = 1, \ldots, N; \\ a_m &= \textstyle\coprod_{|I|=m} a_I : D(m) \to X. \end{aligned}$$

Then each connected component of $D(m)$ is a complex submanifold of X of codimension m. Note that

$$[U] = \sum_m (-1)^m [D(m)] \in \mathrm{K}_0(\mathrm{Var}).$$

Hence

$$\begin{aligned} \chi_{\mathrm{Hdg}}(U) &= \sum_m (-1)^m \chi_{\mathrm{Hdg}}(D(m)); \\ e_{\mathrm{Hdg}}(U) &= \sum_m (-1)^m e_{\mathrm{Hdg}}(D(m)). \end{aligned}$$

2) If X is compact the construction of cubical hyperresolutions $(X_I)_{\varnothing \neq I \subset A}$ of X from [GNPP] leads to the expression

$$[X] = \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} [X_I].$$

and we find:

$$\begin{aligned} \chi_{\mathrm{Hdg}}(X) &= \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} \chi_{\mathrm{Hdg}}(X_I); \\ e_{\mathrm{Hdg}}(X) &= \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} e_{\mathrm{Hdg}}(X_I). \end{aligned}$$

The scissor-relations imply that the inclusion-exclusion principle can be applied to a disjoint union X of locally closed subvarieties $X_1, \ldots, X_m$:

$$\chi_{\mathrm{Hdg}}(X) = \sum_{i=1}^m \chi_{\mathrm{Hdg}}(X_i)$$

and a similar expression holds for the Hodge Euler polynomials.

Example 19.1.6. Let $T^n = (\mathbb{C}^*)^n$ be an n-dimensional algebraic torus. Then $e_{\mathrm{Hdg}}(T^1) = uv - 1$ so $e_{\mathrm{Hdg}}(T^n) = (uv-1)^n$. Consider an n-dimensional toric variety X. It is a disjoint union of T^n-orbits. Suppose that X has s_k orbits of dimension k. Then

$$e_{\mathrm{Hdg}}(X) = \sum_{k=0}^n s_k e_{\mathrm{Hdg}}(T^k) = \sum_{k=0}^n s_k (uv-1)^k.$$

If X has a pure Hodge structure (e.g. if X is compact and has only quotient singularities) then this formula determines the Hodge numbers of X.

19.2 Nearby and vanishing cohomology

In this section we consider a relative stituation. We let X be a complex manifold, $\Delta \subset \mathbb{C}$ the unit disk and $f : X \to \Delta$ a holomorphic map which is smooth over the punctured disk Δ^*. We say that f is a *one-parameter degeneration*. Let us assume that $E = \bigcup_{i \in I} E_i = f^{-1}(0)$ is a divisor with strict normal crossings on X. We have the *specialization diagram*

$$\begin{array}{ccccc} X_\infty & \xrightarrow{k} & X & \xleftarrow{i} & E \\ \big\downarrow{\scriptstyle \tilde f} & & \big\downarrow{\scriptstyle f} & & \big\downarrow \\ \mathfrak{h} & \xrightarrow{\mathrm{e}} & \Delta & \leftarrow & \{0\} \end{array}$$

where $\mathfrak{h}$ is the complex upper half plane, $\mathrm{e}(z) := \exp(2\pi \mathrm{i} z)$ and where

$$X_\infty := X \times_{\Delta^*} \mathfrak{h}.$$

We let e_i denote the multiplicity of f along E_i and choose a positive integer multiple e of all e_i. We let $\tilde f : \tilde X \to \Delta$ denote the normalization of the pull-back of X under the map $\mu_e : \Delta \to \Delta$ given by $\tau \mapsto \tau^e = t$. It fits into a commutative diagram describing the *e-th root* of f:

$$\begin{array}{ccc} \tilde X & \xrightarrow{\rho} & X \\ {\scriptstyle \tilde f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \Delta & \xrightarrow{\mu_e} & \Delta. \end{array}$$

We put

$$D_i = \rho^{-1}(E_i)\,,\; D_J = \bigcap_{i \in J} D_j,\; D(m) = \coprod_{|J|=m} D_J.$$

Then $D_J \to E_J$ is a cyclic cover of degree $\gcd(e_j \mid j \in J)$. The maps $D_J \to E_J$ do not depend on the choice of the integer e and so in particular this is true for the varieties D_J . See e.g. [Ste77] or [B05] for a detailed study of the geometry of this situation. The special fibre $\tilde X_0 = \tilde f^{-1}(0)$ is now a complex variety equipped with the action of the cyclic group of order e. Let us introduce the associated Grothendieck-group:

Definition 19.2.1. We let $\mathrm{K}_0^{\hat\mu}(\mathrm{Var})$ denote the Grothendieck group of complex algebraic varieties with an action of a finite order automorphism modulo the subgroup generated by expressions $[\mathbb{P}(V)] - [\mathbb{P}^n \times X]$ where V is a vector bundle of rank $n+1$ over X with action which is linear over the action on X. See [B05, Sect. 2.2] for details.

As a motivation, we should remark that in the ordinary Grothendieck group the relation $[\mathbb{P}(V)] = [\mathbb{P}^n \times X]$ follows from the fact that any algebraic vector bundle is trivial over a Zariski-open subset and the fact that the scissor-relations hold. The above relation extend to the case where one has a group action.

Definition 19.2.2. Suppose that the fibres of f are projective varieties. Following [B05, Ch. 2] we define the *motivic nearby fibre* of f by

$$\psi_f := \sum_{m \geq 1} (-1)^{m-1} [D(m) \times \mathbb{P}^{m-1}] \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var})$$

and the *motivic vanishing fibre* by

$$\phi_f := \psi_f - [E] \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}) \tag{19.7}$$

Remark 19.2.3. If we let E_I^0 be the open subset of E_I consisting of points which are exactly on the E_j with $j \in I$, D_I^0 the corresponding subset of D_I and $D^0(i) = \coprod_{|I|=i} D_I^0$. We have $D_I = \coprod_{J \supset I} D_J^0$ and the scissor relations imply that

$$\begin{aligned}
\sum_{i \geq 1} (-1)^{i-1} [D(i) \times \mathbb{P}^{i-1}] &= \sum_{i \geq 1} (-1)^{i-1} \sum_{j \geq i} \binom{j}{i} [D^0(j) \times \mathbb{P}^{i-1}] \\
&= \sum_{j \geq 1} [D^0(j)] \times \sum_{i=1}^{j} (-1)^{i-1} \binom{j}{i} [\mathbb{P}^{i-1}] \\
&= \sum_{j \geq 1} (-1)^{j-1} [D^0(j) \times (\mathbb{C}^*)^{j-1}].
\end{aligned}$$

This expression for the motivic nearby fibre has been used in [L].

The motivic nearby fibre turns out to be a relative bimeromorphic invariant:

Lemma 19.2.4. *Suppose that $g : Y \to X$ is a bimeromorphic proper map which is an isomorphism over $X - E$. Assume that $(f \circ g)^{-1}(0)$ is a divisor with strict normal crossings. Then*

$$\psi_f = \psi_{fg}.$$

Proof In [B05], the proof relies on the theory of motivic integration [DL]. We give a different proof, based on the *weak factorization theorem* [AKMW].

This theorem reduces the problem to the following situation: g is the blowing-up of X in a connected submanifold $Z \subset E$, with the following property. Let $A \subset I$ be those indices i for which $Z \subset E_i$. Then Z intersects the divisor $\bigcup_{i \notin A} E_i$ transversely, hence $Z \cap \bigcup_{i \notin A} E_i$ is a divisor with normal crossings in Z.

We fix the following notation. We let $L := [\mathbb{A}^1]$ and $P_m := [\mathbb{P}^m] = \sum_{i=0}^m L^i$. For $J \subset I$ we let $j = |J|$. So, using the product structure in $K_0(\mathrm{Var})$, we have

$$\psi_f = \sum_{\varnothing \neq J \subset I} (-1)^{j-1} [D_J] P_{j-1}.$$

Let $E' = \bigcup_{i \in I'} E'_i$ be the zero fibre of fg. We have $I = I' \cup \{*\}$ where E'_* is the exceptional divisor of g and E'_i is the proper transform in Y of E_i. Form $\rho' : D' \to E'$, the associated ramified cyclic covering. For $J \subset I$ we let $J' = J \cup \{*\}$. Note that E'_* has multiplicity equal to $\sum_{i \in A} e_i$. Without loss of generality we may assume that e is also a multiple of this integer. We have two kinds of j'-uple intersections $D_{J'}$: those which only contain D'_j, $j \neq *$ and those which contain D'_*. So,

$$\psi_{fg} = [D'_*] + \sum_{\varnothing \neq J \subset I} (-1)^{j-1} \left([D'_J] P_{j-1} - [D'_{J'}] P_j \right).$$

We are going to calculate the difference between ψ_{fg} and ψ_f. Let $B \subset I - A$. Let $c = \mathrm{codim}(Z, X)$. Then for all $K \subset A$ we have that $\mathrm{codim}(Z \cap E_{K \cup B}, E_{K \cup B}) = c - k$, and $D'_{K \cup B}$ is the blowing up of $D_{K \cup B}$ with centre $Z \cap E_{K \cup B} = Z \cap E_B =: Z_B$ and exceptional divisor $D'_{K \cup B'}$. Hence if we let $W_B = {\rho'}^{-1}(Z_B)$ we have

$$[D'_{K \cup B}] = [D_{K \cup B}] + [D'_{K \cup B'}] - [W_B] = [D_{K \cup B}] + [W_B](P_{c-k-1} - 1).$$

Hence

$$\psi_{fg} - \psi_f = \sum_B c_B [W_B]$$

for suitable coefficients c_B.

Note that $Z_\varnothing = Z$ and that $D'_* = W_\varnothing \times P_{c-1}$ and so if $B = \varnothing$ we get

$$\begin{aligned} c_\varnothing &= P_{c-1} + \sum_{\varnothing \neq K \subset A} (-1)^{k-1} \left((P_{c-k-1} - 1) P_{k-1} - P_{c-k-1} P_k \right) \\ &= P_{c-1} + \sum_{\varnothing \neq K \subset A} (-1)^k (L^k P_{c-k-1} + P_{k-1}) = P_{c-1} \sum_{K \subset A} (-1)^k = 0. \end{aligned}$$

In case $B \neq \varnothing$ we get

$$\begin{aligned} c_B &= \sum_{K \subset A} (-1)^{k+b-1} \left((P_{c-k-1} - 1)P_{k+b-1} - P_{c-k-1}P_{k+b}\right) \\ &= \sum_{K \subset A} (-1)^{k+b} (L^{k+b} P_{c-k-1} + P_{k+b-1}) = P_{c+b-1} \sum_{K \subset A} (-1)^{b+k} = 0. \quad \square \end{aligned}$$

Let us now pass to the nearby fibre in the Hodge theoretic sense. In [Schm] and [Ste76] a mixed Hodge structure on $H^k(X_\infty)$ was constructed; its weight filtration is the *monodromy weight filtration* which we now explain. The loop winding once counterclockwise around the origin gives a generator of $\pi(\Delta^*, *)$,$* \in \Delta^*$. Its action on the fibre X_* over $*$ is well defined up to homotopy and on $H^k(X_*) \simeq H^k(X_\infty)$ it defines the *monodromy automorphism* T. Let $N = \log T_u$ be the logarithm of the unipotent part in the Jordan decomposition of T. Then W is the unique increasing filtration on $H^k(X_\infty)$ such that $N(W_j) \subset W_{j-2}$ and $N^j : \mathrm{Gr}^W_{k+j} \to \mathrm{Gr}^W_{k-j}$ is an isomorphism for all $j \geq 0$. In fact, from CHECK [Schm, Lemma 6.4] we deduce:

Lemma 19.2.5. *There is a Lefschetz-type decomposition*

$$\mathrm{Gr}^W H^k(X_\infty) = \bigoplus_{\ell=0}^{k} \bigoplus_{r=0}^{\ell} N^r P_{k+\ell},$$

where $P_{k+\ell}$ is pure of weight $k+\ell$. The endomorphism N has $\dim P_{k+m-1}$ *Jordan blocs of size m.*

The Hodge filtration is constructed in [Schm] as a limit of the Hodge filtrations on nearby smooth fibres in a certain sense, and in [Ste76] using the relative logarithmic de Rham complex. With X_t a smooth fibre of f it follows that

$$\dim F^m H^k(X_t) = \dim F^m H^k(X_\infty)$$

and hence

$$e_{\mathrm{Hdg}}(X_\infty)_{|v=1} = e_{\mathrm{Hdg}}(X_t)_{|v=1}. \tag{19.8}$$

We next remark that the nearby cycle sheaf $i^* Rk_* k^* \mathbb{R}_X$ can be used to put a Hodge structure on the cohomology groups $H^k(X_\infty)$, while the hypercohomology of the vanishing cycle sheaf $\phi_f = \mathrm{Cone}[i^*\mathbb{R}_X \to \psi_f \mathbb{R}_X]$ (it is a *complex* of sheaves of real vector spaces on E) likewise admits a Hodge structure. In fact, the spectral sequence of [Ste76, Cor. 4.20] shows that

$$\chi_{\mathrm{Hdg}}(X_\infty) = \chi_{\mathrm{Hdg}}(\psi_f)$$

and then the definition shows that

$$\chi_{\mathrm{Hdg}}(X_\infty) - \chi_{\mathrm{Hdg}}(E) = \chi_{\mathrm{Hdg}}(\phi_f).$$

In fact this formula motivates the nomenclature "motivic nearby fibre" and "motivic vanishing cycle".

As a concluding remark, the semisimple part T_s of the monodromy is an automorphism of the mixed Hodge structure on $H^k(X_\infty)$.

We can use these remarks to deduce information about the Hodge numbers on $H^k(X_\infty)$ from information about the geometry of the central fibre as we shall illustrate now.

Example 19.2.6. 1) Let $F, L_1, \ldots, L_d \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous forms with $\deg F = d$ and $\deg L_i = 1$ for $i = 1, \ldots, d$, such that $F \cdot L_1 \cdots L_d = 0$ defines a reduced divisor with normal crossings on $\mathbb{P}^2(\mathbb{C})$. We consider the space

$$X = \{([x_0, x_1, x_2], t) \in \mathbb{P}^2 \times \Delta \mid \prod_{i=1}^{d} L_i(x_0, x_1, x_2) + tF(x_0, x_1, x_2) = 0\}$$

where Δ is a small disk around $0 \in \mathbb{C}$. Then X is smooth and the map $f : X \to \Delta$ given by the projection to the second factor has as its zero fibre the union $E_1 \cup \cdots \cup E_d$ of the lines E_i with equation $L_i = 0$. These lines are in general position and have multiplicity one. We obtain

$$\psi_f = (d - \binom{d}{2})[\mathbb{P}^1]$$

so

$$e_{\mathrm{Hdg}}(\psi_f) = (1 - \binom{d-1}{2})(1 + uv)$$

and substituting $v = 1$ in this formula we get the formula $g = \binom{d-1}{2}$ for the genus of a smooth plane curve of degree d. The monodromy on $H^1(X_\infty)$ has g Jordan blocks of size 2, so is "maximally unipotent".

2) If we consider a similar example, but replace $\mathbb{P}^2$ by $\mathbb{P}^3$ and curves by surfaces, lines by planes, then the space X will not be smooth but has ordinary double points at the points of the zero fibre where two of the planes meet the surface $F = 0$. There are $d\binom{d}{2}$ of such points, d on each line of intersection. If we blow these up, we obtain a family $f : \tilde{X} \to \Delta$ whose zero fibre $D = E \cup F$ is the union of components $E_i,\ i = 1, \ldots, d$ which are copies of $\mathbb{P}^2$ blown up in $d(d-1)$ points, and components

$F_j, j = 1, \dots, d\binom{d}{2}$ which are copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus

$$e_{\mathrm{Hdg}}(D(1)) = d(1 + (d^2 - d + 1)uv + u^2v^2) + d\binom{d}{2}(1 + uv)^2.$$

The double point locus $D(2)$ consists of the $\binom{d}{2}$ lines of intersections of the E_i together with the $d^2(d-1)$ exceptional lines in the E_i. So

$$e_{\mathrm{Hdg}}(D(2)) = d(d-1)(d + \frac{1}{2})(1 + uv).$$

Finally $D(3)$ consists of the $\binom{d}{3}$ intersection points of the E_i together with one point on each component F_j, so

$$e_{\mathrm{Hdg}}(D(3)) = \binom{d}{3} + d\binom{d}{2} = \frac{1}{3}d(d-1)(2d-1)$$

We get

$$e_{\mathrm{Hdg}}(\psi_f) = \left(\binom{d-1}{3} + 1\right)(1 + u^2v^2) + \frac{1}{3}d(2d^2 - 6d + 7)uv$$

in accordance with the Hodge numbers for a smooth degree d surface:

$$h^{2,0} = h^{0,2} = \binom{d-1}{3}, \quad h^{1,1} = \frac{1}{3}d(2d^2 - 6d + 7)$$

The monodromy on $H^2(X_\infty)$ has $\binom{d-1}{3}$ Jordan blocs of size 3 and $\frac{1}{2}d^3 - d^2 + \frac{1}{2}d + 1$ blocks of size 1.

3) Consider a similar smoothing of the union of two transverse quadrics in $\mathbb{P}^3$. The generic fibre is a smooth K3-surface and after blowing up the 16 double points of the total space we obtain the following special fibre:

- (i) $E(1)$ has two components which are blowings up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 16 points, and 16 components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; hence $e_{\mathrm{Hdg}}(E(1)) = 18(1 + uv)^2 + 32uv$.
- (ii) $E(2)$ consists of the 32 exceptional lines together with the strict transform of the intersection of the two quadrics, which is an elliptic curve; hence $e_{\mathrm{Hdg}}(E(2)) = 33(1 + uv) - u - v$;
- (iii) $E(3)$ consists of 16 points: one point on each exceptional $\mathbb{P}^1 \times \mathbb{P}^1$, so $e_{\mathrm{Hdg}}(E(3)) = 16$.

We get

$$e_{\mathrm{Hdg}}(X_\infty) = 1 + u + v + 18uv + u^2v + uv^2 + u^2v^2$$

Putting $v = 1$ we get $2 + 20u + 2u^2$, in agreement with the Hodge numbers $(1, 20, 1)$ on the H^2 of a K3-surface. The monodromy has two Jordan blocs of size 2 and 18 blocs of size 1.

19.3 Equivariant Hodge number polynomials

We have seen that the mixed Hodge structure on the cohomology of the nearby fibre of a one-parameter degeneration comes with an automorphism of finite order. This leads us to consider the category $\mathfrak{hs}_{\mathbb{R}}^{\hat{\mu}}$ of pairs (H, γ) consisting of a real Hodge structure (i.e. direct sum of pure real Hodge structures of possibly different weights) H and an automorphism γ of finite order of this Hodge structure.
We are going to consider a kind of tensor product of two such objects, which we call *convolution* (see [SchS], where this operation was defined for mixed Hodge structures and called *join*). We will explain this by settling an equivalence of categories between $\mathfrak{hs}_{\mathbb{R}}^{\hat{\mu}}$ and a category $\mathfrak{fhs}$ of so-called *fractional Hodge structures.* (called *Hodge structures with fractional weights* in [L]; it is however not the weights which are fractional, but the indices of the Hodge filtration!

Definition 19.3.1. (See [L]). A *fractional Hodge structure* of weight k is a real vector space H of finite dimension, equipped with a decomposition

$$H_{\mathbb{C}} = \bigoplus_{a+b=k} H^{a,b}$$

where $a, b \in \mathbb{Q}$, such that $H^{b,a} = \overline{H^{a,b}}$. A *fractional Hodge structure* is defined as a direct sum of pure fractional Hodge structures of possibly different weights.

Lemma 19.3.2. *We have an equivalence of categories $G : \mathfrak{hs}^{\hat{\mu}} \to \mathfrak{fhs}$.*

Proof Let (H, γ) be an object of $\mathfrak{hs}^{\hat{\mu}}$ pure of weight k. We define $H_a = \mathrm{Ker}(\gamma - \exp(2\pi \mathrm{i} a); H_{\mathbb{C}})$ for $0 \le a < 1$ and for $0 < a < 1$ put

$$\tilde{H}^{p+a,k+1-a-p} = H_a^{p,k-p}, \quad \tilde{H}^{p,k-p} = H_0^{p,k-p}$$

This transforms (H, γ) into a direct sum $\tilde{H} =: G(H, \gamma)$ of fractional Hodge structures of weights $k + 1$ and k respectively. Conversely, for a fractional Hodge structure $\tilde{H}$ of weight k one has a unique automorphism γ of finite order which is multiplication by $\exp(2\pi \mathrm{i} b)$ on $\tilde{H}^{b,k-b}$.

Note that this equivalence of categories does not preserve tensor products! Hence it makes sense to make

Definition 19.3.3. The *convolution* $(H', \gamma') * (H", \gamma")$ of two objects in $\mathfrak{hs}^{\hat{\mu}}$ is the object corresponding to the tensor product of their images in $\mathfrak{fhs}$:

$$G\left((H', \gamma') * (H'', \gamma'')\right) = G(H', \gamma') \otimes G(H'', \gamma'').$$

Note that the Hodge number polynomial map P_{hn} extends to a ring homomorphism

$$P_{\mathrm{hn}}^{\hat{\mu}} : K_0(\mathfrak{fhs}) \to R := \varprojlim \mathbb{Z}[u^{\frac{1}{n}}, v^{\frac{1}{n}}, u^{-1}, v^{-1}]$$

We denote its composition with the functor G by the same symbol. Hence

$$P_{\mathrm{hn}}^{\hat{\mu}} : K_0^{\hat{\mu}}(\mathfrak{hs}) \to R$$

transforms convolutions into products. We equally have an equivariant Hodge-Grothendieck character

$$\chi_{\mathrm{Hdg}}^{\hat{\mu}} : K_0^{\hat{\mu}}(\mathrm{Var}) \to K_0^{\hat{\mu}}(\mathfrak{hs})$$

and an equivariant Hodge-Euler characteristic

$$e_{\mathrm{Hdg}}^{\hat{\mu}} : K_0^{\hat{\mu}}(\mathrm{Var}) \to R.$$

Let $f : X \to \mathbb{C}$ be a projective morphism where X is smooth, of relative dimension n, with a single isolated critical point x such that $f(x) = 0$. Construct ψ_f by replacing the zero fibre X_0 by a divisor with normal crossings as above. Then the Milnor fibre of F of f at x has the homotopy type of a wedge of spheres of dimension n. Its cohomology is also equipped with a mixed Hodge structure. Recalling definition 19.7 of the motivic vanishing fibre, it can be shown that

$$P_{\mathrm{hn}}^{\hat{\mu}}(\tilde{H}^n(F)) = (-1)^n e_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_f).$$

Write $P_{\mathrm{hn}}^{\hat{\mu}}(\tilde{H}^n(F)) = \sum_{\alpha \in \mathbb{Q},\ w \in \mathbb{Z}} m(\alpha, w) u^{\alpha} v^{w-\alpha}$. In the literature several numerical invariants have been attached to the singularity $f : (X, x) \to (\mathbb{C}, 0)$. These are all related to the numbers $m(\alpha, w)$ as follows:

i) The *characteristic pairs* [Ste77, Sect. 5].

$$\mathrm{Chp}(f, x) = \sum_{\alpha, w} m(\alpha, w) \cdot (n - \alpha, w).$$

ii) The *spectral pairs* [N-S]:

$$\mathrm{Spp}(f, x) = \sum_{\alpha \notin \mathbb{Z}, w} m(\alpha, w) \cdot (\alpha, w) + \sum_{\alpha \in \mathbb{Z}, w} m(\alpha, w) \cdot (\alpha, w + 1).$$

iii) The singularity spectrum in Saito's sense [Sa]:

$$\mathrm{Sp}_{\mathrm{Sa}}(f,x) = P_{\mathrm{hn}}^{\hat{\mu}}(\tilde{H}^n(F))(t,1).$$

iv) The singularity spectrum in Varchenko's sense [Var]:

$$\mathrm{Sp}_{\mathrm{V}}(f,x) = t^{-1}P_{\mathrm{hn}}^{\hat{\mu}}(\tilde{H}^n(F))(t,1).$$

Note that the object ϕ_f depends only on the germ of f at the critical point x. Let us as a final remark rephrase the original Thom-Sebastiani theorem (i.e. for the case of isolated singularities):

Theorem 19.3.4. *Consider holomorphic germs* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *and* $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ *with isolated singularity. Then the germ* $f \oplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, (0,0)) \to \mathbb{C}$ *with* $(f \oplus g)(x,y) := f(x) + g(y)$ *has also an isolated singularity, and*

$$\chi_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_{f\oplus g}) = -\chi_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_f) * \chi_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_g)$$

so

$$e_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_{f\oplus g}) = -e_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_f) \cdot e_{\mathrm{Hdg}}^{\hat{\mu}}(\phi_g) \in R.$$

Remark 19.3.5. This theorem has been largely generalized, for functions with arbitrary singularities, and even on the level of motives. Denef and Loeser [DL] defined a convolution product for Chow motives, and Looijenga [L] defined one on $\mathcal{M}^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var})[L^{-1}]$, both with the property that the Hodge Euler polynomial commutes with convolution and that the Thom-Sebastiani property holds already on the level of varieties/motives.

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