On some criteria of irrationality for series of positive rationals : A survey

Catalin BADEA*

URA 751 au CNRS & UFR de Mathématiques
Université de Lille I, F–59655 Villeneuve d'Ascq, France

Dédié à la mémoire de Roger Apéry

Abstract

This paper is an English and expanded version of a talk delivered at the "Rencontre Arithmétique de Caen", June 1995, dedicated to the memory of Roger Apéry. It is a survey of some general irrationality criteria and other irrationality results. The emphasis is on irrationality of reciprocals of binary recursive sequences and on an open problem of P. Erdős concerning the so-called Sylvester sequence.

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*e-mail address : `badea@gat.univ-lille1.fr`

1 Introduction

There are several known conditions for an infinite convergent series of positive rational numbers to have an irrational (or transcendental) sum. There are also other results about the irrationality or transcendence of particular constants expressed as series of positive rationals. In general, it seems to be hopeless to obtain general irrationality criteria which are sufficiently strong to imply the irrationality of many particular constants.

R. Apéry's wonderful proof [2] of the irrationality of $\zeta(3)$ belongs to the second class. The aim of this paper is to survey some of the general criteria of irrationality and to discuss some irrationality and trancendency results, mainly about series of reciprocals of binary recursive sequences. The only place when we will cite again Apéry's name will be in the next section, when we will mention a result due to André-Jeannin [3]. He used Apéry's method in order to prove the irrationality of the series of reciprocals of Fibonacci numbers. Speaking about Apéry's method, it would be fair to see his proof as belonging to the midle class of particular irrationality assertions yielding some new ideas for irrationality proofs, although we cannot speak yet of Apéry's criterion of irrationality. We refer to [8] and [18] for several developments of Apéry's method.

The irrationality and the trancendence of series of reciprocals of binary recursive sequences is discussed in the third section, while the particular case of Fibonacci and Lucas numbers is considered in the next one. In the last section we deal with some general irrationality criteria related to a conjecture of P. Erdős.

We will make two conventions. The first one is that all series which appear are supposed to be convergent. The second one is the following. Let $(a_n), n \geq 0$, be a sequence of complex numbers and $(s_h), h \geq 0$, be a strictly increasing sequence of integers. When writing

$$\sum_{h=0}^{\infty} \frac{1}{a_{s_h}},$$

we will understand that the sum is in fact taken over those h with $s_h \geq 0$ and $c_{s_h} \neq 0$.

2 Sums of reciprocals of Fibonacci and Lucas numbers.

Let $(F_n), n \geq 0$, be the Fibonacci sequence, defined by

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1} \ (n \geq 1).$$

We define the Lucas sequence by

$$L_n = F_{n-1} + F_{n+1}.$$

Using R. Apéry's method, André-Jeannin [3] proved in 1989 that the series of reciprocals of Fibonacci numbers is an irrational number :

$$\theta_0 = \sum_{n=0}^{\infty} \frac{1}{F_n} \notin \mathbf{Q}.$$

In fact, in [3] there is a more general result implying also the irrationality of reciprocals of more general recursive sequences (cf. the next section). Recently, Bundschuch and Väänänen [14] obtained an irrationality measure for $\sum_{n=0}^{\infty} \frac{1}{F_n}$. Namely, one has :

$$\sum_{n=0}^{\infty} \frac{1}{F_n} = -\frac{5+\sqrt{5}}{2} L_q(\frac{1+\sqrt{5}}{2})$$

where $q = -(3+\sqrt{5})/2 \in \mathbf{Q}(\sqrt{5})$ and

$$zL_q(-z) = \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{z}{q^n - z}$$

and, using this, they obtained $6/(1-(3/\pi^2)) \approx 8.62\ldots$ as a measure of irrationality for θ_0. We refer to [14] for the details. We still don't know if θ_0 is a transcendental number.

Surprising facts are known if the sum is taken not over the whole sequence. For instance, we have

$$\theta_1 = \sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2} \in \mathbf{Q}(\sqrt{5})$$

(cf. Good [28] , Hoggatt and Bicknell [34], [35], and Cuculière [19]). Therefore θ_1 is algebraic. The trancendence of

$$\theta_2 = \sum_{n=0}^{\infty} \frac{1}{n! F_{2^n}}$$

was proved independently by Mignotte [44] and Mahler [42].

P. Erdős and R.L. Graham [23, pp. 64-65] have raised, among many problems, the following ones :

A. *What is the character of*

$$\theta_3 = \sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}} \quad and \quad \theta_4 = \sum_{n=1}^{\infty} \frac{1}{L_{2^n}} \ ?$$

B. *Is it true that if* $(n(k)), k \geq 1$, *is a sequence of positive integers such that there exists a constant* $c > 1$ *with* $n(k+1)/n(k) \geq c$ *for every* k, *then the sum of the sum of the series* $\sum_{k=1}^{\infty} \frac{1}{F_{n(k)}}$ *is irrational ?*

The author [5] proved that θ_3 and θ_4 are irrational, while Bundschuh and Pethö [13] showed that θ_3 is transcendental. André-Jeannin [4] proved that $\theta_4 \notin \mathbf{Q}(\sqrt{5})$. Recently, Becker and Töpfer [11] proved a general theorem (see later) implying that θ_3 and θ_4 are transcendental.

For the second part of the Erdős-Graham's problem, we know [6] the affirmative answer for $c \geq 2$. We will return to this problem in the next section.

3 Sums of reciprocals of binary recursive sequences

Let P and Q be two coprime integers. Let α and β be the roots of the equation $x^2 - Px + Q = 0$. Consider the binary recursive sequences $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ defined, respectively, by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad and \quad V_n = \alpha^n + \beta^n \quad , n \geq 0.$$

Then $U_{n+2} = PU_{n+1} - QU_n, \ n \geq 0$, and the same recurrence relation holds for V_n.

The following formula was obtained in 1878 by Lucas [41, p. 225] :

$$\sum_{n=1}^{\infty} \frac{Q^{2^{n-1}r}}{U_{2^n r}} = \frac{\beta^r}{U_r} \ , \ r \geq 1.$$

This implies that

$$\Theta_0 = \sum_{n=1}^{\infty} \frac{1}{U_{2^n r}} \notin \mathbf{Q}$$

whenever $Q = \pm 1$ and $\Delta = P^2 - 4Q > 0$. Indeed, U_r and V_r are integers, $\alpha - \beta$ is irrational and $\beta^r = (V_r - U_r(\alpha - \beta))/2$ is irrational. Many special cases of this result were re-discovered in the seventies. As we will see a little bit later, Becker and Töpfer [11] showed that algebraic numbers of this kind belong to a explicitely given exceptional set.

In the above section, we mentioned that the second Erdős-Graham's problem for Fibonacci numbers has a positive solution for $c \geq 2$. In fact [6],

$$\Theta_1 = \sum_{n=1}^{\infty} \frac{1}{U_{n(k)}} \notin \mathbf{Q}$$

whenever $n(k+1) \geq 2n(k) - 1$ for all sufficiently large k for $P > 0$ and $Q < 0$. Wayne McDaniel [43] assumed that $\Delta > 0$ and proved that Θ_1 is irrational if $n(k+1) \geq 2n(k)$ for large k, for all sequences U_n with $P > 0, (P, Q) = 1$ and $P^2 - 4Q > 0$. He also proved that if $n(k+1) \geq 2n(k) - 1$ for all large k and $n(k)$ is even, then the result holds for all such positive parameters P and Q. Similar results hold for the sequences V_n. André-Jeannin [4] has shown that, if $P > 0$ and $Q = \pm 1$, then

$$\Theta_2 = \sum_{n=1}^{\infty} \frac{1}{U_n} \notin \mathbf{Q}.$$

The following result was proved recently by Becker and Töpfer [10]: we have

$$\Theta_3 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{V_{2^n}} \notin \mathbf{Q}$$

whenever $\varepsilon = \pm 1$, the roots α and β are distinct, not necessarily real, $|\alpha| \geq |\beta|$, and α/β is not a root of unity. In fact, if $\Delta > 0$, not a perfect square, Θ_3 is even a trancendental number [11]. In the same paper, the authors were able to carry out a complete study of similar trancendency problems for binary recursive sequences with irreducible companion polynomial of positive discriminant. The proofs are based upon Mahler's method for transcendency.

Theorem 1 (Becker and Töpfer [11]) *Let* $(R_n),\ n \geq 0$, *be a sequence of integers which is not eventually periodic and satisfies the recurrence relation*

$$R_{n+2} = PR_{n+1} - QR_n \ (n \geq 0) \ ,$$

with integers P *and* Q *satisfying* $P \neq 0$, $\Delta = P^2 - 4Q > 0$. *Suppose that* Δ *is not a perfect square.*

Let $(b_h),\ h \geq 0$, *be a periodic sequence of algebraic numbers which is not indentically zero and let* $d,\ k$, *and* l *be integers with* $d \geq 2$ *and* $k \geq 1$.

Then

$$\Theta_4 = \sum_{h=0}^{\infty} \frac{b_h}{R_{d^h k + l}}$$

is algebraic if and only if (b_h) *is a constant sequence,* $d = 2$, $|Q| = 1$, *and* $R_l = 0$. *Moreover, if* Θ_4 *is algebraic, then* $\Theta_4 b_0^{-1} \in \mathbf{Q}(\sqrt{\Delta}) \setminus \mathbf{Q}$.

For other results of this type, we refer the interested reader to [38], [40], [45], [13], [31], [10], [11].

4 The Sylvester sequence and a problem of Erdős

In proposition 20 of Book IX of his *Elements*, Euclid gave a proof like the following that there are infinitely many primes. Suppose that $p_1, \ldots, p_n$ are all the primes we know about. Let

$$P_n = \prod_{i=1}^{n} p_i \ .$$

Then $1 + P_n$ is not divisible by any of the primes $p_1, \ldots, p_n$, so the prime factors of $1 + P_n$ are new to us. Hence, the number of primes is unbounded. If we "discover" just the smallest prime factor of $1 + P_n$ and if we begin with 2, then we are lead in a natural way to the sequence

$$2,\ 3,\ 7,\ 43,\ 13,\ etc.$$

Shanks [53] has conjectured that this sequence contains all primes and he gave a heuristic argument which makes this conjecture plausible. For this and other similar Euclid sequences we also refer to Guy and Nowakowski [30] and Wagstaff [58].

If one feels that *all* prime factors of 1 plus the product of those found so far are “discoverd”, then one is lead to the sequence

$$S_1 = 2,\ S_{n+1} = 1 + S_1 \cdots S_n.$$

The terms of this sequence can be computed without any factoring since

$$S_{n+1} = S_n^2 - S_n + 1 = S_n(S_n - 1) + 1\ .$$

We refer to Guy and Nowakowski [30] and Odoni [46] (and the references cited therein) for the study of the primes of this sequence.

It seems that this sequence (#331 in [54]) was first mentioned by J.J. Sylvester in 1880 [55], although some authors attribute it to E. Lucas. We will call it the *Sylvester sequence*. It may be worthwile to mention that the Sylvester sequence appears in many different contexts : see the list of references for several of the many papers mentioning the Sylvester sequence.

For irrationality assertions, the following greedy property of the Sylvester sequence may be important : for each N, the first N terms of the Sylvester sequence are known [37], [20], [36] to give the smallest positive value of

$$1 - \sum_{i=1}^{N} \frac{1}{a_i}$$

among all choices of positive integers $a_1, \ldots, a_N$. In particular, the sum

$$\sum_{i=1}^{\infty} \frac{1}{S_n} = 1$$

is rational. The following open problem due to Erdős conjectures that this is essentially the only possibility among sequences satisfying $a_{n+1}/a_n^2 \sim 1$.

Conjecture 2 (Erdős [23]) *Let $(a_n), n \geq 1$, be a sequence of positive integers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n^2} = 1.$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \in \mathbf{Q}$$

implies $a_{n+1} = a_n^2 - a_n + 1$ *for all large* n.

The following are some partial results towards this conjecture.

Theorem 3 (Erdős and Straus [24]) *Let* $(a_n), n \geq 1$, *be an increasing sequence of positive integers such that*

1. $\limsup_{n\to\infty} a_n^2/a_{n+1} \leq 1$;
2. *the sequence* $[a_1, \ldots, a_n]/a_{n+1}$ *is bounded.*

Then the same conclusion as in Conjecture 4.1 holds.

In the above theorem, $[a_1, \ldots, a_n]$ denotes the least common multiple of $a_1, \ldots, a_n$.

The following result is a consequence of a more general result [6].

Theorem 4 ([6]) *Let* $(a_n), n \geq 1$, *be a sequence of positive integers such that*

$$a_{n+1} \geq a_n^2 - a_n + 1.$$

Then the same conclusion as in Conjecture 4.1 holds.

A generalization of this result to irrationality of series of rationals can be found in [6], while a generalization of Erdős-Straus's result to more general series was obtained by Oppenheim [48]. A similar result holds for infinite products [47].

The following variation of Theorem 4.2 can be proved.

Theorem 5 ([7]) *Let* $(a_n), n \geq 1$, *be an increasing sequence of positive integers such that*

1. $a_n^2/a_{n+1} \geq 1$;
2. $\sum_{n=1}^{\infty} \left(\frac{a_n^2}{a_{n+1}} - 1 \right) < \infty$.

Then the same conclusion as in Conjecture 4.1 holds.

If one admits also alternating series, then we should mention that

$$C_0 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{S_n}$$

is transcendental. Therefore, Cahen's [15] constant

$$C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{S_n - 1}$$

is also a transcendental number since $2C = C_0 + 1$. These results were first proved by Davison and Shallit [21], [52], who also obtained their continued fractions development. Note that C was also mentioned by Remez [49] and that the transcendency of Cahen's constant was also proved, as a corollary of a more general result, by Becker [9]. A trancendency measure for C was recently obtained by Töpfer [57].

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