# Literature about manifolds of special holonomy (and in particular $G_2$-manifolds)

This list is supposed to give a rough idea and is by no means exhaustive – it is also worthwhile checking out further references contained in the quoted material or by the authors below.

## Contents



## 1 $G$-structures, principal fibre bundles and holonomy

The foundational article of the theory of special holonomy is

M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83**, 279–330, 1955.

In my opinion by far the best book covering aspects of group actions and the underlying linear algebra is Harvey's book

F. R. HARVEY. *Spinors and Calibrations*, vol. **9** in Perspectives in Mathematics, Academic Press Inc., Boston, MA, 1990.

A very intuitive, but still fairly accurate introduction is given by

M. NAKAHARA. *Geometry, topology and physics*, 2nd edition, Graduate Student Series in Physics, Bristol: Institute of Physics, 2003,

which is widely used among physicists, but is also useful for mathematicians. A quick and good introduction, focusing on the topological aspects of principal $G$-fibre bundles, is

S. J. AVIS AND C. J. ISHAM, *Quantum field theory and fiber bundles in a general space-time*, Recent developments in gravitation - Cargese 1978 : proceedings, vol. **44** in NATO Advanced Study Institute, Series B: Physics, 347–401, edited by M. Levy and S. Deser, Plenum Press, 1979.

A still fairly basic, but technically very accurate and clearly written introduction to principal $G$- and associated fibre bundles is provided by the second and third chapter of

D. D. JOYCE. *Compact manifolds with special holonomy*, Oxford University Press, Oxford Mathematical Monographs, Oxford, 2000.

The remainder of the book is devoted to study special holonomy of compact manifolds. As such it became (together with Salamon's book below) **the** reference for special holonomy. It also contains a detailed account of the author's first construction of compact irreducible $G_2$- and $Spin(7)$-manifolds. It also contains an extensive reading list about various aspects of special holonomy issues. The standard reference which gives a full account of the principal $G$-fibre bundle formalism is still the classic

S. KOBAYASHI AND K. NOMIZU. *Foundations of differential geometry I and II*, John Wiley and Sons, New York-London, 1963.

Another highly recommendable classic is

H. B. LAWSON AND M.-L. MICHELSOHN, *Spin geometry*, vol. **38** in Princeton Mathematical Series, Princeton University Press, Princeton, N. J., 1989.

which focuses on the reformulation of special holonomy in terms of spinors. A good introduction to holonomy theory is also provided by chapter 10 in

A. L. BESSE. *Einstein manifolds*, vol. **10** in Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

A very difficult, yet highly recommended reference is

S. M. Salamon. *Riemannian geometry and holonomy groups*, vol. **201** in Pitman Research Notes in Mathematics Series, Longmann Scientific & Technical, Harlow, 1989.

This book comes closest to the spirit of this lecture. Integrability issues are tackled in

V. W. Guillemin. *The integrability problem for G-structures*, Trans. Am. Math. Soc. **116**, 544–560, 1965.

A short summary of the essential idea is contained in the first sections of Bryant's famous article

R. Bryant. *Metrics with exceptional holonomy*, Ann. Math. **126**, 525–576, 1987.

which gave the first *local* examples of irreducible $G_2$- and $Spin(7)$-manifolds. The first half is highly recommended. The second half goes **very** deep into linear algebra and is not crucial for the general understanding.

# 2 $G_2$-manifolds

## 2.1 Early work (until end-nineties)

The early work is mostly devoted to existence questions of 1-flat $G_2$-structures. The first *local* examples were constructed in the above mentioned work of Bryant's. Shortly afterwards, *complete* examples were published in

R. L. Bryant and S. M. Salamon. *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** no. 3, 829–850, 1989.

In the mid-nineties, Joyce gave a construction method for *compact* 1-flat $G_2$-manifolds where he uses a heavy analytical machinery. Instead of the original papers, the reader should refer to his book about special holonomy. While Bryant uses an extension of the classical Frobenius theorem known as Cartan-Kähler theory which is also rather technical, the article of Bryant-Salamon contains a very geometric (and readable) approach to $G_2$-manifolds by considering certain rank 3-bundles over a 4-manifold with special geometric properties.

Another aspect which gained a lot of attention were the so–called *calibrated submanifolds.* This is a natural notion of a "preferred" submanifold for a given geometric structure. In complex geometry, these would be complex submanifolds while in symplectic geometry, these would be Lagrangian submanifolds. In the case of $G_2$, there are the so–called *associative* and *coassociative* submanifolds. The best introduction to that subject is presumably still the original article

F. R. Harvey and H. B. Lawson. *Calibrated geometries*, Acta Math. **148**, 47–157, 1982, as well as Harvey's book mentioned earlier. The deformation theory of calibrated submanifolds is described in

R. C. McLean. *Deformations of calibrated submanifolds*, Commun. Anal. Geom. **6**, no. 4, 705–747, 1998.

## 2.2 Recent work

Recent work about $G_2$ is mostly centered around topological $G_2$-structures with weaker integrability conditions than 1-flatness, i.e. we have non-trivial torsion classes. The original idea however, which clearly was ahead of its time, was presented in

A. Gray. *Weak holonomy groups*, Math. Z. vol. **123**, 290–300, 1971.

In the paper

M. Fernandez and A. Gray. *Riemannian manifolds with structure group* $G_2$, Ann. Mat. Pura Appl. (4) vol. **132**, 1982, 19–45,

the authors use the standard representation theoretic machinery to show the equivalence of $\nabla^{LC}\varphi$ with $d\varphi = 0$, $d{\star}\varphi = 0$. As an example of a $G_2$-structure with non-trivial torsion, I recommend

T. Friedrich, I. Kath, A. Moroianu and U. Semmelmann. *On nearly parallel* $G_2$*-structures*, J. Geom. Phys. **23**, 1997, 259–286,

which also exposes very clearly the relationship between the standard form definition of $G_2$ and the spinorial definition (as the stabiliser of a non-zero spinor). Examples for all existing torsion classes can be found in

F. M. Cabrera, M. D. Monar and A. F. Swann. *Classification of* $G_2$*-structures*, J. Lond. Math. Soc. II. Ser. **53** no. 2, 407–416, 1996.

The accrued interest of physicists in structures with torsion triggered active research in this area. Good articles are

T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** no. 2, 303–335, 2002, math.DG/0102142,

T. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable* $G_2$*-manifolds*, J. Geom. Phys. 48 no. 1, 1–11, 2003, math.DG/0112201

which considers $G_2$-structures with skew-symmetric torsion or

S. Chiossi and S. Salamon. *The intrinsic torsion of* SU(3) *and* $G_2$ *structures*, Differential geometry Valencia 2001, 115–133, World Sci. Publishing, River Edge, N. J., 2002, math.DG/0202282,

which discusses the torsion of $G_2$-structures that are induced by an underlying $SU(3)$-structure (in the vein of the lectures' example of $M^7 = CY^3 \times S^1$).

A good discussion is also contained in

R. Bryant. *Some Remarks on* $G_2$*-Structures*, 2003, math.DG/0305124.

Other new developments include Hitchin's variational principle

N. J. Hitchin. *The geometry of three-forms in six dimensions*, J. Differential Geom. **55** no. 3, 547–576, 2000, math.DG/0010054,

N. J. Hitchin. *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), edited by M. Fernández and J. A. Wolf, no. 288 Contemp. Math., 70–89, 2001, math.DG/0107101.

This leads to a very geometric approach to the moduli space of 1-flat $G_2$-structures (though the results were already known to Joyce) and yields a further construction method of 1-flat $G_2$-structures by solving the so–called *Hitchin flow equations.* Arising out of a generalisation of Hitchin's principle is the notion of a generalised $G_2$-manifold

F. Witt. *Generalised* $G_2$*-manifolds*, math.DG/0411642.

Here, we have *two* topological $G_2$-structures with skew-symmetric, *closed* torsion $\pm T$. For more details, as well as an introduction in the basic tricks of trade in $G_2$-geometry, consult with my thesis

F. Witt. *Closed forms and special metrics*, PhD thesis, Oxford University, 2004, math.DG/0502443.

# 3 $G_2$-manifolds in physics

Physicists interest focused around $G_2$-manifolds with torsion (which arise as solutions to the supersymmetry variations after a suitable compactification) and calibrated submanifolds (arising as $D$-branes).

S. Gukov. *M-theory on manifolds with exceptional holonomy*, Fortschr. Phys. **51** no.7-8, 719–731,

B. S. Acharya and S. Gukov, *M theory and Singularities of Exceptional Holonomy Manifolds*, Phys. Rept. **392**, 121–189, 2004, hep-th/0409191,

provides a good idea why these structures are important in physics and what they are supposed to describe. A (for a mathematician) very readable account of special holonomy issues in string theory is

J. P.Gauntlett, D. Martelli, S. Pakis and D. Waldram. *G-Structures and Wrapped NS5-Branes*, Comm. Math. Phys. **247**, 421–445, 2004, hep-th/0205050,

J. P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with Intrinsic Torsion*, Phys. Rev. **D69**, 2004, hep-th/0302158,

For applications of generalised $G_2$-manifolds in physics, see

C. Jeschek and F. Witt. *Generalised $G_2$-structures and type IIB superstrings*, hep-th/0412280, to appear in JHEP.

The variational principle has drawn a lot of attention recently. The appeal lies precisely in the non-linearity of the resulting equations, that is, we do not fix the metric beforehand as it is determined by the critical point itself. This could potentially lead to a “topological (i.e. without prior specification the metric) M-theory”. References are

R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa. *Topological M-theory as Unification of Form Theories of Gravity*, hep-th/0411073,

V. Pestun and E. Witten. *The Hitchin functionals and the topological B-model at one loop*, hep-th/0503083.