Homework problems, Set 9

1. Given a morphism $X \to Y$ and a point $y \in Y$, set $Z = \operatorname{Spec}(k(y))$. Show that $\underline{f^{-1}(y)}(T) = \underline{f}^{-1}(\underline{Z}(T)) \subseteq \underline{X}(T)$, where $\underline{f}\colon \underline{X} \to \underline{Y}$ is the natural transformation of functors induced by f, and $\underline{Z}(T)$ is identified with a subset of $\underline{Y}(T)$ by virtue of the fact that $Z \to Y$ is a monomomorphism. [This problem is essentially a tautology once you unravel the definitions properly.]

2. Let X, T be schemes over S. Let $T = \bigcup_\alpha U_\alpha$ be an affine covering, and for each α, β, let $U_\alpha \cap U_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ be an affine covering.

(a) Using Problem 1 from Problem Set 6, show that $\underline{X}(T)$ is the projective limit of the system formed by the sets $\underline{X}(U_\alpha)$, $\underline{X}(W_{\alpha\beta\gamma})$ and the the maps induced by the inclusions $W_{\alpha\beta\gamma} \hookrightarrow U_\alpha, U_\beta$.

(b) Deduce that the functor $\underline{X}$ is determined by its restriction to affine schemes, or equivalently, by the covariant functor $\underline{X}(R)$ from rings (equipped with a morphism $\operatorname{Spec}(R) \to S$) to sets. Explicitly, every natural transformation $\underline{X} \to \underline{Y}$ between two such restricted functors extends to a unique natural transformation between the full functors, hence is induced by a unique S-morphism $X \to Y$.

3. Let $G_m = \operatorname{Spec}(k[z, z^{-1}])$, regarded as a group scheme over k in the usual way, act on $\mathbb{A}^2_k$ by the formula $(z) \cdot (x, y) = (zx, z^{-1}y)$; in other words, the morphism $\rho\colon G_m \times \mathbb{A}^2 \to \mathbb{A}^2$ which defines the action corresponds to the k-algebra homomorphism $\phi\colon k[x, y] \to k[z, z^{-1}] \otimes k[x, y] = k[z, z^{-1}, x, y]$ that maps $x \mapsto zx$, $y \mapsto z^{-1}y$.

(a) Let X be the open subscheme $\mathbb{A}^2 \setminus V(x, y)$. Show that X is G_m-invariant, *i.e.*, the action ρ restricted to $G_m \times X$ has image X.

(b) Let Y be the non-separated scheme obtained by gluing two copies of $\mathbb{A}^1$ along the identity map on the open subscheme $U = \mathbb{A}^1 \setminus V(x)$. Construct a surjective morphism $f\colon X \to Y$ which is equivariant for the above G_m action on X and the trivial G_m action on Y.

(c) Show that G_m acts freely on X with quotient $X/G_m = Y$, in the following precise sense: there is a covering of Y by open subschemes V such that $f_V\colon f^{-1}(V) \to V$ makes $f^{-1}(V)$ isomorphic to $G_m \times V$ as a scheme over V with G_m-action (where G_m acts on $G_m \times V$ by the left action of G_m on itself and the trivial action on V).

Note that this example is not just an artificial pathology having to do with the definition of schemes. If you like, you can take $k = \mathbb{C}$, so all schemes here are classical algebraic varieties. Then the group variety $\mathbb{C}^*$ acts freely on the open subvariety $X = \mathbb{C}^2 \setminus \{(0, 0)\}$, the orbits are the hyperbolas $xy = c$ for $c \neq 0$, plus the two components of $xy = 0$, and the quotient is the genuinely non-separated variety "$\mathbb{C}^1$ doubled at $\{0\}$."

4. Let $X = Y = \mathbb{C}^*$ and consider the map $f\colon X \to Y$, $z \to z^2$. The two-element group Z_2 acts on X by $z \to -z$, and the action commutes with f (where Z_2 acts trivially on Y).

In the analytic topology, f makes X a principal Z_2-bundle over Y, that is, we can cover Y by open sets U such that $f^{-1}(U) \to U$ is isomorphic to the projection $Z_2 \times U \to U$, as a space over U with Z_2 action.

(a) Show that X, Y and Z_2 can be identified with the sets of closed points of (affine) schemes of finite type over $\mathbb{C}$, so that f is a morphism, and Z_2 is a group scheme which acts on X as a scheme over Y.

(b) Show that for every closed point $y \in Y$, the scheme-theoretic fiber $f^{-1}(y)$ is isomorphic to Z_2 with its usual left action on itself.

(c) Show that X is *not* a principal Z_2-bundle in the Zariski topology, in fact there is no non-empty open subscheme $U \subseteq Y$ such that $f^{-1}(U)$ is isomorphic to $Z_2 \times U$ as a scheme over U. [The easiest way to see this is by considering generic points. Note, however, that by the equivalence between X_{cl} and X for Jacobson schemes X, it follows that the result also holds if we omit the generic points. So this is really about the difference between the Zariski topology and the analytic topology on the classical points.]

(d) Show that X is a principal Z_2-bundle in the following weaker sense: there exists a surjective morphism $U \to Y$ (U also of finite type over $\mathbb{C}$) such that $U \times_Y X$ is isomorphic to $Z_2 \times U$ as a scheme over U with Z_2 action.

5. Let k be a reduced ring and let I be the ideal in the polynomial ring $k[a_{1,1}, \ldots, a_{m,n}]$ generated by all the 2×2 minors of the $m \times n$ matrix with entries $a_{i,j}$. Prove that $I = \sqrt{I}$ by parametrizing the variety $V(I)$, generalizing the case $m = 2$ that we did in class. Also prove that I is prime if k is an integral domain. [For a harder version, try to do the same for the ideal generated by all $r \times r$ minors, for any r.]

6. Suppose you wanted to prove that the ideal $I = (xz - y^2, x(x-1)^2 - z^2)$ in $\mathbb{C}[x, y, z]$ is prime. The solution locus $C = V(I)$ is a curve, but not a rational curve, *i.e.*, it is not possible to parametrize it algebraically in terms of a single parameter t. Try to find a parametrization of C by a plane curve in $\mathbb{C}^2$ defined by an irreducible polynomial.

[This problem can be solved more easily using other methods; I'll assign it again later after we discuss those.]

7. Let $f \colon X \to Y$ be a morphism such that the scheme-theoretic closed image $\overline{f(X)}$ is equal to Y. Prove that f is an epimorphism in the category of *separated* schemes; more generally, given two morphisms $g_1, g_2 \colon Y \to Z$ with Z separated, show that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$, even if you do not assume X, Y separated.

8. (a) Let $f \colon X \to Y$ be a morphism such that $\overline{f(X)} = Y$ as in Problem 7, and the only open subset of Y that contains $f(X)$ is Y itself. [The earlier, incorrect, version of this problem only had you assume that $f(X)$ contains every closed point of Y. That suffices if Y is locally Noetherian, but not in general.] Prove that f is an epimorphism in the category of all (possibly non-separated) schemes.

(b) Show that if Y is reduced and locally of finite type over a field, then the obvious morphism from $X = \coprod_{y \in Y_{\text{cl}}} \operatorname{Spec}(k(y))$ to Y is an epimorphism. This gives many examples of non-surjective epimorphisms.

9. Prove EGA (5.3.5), (5.3.10) and (5.4.2).