Spectral Ambiguity of Allan Variance

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***Abstract*— The phase-noise spectrum determines the Allan variance by a well-known integral formula. It is shown here that unique inversion of this formula is not possible in general because the mapping from spectrum to Allan variance is not one-to-one. A necessary and sufficient condition for two distinct phase spectra to have the same Allan variance is given.**

***Index Terms*—Covariance functions, finite difference methods, Fourier transforms, frequency stability, integral equations, phase noise, spectral analysis, time domain analysis.**

I. Introduction

THIS paper addresses the question "To what extent does the Allan variance of a noise process determine its power spectrum?" Devised to satisfy a need for characterizing nonstationary phase and frequency noise in clocks and oscillators [1], [2], conventional Allan variance is the most often-used method for reducing a clock-noise time series to a statistical summary of frequency stability; its use has also spread to other fields of science as a tool for studying low-frequency spectral behavior of physical processes. Because Allan variance $\sigma_y^2(\tau)$ acts as an approximate highpass spectral analyzer on phase modulation (PM) noise and an approximate octave bandpass analyzer on frequency modulation (FM) noise, linear regions in the log-log "sigma-tau" plot, $\sigma_y(\tau)$ versus averaging time τ, are associated with regions of power-law behavior f^β in the PM spectrum $S_x(f)$, which maps to $\sigma_y^2(\tau)$ by the straightforward formula

$$\sigma_y^2(\tau) = \frac{8}{\tau^2}\int_0^\infty \sin^4(\pi f\tau) S_x(f)\, df \tag{1}$$

[3]. The identification of power-law components of $S_x(f)$ from those of $\sigma_y^2(\tau)$ is an example of a *parametric* inversion of (1) from Allan variance to spectrum. There is no problem with this practice if the actual spectrum is known to have the desired parametric form.

On the other hand, claims of *nonparametric* inversion formulas have been published. Lindsey and Chie [4] give formulas for $S_x(f)$ or its generalized Fourier transform in terms of $\sigma_y^2(\tau)$ or other finite-difference variances. Van Vliet and Handel [5], regarding (1) as an integral transform that generalizes the Fourier transform, assert that $\sigma_y^2(\tau)$ *uniquely* determines $S_x(f)$ and give an inversion formula involving Laplace and Mellin transforms. The principal claim of the present paper is that Allan variance does not always determine a unique PM spectrum. Moreover, the ambiguity is centered at the most interesting case, namely, flicker-FM noise, $S_x(f) \propto f^{-3}$, whose Allan variance is constant. It is shown that Allan variance is totally insensitive to a certain class of log-periodic modulations of the spectrum by octaves (see Fig. 1 for examples). Consequently, an inversion algorithm for (1) must be one-sided: starting from a given Allan variance the algorithm does not necessarily arrive at the correct spectrum, but only at *some* spectrum that has the same Allan variance.

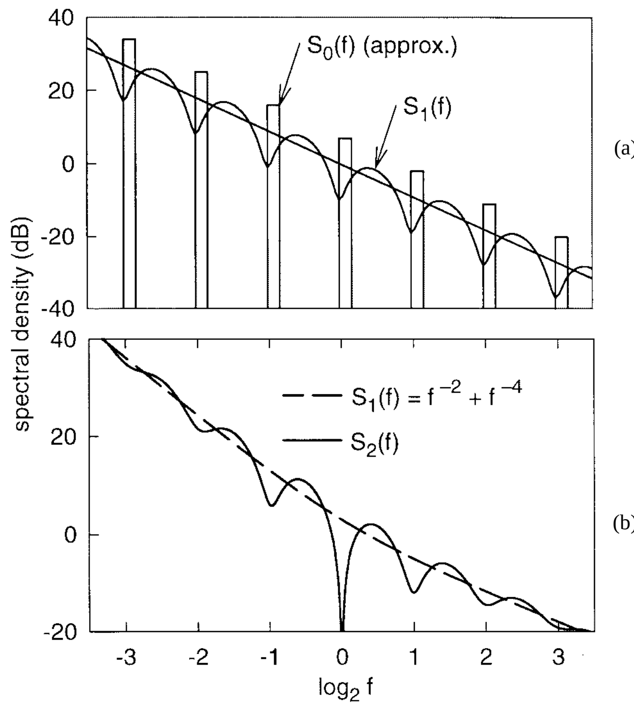

Fig. 1. Examples of spectra with the same Allan variance. (a) Examples for Theorem 1: three spectra with the same constant Allan variance. The straight line is f^{-3}. The rectangles approximate a delta-function spectrum. (b) Examples for Theorem 2: two spectra with the same nonconstant Allan variance.

This paper has two main results. The crux of the matter is contained in Theorem 1, which characterizes the infinite set of PM spectra whose Allan variance equals a given constant. This theorem leads immediately to Theorem 2, which gives a necessary and sufficient condition for any two spectra to have the same Allan variance. An alternate derivation of this result is carried out from the formalism of Van Vliet and Handel. Some additional argument yields a class of spectra that does enjoy unique inversion of (1). The Appendix gives a proof of Theorem 1.

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II. Notation and Terminology

Allan variance is defined for a PM (or time deviation) process $x(t)$ with stationary second differences by

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \operatorname{var}[x(t) - 2x(t-\tau) + x(t-2\tau)] \qquad (2)$$

(where var = variance). The one-sided power spectral density $S_x(f)$ of the process $x(t)$ maps directly to $\sigma_y^2(\tau)$ according to (1). In this exposition, which deals mainly with spectra, not with the processes themselves, a "PM spectrum" is defined to be a nonnegative measurable function[1] $S_x(f)$ for $f > 0$ that satisfies

$$\int_1^\infty S_x(f)\, df < \infty, \int_0^1 S_x(f) f^{2d}\, df < \infty \qquad (3)$$

for some nonnegative integer d. These spectra have finite power at high frequencies and diverge in a controlled way at low frequencies. The smallest such d is called the *degree* of S_x, written $\deg S_x$. The degree is also the smallest number of difference operations that have to be applied to the PM process $x(t)$ to produce a stationary process. A PM spectrum's Allan variance is finite if and only if the spectrum has degree ≤ 2.

It is convenient to embed the set of PM spectra in a vector space of real-valued functions. A *signed PM spectrum* $\Phi(f)$ is defined to be a measurable function such that $|\Phi(f)|$ is a PM spectrum. Its degree is defined to be that of $|\Phi(f)|$. Let us extend the notion of Allan variance to a linear mapping on the subspace of signed PM spectra of degree ≤ 2 by

$$V_{\mathrm{A}}(\tau; \Phi) = \frac{8}{\tau^2} \int_0^\infty \sin^4(\pi f \tau) \Phi(f)\, df \qquad (4)$$

which will still be called Allan "variance" even though it can assume any real value, including zero.

III. Results

The first result says that the most general PM spectrum with a constant Allan variance is obtained from a log-periodic modulation of an f^{-3} spectrum by octaves. The result is established here for signed PM spectra so that it can easily be applied to the proof of Theorem 2.

Theorem 1: A signed PM spectrum $\Phi(f)$ has a constant Allan variance V_{A} if and only if

$$\Phi(2f) = \frac{\Phi(f)}{8} \quad \text{a.e.} \qquad (5)$$

In this case

$$V_{\mathrm{A}} = 8\pi^2 \int_1^2 f^2 \Phi(f)\, df. \qquad (6)$$

(a.e. = almost everywhere with respect to Lebesgue measure.)

Some remarks and examples follow.

1) The condition (5) on $\Phi(f)$ is equivalent to the representations

$$\Phi(f) = \frac{\phi(f)}{f^3} = \frac{\psi(\log_2 f)}{f^3} \quad \text{a.e.} \qquad (7)$$

where $\phi(2f) = \phi(f)$ for all f, and $\psi(x)$ is a function with period on, integrable over a period. Then (6) becomes

$$V_{\mathrm{A}} = 8\pi^2 \ln 2 \int_0^1 \psi(x)\, dx. \qquad (8)$$

2) The range of integration in (6) can be any octave $a < f < 2a$. That is so because the interval of integration in (8) can be replaced by any interval of length one.

3) Any locally integrable function $\Phi(f)$ that satisfies (5) is a signed PM spectrum of degree two, or is identically zero a.e. This can be shown by expressing the integrals of $S_x(f) = |\Phi(f)|$ in (3) as sums of integrals over octaves $[2^n, 2^{n+1}]$ for integers n.

Examples of PM spectra with the same constant Allan variance $8\pi^2 \ln 2$ are shown in Fig. 1(a). The straight line is just f^{-3}. The PM spectrum $S_1(f)$ is given by

$$S_1(f) = f^{-3}[1 - 0.9\cos(2\pi \log_2 f)]. \qquad (9)$$

The series of rectangles is an approximation to the pure delta-function spectrum

$$S_0(f) = \ln 2 \sum_{n=-\infty}^{\infty} 4^{-n} \delta(f - 2^n). \qquad (10)$$

which lies slightly outside the mathematical framework given here. The approximating rectangles have height proportional to 8^{-n} but area proportional to 4^{-n}. Among PM spectra with constant Allan variance, this one is the most extreme[2] in that all the power in each octave is concentrated at one frequency.

The proof that $S_0(f)$ has the same constant Allan variance as f^{-3} is short and instructive. By (4)

$$V_{\mathrm{A}}\left(\frac{x}{\pi}; S_0\right) = 8\pi^2 \ln 2 \sum_{n=-\infty}^{\infty} \frac{\sin^4(2^n x)}{4^n x^2}. \qquad (11)$$

Because of the critical identity

$$\sin^4 x = \sin^2 x - \tfrac{1}{4}\sin^2 2x \qquad (12)$$

the summation in (11) equals

$$\lim_{n \to -\infty} \sum_{k=n}^{\infty} \left[\frac{\sin^2(2^k x)}{4^k x^2} - \frac{\sin^2(2^{k+1} x)}{4^{k+1} x^2}\right]. \qquad (13)$$

For each n, the series in (13) telescopes to the single term $4^{-n} x^{-2} \sin^2(2^n x)$, whose limit as $n \to -\infty$ is one. Hence $V_{\mathrm{A}}(\tau; S_0) = 8\pi^2 \ln 2$.

The second main result, the characterization of the spectral ambiguity of Allan variance, is an immediate corollary of Theorem 1. Two PM spectra have the same Allan variance if and only if their difference is a signed PM spectrum with zero Allan variance; thus Theorem 1 applies with $V_{\mathrm{A}} = 0$.

[1] This theory can also be carried out in the context of general spectral measures, which include delta functions and other singular measures. Indeed, one of the examples below consists entirely of delta functions.

[2] But canonical in the sense that it generates all others by a logarithmic convolution operation

Theorem 2: Two PM spectra $S_1(f)$ and $S_2(f)$ of degree ≤ 2 have the same Allan variance if and only if the signed PM spectrum $\Phi(f) = S_1(f) - S_2(f)$ satisfies $\Phi(2f) = \Phi(f)/8$ a.e. and

$$\int_1^2 f^2\Phi(f)\,df = 0. \tag{14}$$

Remarks 1) and 2) above hold here also; in particular, the integral of $f^2\Phi(f)$ over any octave is zero.

If $f^3S_x(f) \geq a > 0$ for all f, then one can obtain other PM spectra with the same Allan variance as $S_x(f)$ by adding a variety of log-periodic "modulations" of form $f^{-3}\psi(\log_2 f)$ with $\psi(x+1) = \psi(x)$ and $\int_0^1 \psi(x)\,dx = 0$. This is illustrated in Fig. 1(b), which shows the two PM spectra

$$S_1(f) = f^{-2} + f^{-4} \tag{15}$$

$$S_2(f) = S_1(f) - 2f^{-3}\cos(2\pi\log_2 f) \tag{16}$$

both of which have Allan variance $2\pi^2/\tau + 8\pi^4\tau/3$. Just enough of the modulation has been added to make $S_2(1) = 0$ while keeping $S_2(f) \geq 0$. Suitably scaled in amplitude and frequency, the PM spectrum $S_1(f)$, which is just white FM plus random-walk FM, is often used as a noise model for rubidium or cesium-beam frequency standards.

A. The Octave Variance

This name is given here to the expression

$$V_o(\tau;\Phi) = 8\pi^2\int_{1/(4\tau)}^{1/(2\tau)} f^2\Phi(f)\,df \tag{17}$$

which was introduced by Percival [6] as an ideal version of a bandpass variance of Rutman [7]. Again, this "variance" can assume any real value on signed PM spectra. It leads to a reformulation of Theorems 1 and 2. Since the derivative of $\int_a^{2a} f^2\Phi(f)\,df$ with respect to a equals $8a^2\Phi(2a) - a^2\Phi(a)$ a.e., it follows that $V_o(\tau;\Phi)$ is constant if and only if $\Phi(f)$ satisfies the condition (5) of Theorem 1. Thus, Theorem 1 says that a signed PM spectrum $\Phi(f)$ has a constant Allan variance if and only $\Phi(f)$ has a constant octave variance, i.e., the corresponding signed FM spectrum $4\pi^2f^2\Phi(f)$ gives equal (signed) power to every octave $a < f < 2a$ for $a > 0$; in this case, the two variances V_A and V_o are equal. Theorem 2 says that two PM spectra of degree ≤ 2 have the same Allan variance if and only if their difference has octave variance zero, i.e., the corresponding FM spectra give the same power to every octave. The nullspaces of the V_A and V_o operators turn out to be the same.

B. Another Derivation of the Ambiguity

Although Van Vliet and Handel [5] claim unique inversion of Allan variance to spectrum, their method actually leads to another derivation of the nonuniqueness condition of Theorem 2. After taking the Laplace transform of both sides of (4), they solve the resulting integral equation by complex Mellin transforms. The solution for $\Phi(f)$ contains an additive term

$$\oint_C F(p)\frac{\cos(p\pi/2)}{1-2^{p-3}}\,\frac{dp}{\omega^p} \tag{18}$$

(where $\omega = 2\pi f$) that represents the general solution of the homogeneous equation, i.e., the spectral ambiguity. Here, C is a contour and $F(p)$ some analytic function in the strip $1 < \mathrm{Re}\ p < 5$. Because the integrand has a simple pole at $p_n = 3 + i2\pi n/\ln\ 2$ for each nonzero integer n, the sum of the residues takes the form

$$\sum_{n\neq 0} c_n\omega^{-p_n} = \omega^{-3}\sum_{n\neq 0} c_n\exp(-i2\pi n\log_2\omega) \tag{19}$$

which is of form $f^{-3}\psi(\log_2 f)$ with $\psi(x)$ of period one and integral zero over a period, as specified in Theorem 2.

C. Stationary FM

From the results already given, one can deduce that unique inversion of the Allan variance formula (1) is indeed possible if the FM noise is stationary. Suppose that $S_1(f)$ and $S_2(f)$ are *distinct* nonnegative PM spectra of degree ≤ 2 with the same Allan variance. Then both their degrees must be two. Proof: according to Theorem 2, $S_1(f) = S_2(f) + \Phi(f)$, where $\Phi(f)$ satisfies the conditions given there. Since $S_1(f) \geq 0, S_2(f) \geq 0$, it follows that

$$S_1(f) \geq \Phi_+(f) \tag{20}$$

where $\Phi_+(f) = \max(\Phi(f), 0)$. Since $f^2\Phi(f)$ is not a.e. zero on an octave but integrates to zero there, the PM spectrum $\Phi_+(f)$ cannot be a.e. zero. By remark 3) following Theorem 1, $\deg\Phi_+ = 2$. By (20), $\deg S_1 = 2$. By a similar argument, $\deg S_2 = 2$. One concludes that *unique inversion of the Allan variance formula (1) is possible for PM spectra of degree* ≤ 1. Examples include power laws $f^\beta, -3 < \beta \leq 0$ (with a high-frequency rolloff if $\beta \geq -1$) and integrated Lorentzians $f^{-2}(f^2 + f_0^2)^{-1}$. These processes are already characterized by the *first*-difference variance $D_1(\tau) = \mathrm{var}[x(t) - x(t-\tau)]$, for which unique inversion to spectrum is known. Even so, the inversion problem is ill-posed: for example, $\sigma_y^2(\tau)$ and $D_1(\tau)$ both distinguish the flicker PM spectrum f^{-1} (rolled off above f_h) from the white PM spectrum f^0 by a factor of order $\ln(f_h\tau)$, which is hard to see in practice. This was the main reason for introducing the modified Allan variance [8].

IV. Does It Matter?

These spectral nonuniqueness results have been given here, not to discourage the conventional use of Allan variance for analyzing time series, but simply to expose a previously unknown limitation of the technique. One can object that these results are irrelevant because the log-periodic spectral modulations that constitute the ambiguity do not arise from any known physical theory; consequently, any spectral disturbances lying in the nullspace of the integral operator given by (4) can be excluded on physical grounds. Naturally, if one of the spectra belonging to a spectrally ambiguous Allan variance has a simple parametric form, as in the examples of Fig. 1, then it is reasonable to exclude the other spectra; the example of Fig. 1(b) is intended only to show how the ambiguity works. In the general nonparametric case, unless one has an objective criterion for a physically relevent spectrum, it is hard to choose which spectrum is the right one or to decide whether a proposed inversion algorithm introduces a physically objectionable spectral disturbance.

Even if the previous objection is granted, one can argue that users of Allan variance still ought to be aware of the facts that are proved here. First, since other researchers have asserted unique invertibility, the record needs to be set straight. The operation (4) ought not to be regarded as an integral transform that extends the Fourier transform. Second, the outputs of artificial $1/f$ noise generators built from ladders of first-order analog or digital filters [9], [10] have just this kind of modulated spectrum, although the frequency ratio of the ladder is not usually designed to be two, and there are only a few filter stages in practice. Nevertheless, it is unsafe to use flatness of Allan variance of the integrated output as a test of the spectral accuracy of such generators. Finally, these results put a fundamental limitation on what can be learned about a noise process from examination of its Allan variance, which, in general, does not completely characterize the covariance properties of the noise. Although the Allan-variance statistic remains useful for revealing broad spectral trends, the extraction of spectral details by this means is difficult, if not impossible.

Appendix
Proof of Theorem 1

The "if" part of Theorem 1 and the formula (6) for V_{A} can be proved by generalizing the derivation of the Allan variance of the delta-function spectrum (10). Assume merely that $\Phi(f)$ is a locally integrable function satisfying (5). By Remark 3) following Theorem 1, $\Phi(f)$ is a signed PM spectrum of degree two (or vanishes a.e.). Then, for any $\tau > 0, \sin^4(\pi f\tau)\Phi(f)$ is integrable, and $V_{\mathrm{A}}(\tau;\Phi) = 8\tau^{-2} \lim_{n\to-\infty} I_n(\tau)$, where

$$I_n(\tau) = \int_{2^n}^{\infty} \sin^4(\pi f\tau)\Phi(f)\, df. \tag{21}$$

Use the trigonometric identity (12) to express $I_n(\tau)$ as the difference of two integrals, in the second of which make the change of variable $f' = 2f$ and apply (5). The two integrals, now having the same integrand, recombine to give

$$I_n(\tau) = \int_{2^n}^{2^{n+1}} \sin^2(\pi f\tau)\Phi(f)\, df \tag{22}$$

which, via another change of variable and (5) again, gives

$$I_n(\tau) = \pi^2\tau^2 \int_1^2 \left[\frac{\sin(2^n\pi\tau f)}{2^n\pi\tau f}\right]^2 f^2\Phi(f)\, df. \tag{23}$$

As $n \to -\infty$, the sinc function tends uniformly to one, and $I_n(\tau)$ therefore tends to $\pi^2\tau^2 \int_1^2 f^2\Phi(f)\, df$.

The proof of the "only if" part of Theorem 1 depends on properties of the *generalized autocovariance* (gacv) function [11], defined for signed PM spectra of degree d by

$$R(t;\Phi) = \int_0^{\infty} C_d(2\pi f, t)\Phi(f)\, df \tag{24}$$

where

$$C_d(\omega,t) = \cos(\omega t) - \frac{1}{1+\omega^{2d}} \sum_{j=0}^{d-1} \frac{(-1)^j(\omega t)^{2j}}{(2j)!}. \tag{25}$$

If $d = 0$, then the sum in (25) is omitted, and $R(t;\Phi)$ is just the cosine transform of $\Phi(f)$. The integral in (24) exists absolutely because of (3) and (31) below.

Some facts about the gacv will be developed in a sequence of propositions, the first of which deals with scaling and linearity.

Proposition 1: Let Φ, Φ_1, Φ_2 be signed PM spectra, a, a_1, a_2 real numbers, $a > 0$. The expressions

$$aR[t;\Phi(a\cdot)] - R(t/a;\Phi) \tag{26}$$

$$R(t; a_1\Phi_1 + a_2\Phi_2) - a_1R(t;\Phi_1) - a_2R(t;\Phi_2) \tag{27}$$

are polynomials, where $\Phi(a\cdot)$ denotes the function $f \to \Phi(af)$.

This can be shown by straightforward manipulations of (24). It is necessary to observe that if $\deg\Phi = d_1 < d$, then the right side of (24) differs from $R(t;\Phi)$ by a polynomial. Therefore, when evaluating the members of (27), one can take $d = \max(\deg\Phi_1, \deg\Phi_2)$ in (24).

It is now to be shown that R and Φ form a generalized Fourier-transform pair with respect to a certain space of test functions. For this purpose, a "test function" is defined to be a complex-valued function $\nu(t)$, defined for all real t, such that $\nu(t)$ is the inverse Fourier transform of a function $\hat{\nu}(f)$ that is infinitely differentiable and vanishes outside some closed, bounded subinterval of the positive real line. Let $\nu(t)$ be a test function. Then $\nu(t)$ is bounded and $\int_{-\infty}^{\infty} \nu(t)\, dt = \hat{\nu}(0) = 0$. Integrating the inverse Fourier relationship repeatedly by parts, one finds that $t^n\nu(t)$ is also a test function for any positive integer n. Therefore, for any polynomial $p(t), p(t)\nu(t)$ is integrable and $\int_{-\infty}^{\infty} p(t)\nu(t)\, dt = 0$, i.e, test functions "kill polynomials".

Proposition 2: If $\Phi(f)$ is a signed PM spectrum, then

$$\int_{-\infty}^{\infty} \nu(t)R(t;\Phi)\, dt = \tfrac{1}{2} \int_0^{\infty} \hat{\nu}(f)\Phi(f)\, df \tag{28}$$

for all test functions $\nu(t)$.

Proof: Let $\deg\Phi = d$. By (24), the left side of (28) equals

$$\int_{-\infty}^{\infty} dt\, \nu(t) \int_0^{\infty} df\, \Phi(f)C_d(2\pi f, t). \tag{29}$$

Writing

$$C_d(\omega,t) = \frac{1}{1+\omega^{2d}}\left[\cos\omega t - \sum_{j=0}^{d-1}\frac{(-1)^j(\omega t)^{2j}}{(2j)!}\right] + \cos\omega t\frac{\omega^{2d}}{1+\omega^{2d}} \tag{30}$$

one sees from Taylor's formula with remainder that

$$|C_d(\omega,t)| \le \frac{\omega^{2d}}{1+\omega^{2d}}\left(\frac{t^{2d}}{(2d)!}+1\right). \tag{31}$$

Consequently, the iterated integral (29), with the integrands replaced by their absolute values, is bounded by

$$\int_{-\infty}^{\infty} dt|\nu(t)|\left(\frac{t^{2d}}{(2d)!}+1\right)\int_0^{\infty}\frac{d\omega}{2\pi}\left|\Phi\left(\frac{\omega}{2\pi}\right)\right|\frac{\omega^{2d}}{1+\omega^{2d}} \tag{32}$$

which is finite. By Fubini's theorem (the integrand being jointly measurable), the integral (29) exists and the order of integration can be interchanged, giving

$$\int_{-\infty}^{\infty} \nu(t)R(t;\mu)\,dt = \int_0^{\infty} df\;\Phi(f)\int_{-\infty}^{\infty} dt\;\nu(t)C_d(2\pi f,t). \tag{33}$$

Because $\nu(t)$ kills polynomials

$$\int_{-\infty}^{\infty} \nu(t)C_d(2\pi f,t)\,dt = \int_{-\infty}^{\infty} \nu(t)\cos(2\pi ft)\,dt \tag{34}$$

$$= \tfrac{1}{2}[\hat{\nu}(f)+\hat{\nu}(-f)] = \tfrac{1}{2}\hat{\nu}(f) \tag{35}$$

for $f>0$. ■

Proposition 3: If $\Phi(f)$ is a signed PM spectrum for which $R(t;\Phi)$ is a polynomial, then $\Phi(f)=0$ a.e. (and hence $R(t;\Phi)$ is actually zero).

Proof: If $R(t;\Phi)$ is a polynomial, then it is killed by all test functions $\nu(t)$. By Proposition 2, $\Phi(f)$ is orthogonal to all the test-function transforms $\hat{\nu}(f)$. Because the indicator function of any open interval $]a,b[$, where $0<a<b<\infty$, is the limit of an increasing sequence of such transforms, it follows that $\Phi(f)$ integrates to zero over all such intervals, and therefore vanishes a.e. ■

The last proposition gives a formula for Allan variance in terms of gacv.

Proposition 4: If $\Phi(f)$ is a signed PM spectrum of degree ≤ 2 with gacv $R(t)$, then

$$\tau^2 V_{\mathrm{A}}(\tau;\Phi) = 3R(0) - 4R(\tau) + R(2\tau). \tag{36}$$

Proof: Let Δ_τ^2 be the backward second-difference operator: $\Delta_\tau^2 x(t) = x(t) - 2x(t-\tau) + x(t-2\tau)$ for any function $x(t)$. This operator kills polynomials of degree ≤ 1 and reduces the degree of other polynomials by two; thus, the product operator $\Lambda = \Delta_\tau^2\Delta_{-\tau}^2$ kills polynomials of degree ≤ 3. Applying Λ to both sides of (25) as functions of t for fixed $\omega = 2\pi f$ and $d \le 2$ gives

$$\Lambda C_d(2\pi f,t) = 16\sin^4(\pi f\tau)\cos(2\pi ft). \tag{37}$$

Finally, applying Λ to both sides of (24) and setting $t=0$ gives [in view of (4)]

$$R(-2\tau) - 4R(-\tau) + 6R(0) - 4R(\tau) + R(2\tau) = 2\tau^2 V_{\mathrm{A}}(\tau;\Phi) \tag{38}$$

which is the same as (36) because $R(t;\Phi)$ is even. ■

Although the formula (36) is easy to derive from (2) for a stationary process with autocovariance $R(t)$, the present theory applies to a process with stationary second differences and gacv $R(t)$.

The proof of Theorem 1 can now be completed. Assume that $V_{\mathrm{A}}(\tau;\Phi)$ is constant. Then $\deg\Phi \le 2$. Let its gacv be $R(t)$. From (36) one sees that $R(2t) - 4R(t)$ is a polynomial. By Proposition 1, the gacv of $\Phi(f/2) - 8\Phi(f)$ equals $2R(2t) - 8R(t)$ plus a polynomial, and is therefore also a polynomial. By Proposition 3, $\Phi(f/2) - 8\Phi(f) = 0$ a.e.

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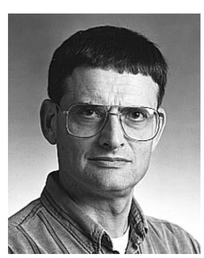

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