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On Candido's Identity

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Giacomo Candido [**1**] (1871–1941) proved the equality

$$[F_n^2 + F_{n+1}^2 + F_{n+2}^2]^2 = 2[F_n^4 + F_{n+1}^4 + F_{n+2}^4],$$

where F_n denotes the nth Fibonacci number, by observing that for all reals x, y one has the curious identity

$$[x^2 + y^2 + (x+y)^2]^2 = 2[x^4 + y^4 + (x+y)^4]. \tag{1}$$

Candido's identity (1) can be easily shown to be true not only in $\mathbb{R}^+ := [0, \infty)$ but also in any commutative ring and admits a clear visual description as presented recently in [**3**]. This identity raises the question: is (1) a characteristic property of the polynomial function $y = x^2$ in $\mathbb{R}^+$? In order to answer this we reformulate (1) as follows. Let f be a function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$f\,(f(x) + f(y) + f(x+y)) = 2\,[f(f(x)) + f(f(y)) + f(f(x+y))]\,. \tag{2}$$

In general (2) admits trivial solutions like $f \equiv 0$ as well as many bizarre, highly discontinuous solutions. For example, define f to be any function from $\mathbb{R}^+$ to $\mathbb{R}^+$ with the property that $f(x) = 0$ whenever x is rational and $f(x)$ is rational (but arbitrary!) whenever x is irrational. It is an exercise (try it) to show that every possible combination of rational or irrational values for the inputs x and y reduces (2) to the identity $0 = 0$. But if we require f to be a continuous surjection on $\mathbb{R}^+$ with $f(0) = 0$, then we shall show that f can differ from the squaring function only by a multiplicative constant.

LEMMA. *For any two positive real numbers a and b with $0 < a < b$, there are integers m and n such that $a < 2^m 3^n < b$.*

Proof. We consider three cases.

Case 1. If $1 \le a < b$ then $0 \le \log_2(a) < \log_2(b)$ and it follows that $\log_2(a)/3^n < \log_2(b)/3^n < 1$ for a sufficiently large positive integer n. Since $2^p \ne 3^q$ for all integers p, q such that $p, q \ne 0$, we deduce $p \log 2 \ne q \log 3$, i.e., $\log_2(3) = \log 3/\log 2$ is clearly irrational (see, e.g., [**2**]). So it follows from the equidistribution theorem [**4**, Theorem 6.2, p. 72] that the sequence $\log_2(3)$, $2\log_2(3)$, $3\log_2(3)$, ... is uniformly distributed modulo 1, i.e., there is some positive integer m such that

$$\log_2(a)/3^n < \log_2(3^m) - \lfloor \log_2(3^m) \rfloor < \log_2(b)/3^n,$$

where $\lfloor x \rfloor$ denotes the greatest integer $k \leq x$. Let $r = \log_2(3^m)$ and let $s = r - \lfloor r \rfloor$. Then since $2^r = 3^m$, it follows that $2^s = 3^m/2^{\lfloor r \rfloor}$. With this notation

$$\log_2(a) < 3^n s < \log_2(b)$$

i.e., $a < 2^{(3^n s)} < b$, whence $a < (3^m/2^{\lfloor r \rfloor})^{3^n} < b$. This shows that there is an integral power of 2 times an integral power of 3 between a and b.

Case 2. If $a < 1 < b$ we can use $n = m = 0$.

Case 3. If $0 < a < b \leq 1$ we will have $1 \leq 1/b < 1/a$ so by case 1 there exist integers m, n such that $1/b < 2^m 3^n < 1/a$ and therefore $a < 2^{-m}3^{-n} < b$. ■

Now we prove the following:

THEOREM. *A continuous surjective function f from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $f(0) = 0$ satisfies Candido's equation (2) if and only if*

$$f(x) = kx^2, \tag{3}$$

where $k > 0$ is an arbitrary constant.

Proof. From Candido's equality (1), it follows that (3) satisfies (2). Conversely, assume that f is a solution of (2) satisfying the above conditions. Since $f(0) = 0$ the substitution $y = 0$ into (2) yields that for all $x \geq 0$: $f(2f(x)) = 4f(f(x))$. Since f is surjective, $f(x)$ ranges throughout $\mathbb{R}^+$ as x ranges throughout $\mathbb{R}^+$, so that if we let $z = f(x)$, we have $f(2z) = 4f(z)$ for all z in $\mathbb{R}^+$. It follows by induction

$$f(2^n z) = (2^n)^2 f(z), \tag{4}$$

for all integers $n \geq 0$.

Since $f(z) = f(2^n(z/2^n)) = (2^n)^2 f(z/2^n)$ we get

$$f(2^{-n}z) = (2^{-n})^2 f(z) \tag{5}$$

for all integers $n \geq 1$. Thus from (4) and (5) we can conclude

$$f(2^n z) = (2^n)^2 f(z), \tag{6}$$

for all integers n. Next, set $y = x$ in (2) to obtain

$$f(2f(x) + f(2x)) = 4f(f(x)) + 2f(f(2x)),$$

and by virtue of (6), using $f(2x) = 4f(x)$, we get:

$$4f(3f(x)) = f(6f(x)) = 4f(f(x)) + 2 \cdot 4^2 \cdot f(f(x)) = 36f(f(x)),$$

i.e., with $f(x) = z \geq 0$ arbitrary, $f(3z) = 3^2 f(z)$ and by induction $f(3^m z) = (3^m)^2 f(z)$, whenever $m \geq 0$. As above, $f(z) = f(3^m(z/3^m)) = (3^m)^2 f(z/3^m)$ so $f(3^{-m}z) = (3^{-m})^2 f(z)$ and therefore

$$f(3^m z) = (3^m)^2 f(z), \tag{7}$$

for all integers m. By means of (6) and (7), we obtain that for all integers m, n:

$$f(2^n 3^m) = (2^n 3^m)^2 f(1). \tag{8}$$

By our previous lemma any real numbers in $[0, \infty)$ may be approximated by a sequence in the set $\{2^n 3^m | n, m \text{ integers }\}$ so from (8) and the continuity of f we can conclude that for all x in $\mathbb{R}^+$, $f(x) = kx^2$, with $k = f(1) > 0$ an arbitrary constant. ■

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Monotonic Convergence to *e* via the Arithmetic-Geometric Mean

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Recently, Hansheng Yang and Heng Yang [**3**], by using only the arithmetic-geometric inequality, have proved the monotonicity of the sequences (x_n), (y_n), related to the number e:

$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad y_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad (n = 1, 2, \dots)$$

Such a method probably is an old one and has been applied e.g. in [**1**], or [**2**].

We want to show that the above monotonicities can be proved much easier than in [**3**].

Recall that the arithmetic-geometric inequality says that for $a_1, \dots, a_k > 0$, and

$$G_k = G_k(a_1, \dots, a_k) = \sqrt[k]{a_1 \dots a_k},$$

$$A_k = A_k(a_1, \dots, a_k) = \frac{a_1 + \dots + a_k}{k},$$

we have

$$G_k \le A_k, \tag{1}$$

with equality only when all a_i are equal.