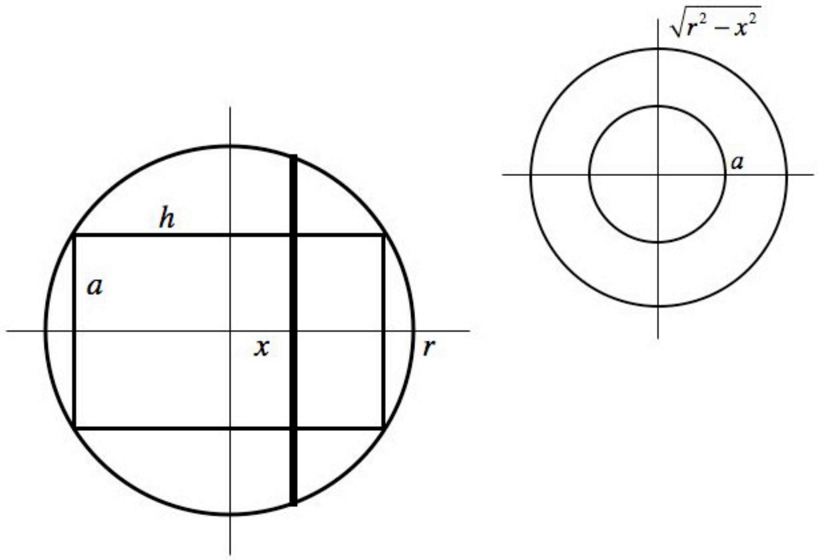
Calculus-free solution to the napkin-ring problem



Consider a slice through the napkin ring a distance x from the origin, where $-h \le x \le h$, shown as a dark vertical line in the cross-section diagram to the left. Face on, this section will be an annulus, shown in the diagram on the right. The radius of the inner boundary of the annulus is a, the diameter of the cylinder. The radius of the outer boundary is, by Pythagoras' theorem, $\sqrt{r^2 - x^2}$. The area of the annulus is thus:

$$\pi(r^2 - x^2) - \pi a^2 = \pi(r^2 - a^2 - x^2) = \pi(h^2 - x^2)$$

where the last equality follows from a second application of Pythagoras' theorem. But for each x ($-h \le x \le h$), this is exactly the same as the area of the cross section of the sphere of radius h centered at the origin. Since the napkin ring and the sphere of radius h have cross sections with identical areas, they must have the same volume.

But we know the volume of the sphere of radius h, it is $\frac{4}{3}\pi h^3$. Hence that is the volume of the napkin ring.

Given the formula for the volume of a sphere, all we used was Pythagoras' theorem. No use of calculus whatsoever.

Of course, the fact that the volume of the ring depends only on its height, not its diameter or the radius of the sphere that forms its outer surface, reflects the fact that there is a functional relationship between h, a, and r. The above proof shows just what that relationship is. The routine solution by integration that I gave last time gives us the answer, but does not explain what is going on, and hence the result surprises us. The alternative proof given here helps us to understand what is going on.

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