Results

Discussion

To solve the problem of extending tetration to non-integer hyper-exponents once and for all, neither property (1) nor property (3) is sufficient alone. Both must be combined in order to get a unique extension of tetration. There is already one way to ensure that property (1) be satisfied, and that is to use a piecewise extension. Now we need a way of ensuring that property (3) is satisfied. To do this we need a way of ensuring infinite differentiability. The problem is that the derivatives of some previous piecewise extensions were not continuous, so there would be a difference between the limit to a point from the left and the limit to the same point from the right. We need a way of ensuring that there is no difference between these two limits. Expressed formally:

Definition 6. *Piecewise error transform*

$$\underset{x \to c}{PWET_k}\{f(x)\} = \lim_{x \to c^+} \boldsymbol{D}^k f(x) - \lim_{x \to c^-} \boldsymbol{D}^k f(x)$$

which should be zero for all k if $f(x)$ is $\boldsymbol{C}^\infty$. To become familiar with the piecewise error transform, lets apply it to the piecewise extensions we have covered so far. First lets apply it to the piecewise extension of tetration as y approaches zero:

$$\underset{y \to 0}{PWET_0}\{{}^y x\} = x^{t(x,-1)} - t(x,0) = 1 - 1 = 0$$
$$\underset{y \to 0}{PWET_1}\{{}^y x\} = x^{t(x,-1)} \log(x) \boldsymbol{D}_y t(x,-1) - \boldsymbol{D}_y t(x,0) = \log(x) - 1$$
$$\underset{y \to 0}{PWET_2}\{{}^y x\} = \log(x)^2$$
$$\ldots$$
$$\underset{y \to 0}{PWET_k}\{{}^y x\} = \boldsymbol{D}_y^k[x^{t(x,y-1)}]_{y=0} - \boldsymbol{D}_y^k[t(x,y)]_{y=0}$$

where the piecewise extension with linear critical function is used to evaluate tetration. The first line indicates that there is no difference between the left and right limits to zero of the function itself, whereas the second line indicates that there is a difference between the left and right limits to zero of the derivative of the function with respect to y. This could be seen in a graph as a discontinuity in the derivative, but the graph in appendix B is for $x = \boldsymbol{e}$, which makes the second line zero. This can be seen as the derivative being continuous. Although the second line can equal zero when $x = \boldsymbol{e}$, the third line will be one, and this would show itself in the graph as a discontinuity in the second derivative. From this we can determine that this extension is not $\boldsymbol{C}^{\infty}$, which makes it a non-analytic extension.

The other piecewise extension that was presented was that of the super-logarithm. Now if we apply the piecewise error transform to the piecewise extension of the super-logarithm as z approaches zero, we get: the following:

$$
\begin{aligned}
&\underset{z\to 0}{PWET_0}\{\operatorname{slog}_x(z)\} = 1 + s(x,0) - s(x,1) = 0\\
&\underset{z\to 0}{PWET_1}\{\operatorname{slog}_x(z)\} = \boldsymbol{D}_z s(x,0) - \log(x)\,\boldsymbol{D}_z s(x,1) = 1 - \log(x)\\
&\underset{z\to 0}{PWET_2}\{\operatorname{slog}_x(z)\} = -\log(x)^2\\
&\ldots\\
&\underset{z\to 0}{PWET_k}\{\operatorname{slog}_x(z)\} = \boldsymbol{D}_z^k[s(x,z)]_{z=0} - \boldsymbol{D}_z^k[s(x,x^z) - 1]_{z=0}
\end{aligned}
$$

where the piecewise extension with linear critical function is used to evaluate the super-logarithm. Again, these expressions can be seen in the graph in appendix B as a continuous red line for any x. The green line, which represents the first derivative, will be discontinuous for all x except $x = \boldsymbol{e}$. For $x = \boldsymbol{e}$, the green line is continuous, because the second expression above is zero. There is no number that makes the third expression zero, so this will be seen as a discontinuous blue line for any x.

To make an extension of tetration that satisfies property (1) and property (3), we can combine the general extensions found in definition (1) and definition (4), by using the series as the critical function $t(x,y)$, keeping the coefficient functions $\beta_k(x)$ unknown. We can restrict those coefficients to satisfy:

$$\underset{y\to 0}{PWET_k}\{{}^{y}x\} = 0$$

for all nonnegative integer k. If we only require that this is true up to some integer n, where $0 \le k < n$ then we get a rather small system of equations, say for $n = 2$:

$$\begin{Bmatrix} -1 + x \cdot x^{\beta_1(x)} \cdot x^{\beta_2(x)/2} = 0 \\ -\beta_1(x) + x \cdot x^{\beta_1(x)} \cdot x^{\beta_2(x)/2} \cdot \log(x) \cdot (\beta_1(x) - \beta_2(x)) = 0 \end{Bmatrix}$$

which are nonlinear equations, and in general, are hard to solve, even with a computer. The reason why only two unknown terms were used in the equations above, is that the zeroth term, or $\beta_0(x) = {}^0x = 1$ is already known, and we only have two equations. Solving systems of equations works best when the number of unknowns is the same as the number of equations, so given two equations we should be able to solve for two unknowns. The solutions to equations obtained in this way for an extension of tetration, generally have very extreme values, and increasing the degree to $n = 3$, for example, will produce solutions for $\beta_k(x)$ that differ greatly. Producing this kind of system of equations for the super-logarithm, on the other hand, is more well-behaved.

Starting with definition (2) and definition (5) instead, we will use the series extension as the critical function of the piecewise extension of the super-logarithm, letting the coefficients $v_k(y)$ of the series remain unknown. We already know the zeroth coefficient: $v_0(x) = \text{slog}_x(0) = -1$ from integer tetration, so we will be solving for the coefficients $v_{k+1}(x)$ where $0 \le k < n$ in the equations generated by letting the piecewise error transform of the super-logarithm equal zero. For $n = 2$:

$$\begin{Bmatrix} 1 - v_1 - \frac{1}{2} v_2 = 0 \\ v_1 - \log(x) v_1 - \log(x) v_2 = 0 \end{Bmatrix}$$

For $n = 3$:

$$\begin{Bmatrix} 1 - v_1 - \frac{1}{2} v_2 - \frac{1}{6} v_3 = 0 \\ v_1 - \log(x) v_1 - \log(x) v_2 - \frac{1}{2} \log(x) v_3 = 0 \\ -\log(x)^2 v_1 + v_2 - 2\log(x)^2 v_2 - \frac{3}{2} \log(x)^2 v_3 = 0 \end{Bmatrix}$$

where $v_k = v_k(x)$. This time, however, the equations are linear, so there are many more methods at our disposal for determining if the equations are solvable, and finding the solution. One such way is finding the determinant of the matrix associated with the equations. Before we can turn the system of equations into a matrix we must put only the unknowns on the left, and constants on the right. As you can see in the systems above the only constant is the 1 in the first equation in each system. For the equations found above for $n = 2$, we can group them like this:

$$\begin{Bmatrix} & -v_1 & -\frac{1}{2}v_2 = & -1 \\ (1-\log(x))\,v_1 & & -\log(x)\,v_2 = & 0 \end{Bmatrix}$$

Also, in the interest of readability all equations will be divided by a power of $\log(x)$, because as you can see above, each equation has an increasing power of $\log(x)$ in it. Also, because the majority of the coefficients are negative we can also reverse the sign of the equations. Reversing the sign, and dividing by a power of $\log(x)$ will make:

$$\begin{Bmatrix} & v_1 & +\frac{1}{2}v_2 = & 1 \\ \left(1-\frac{1}{\log(x)}\right)v_1 & & +v_2 = & 0 \end{Bmatrix}$$

but, before we can represent the above set of equations as a matrix we must define a basis. As we stated earlier, the unknowns we are solving for are related to the coefficients of the series in definition (5). These unknowns are also the derivatives of the super-logarithm at $z = 0$, but as a basis, they are not merely numbers or vectors, they are functions of x. Using the sequence notation $\langle\cdot\rangle_{k=i}^{n}$, the basis we will be using is:

$$\mathbf{v} = \langle v_k \rangle_{k=1}^{n} \text{ where } v_k = v_k(x) = \boldsymbol{D}_z^k[s(x,z)_n]_{z=0}$$

Using this basis, the above system of equations has the matrix equation for $n = 2$:

$$\left\langle \frac{-\underset{z\to 0}{PWET}_k\{\text{slog}_x(z)_2\}}{\log(x)^k} \right\rangle_{k=0}^{1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1-\frac{1}{\log(x)} & 1 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $n = 3$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_3\}}{\log(x)^k} \right\rangle_{k=0}^{2} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $n = 4$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_4\}}{\log(x)^k} \right\rangle_{k=0}^{3} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} & \frac{1}{6} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} & \frac{2}{3} \\ 1 & 4 & \frac{9}{2}-\frac{1}{\log(x)^3} & \frac{8}{3} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $n = 5$:

$$\left\langle \frac{\underset{z\to 0}{-PWET_k}\{\mathrm{slog}_x(z)_5\}}{\log(x)^k} \right\rangle_{k=0}^{4} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 1-\frac{1}{\log(x)} & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 1 & 2-\frac{1}{\log(x)^2} & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} \\ 1 & 4 & \frac{9}{2}-\frac{1}{\log(x)^3} & \frac{8}{3} & \frac{25}{24} \\ 1 & 8 & \frac{27}{2} & \frac{32}{3}-\frac{1}{\log(x)^4} & \frac{125}{24} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mathrm{slog}_x(z)_n$ is the piecewise extension of the super-logarithm with an analytic extension as its critical function whose coefficients are obtained from the system of n equations in n unknowns, generated by letting the piecewise error transform equal zero. The equation matrices above can be computed without finding the derivatives of the super-logarithm. An alternate way of generating the above matrix is:

$$\left\langle \left\langle \frac{m^k}{m!} - \delta_{mk} \log(x)^{-k} \right\rangle_{m=1}^{n} \right\rangle_{k=0}^{n-1}$$

where δ_{jk} is the Kronecker delta (1 if $j = k$, 0 otherwise), usually used with tensors.

To indicate that the critical function used by the general piecewise extension is found at a certain value of *n*, the notation: $s(x,z)_n$ will be used. Now that we can solve for an extension of the super-logarithm, what do the solutions look like? First, the solution when *n* = 1 is actually the same as extension (2):

$$v_1(x)_1 = 1$$
$$s(x,z)_1 = -1 \cdot z^0 + v_1(x)_1 \cdot z^1$$
$$s(x,z)_1 = -1 + z$$

For *n* = 2:

$$s(x,z)_2 = -1 + \frac{2\log(x)}{1+\log(x)} z - \frac{1-\log(x)}{1+\log(x)} z^2$$

For *n* = 3:

$$s(x,z)_3 = -1 + \frac{6[\log(x)+\log(x)^3]z + 3[3\log(x)^2 - 2\log(x)^3]z^2 + 2[1-\log(x)-2\log(x)^2+\log(x)^3]z^3}{2+4\log(x)+5\log(x)^2+2\log(x)^3}$$

Assuming that the system of equations are always linear, we can find any degree solution we want, given enough time. What exactly have we found, though? Solving for the super-logarithm using the piecewise error transform ensures that the function found will be differentiable up to a point.

Lemma 1. If $\underset{x \to c}{PWET_k}\{f(x)\} = 0$ for $0 \le k < n$, then $f(x)$ is $\boldsymbol{C}^{n-1}$.

Even if we do find that the functions we get from the solutions to the system of equations are $\boldsymbol{C}^n$, we still do not know if the solution is unique. To find out whether the solution is unique we can use the determinant of the matrix that expresses the system of equations. When the determinant is zero, then the solution is not unique, when the determinant is not zero, then there must be one and only one solution. Here are the determinants of the simplified matrices obtained from the systems of equations generated by letting the piecewise error transform of the super-logarithm equal zero, for *n* = 2:

$$\det\left\langle \frac{-\underset{z\to 0}{PWET}_k\{\operatorname{slog}_x(z)_2\}}{\log(x)^k} \right\rangle_{k=0}^{1} = \frac{1}{2} + \frac{1}{2\log(x)}$$

For $n = 3$:

$$\det\left\langle \frac{-\underset{z\to 0}{PWET}_k\{\operatorname{slog}_x(z)_3\}}{\log(x)^k} \right\rangle_{k=0}^{2} = \frac{1}{6} + \frac{5}{12\log(x)} + \frac{1}{3\log(x)^2} + \frac{1}{6\log(x)^3}$$

In order to use these expressions to find when the systems of solutions are solvable and have unique solutions, we need to find when these expressions are equal to zero. When $x > 1$, $\log(x)$ is positive, and since all the coefficients in the determinants are positive, the only way the whole determinant will be zero is if $\log(x)$ is negative. Since this only happens when $0 < x < 1$, the determinant is nonzero for $x > 1$. Some of the roots of these determinants for different n are given here to illustrate:

n	***x***
2	0.367879
3	0.190653
4	0.126582, 0.494301
5	0.099918, 0.323049

and as you can see from the table, all roots seem to be between zero and one.

Lemma 2. $\det\left(\left\langle \underset{z\to 0}{PWET}_k\{\operatorname{slog}_x(z)_n\}\right\rangle_{k=0}^{n-1}\right) = 0$ implies $0 < x < 1$.

Going back to the piecewise error transform, implicitly declared in letting its application on the super-logarithm equal zero is the relationship:

$$v_k(x) = \boldsymbol{D}_z^k[s(x,z)]_{z=0} = \boldsymbol{D}_z^k[s(x,x^z) - 1]_{z=0}$$

which can be used to simplify the series expansion of the critical function used by the piecewise definition. We can now define an extension of the super-logarithm as: