

A COMPARISON OF METHODS USED IN RECTANGULAR TO GEODETIC COORDINATE TRANSFORMATIONS¹

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INTRODUCTION

The purpose of this paper is to gather and review the different methods of transforming geocentric coordinates to geodetic coordinates. The direct procedure is fairly simple and straightforward. Formulas for this transformation will be given in the next section. Problems exist for the reverse transformation since no pure inverse relationship exists. The methods presented can be broken down into two general categories: iterative and direct or non-iterative methods.

BASIC CONCEPTS

Before beginning the evaluation, some basic concepts of geometric geodesy are presented. For a comprehensive treatment of the topic the readers are encouraged to read Rapp [1998], Torge [2001], Vanicek and Krakiwsky [1991], or a number of other introductory geodesy textbooks. The following definitions will be used.

a = semi-major axis of the ellipse (equatorial axis)

b = semi-minor axis of the ellipse (polar axis)

C = center of curvature of the meridian ellipse

f = flattening: $f = \frac{a-b}{a}$

e = first eccentricity: $e^2 = \frac{a^2 - b^2}{a^2}$

e' = second eccentricity: $e'^2 = \frac{a^2 - b^2}{b^2}$

E_h = error in height

E_φ = error in latitude

h = geodetic height

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M = radius of curvature in the meridian: $M = \frac{a(1-e^2)}{(1-e^2 \sin^2 \varphi)^{3/2}} = \frac{a(1-e^2)}{W^3}$

N = radius of curvature in the prime vertical: $N = \frac{a}{\sqrt{1-e^2 \sin^2 \varphi}} = \frac{a}{W}$

p = distance from the polar axis to the point: $p = \sqrt{X^2 + Y^2} = (N + h) \cos \varphi$

R = distance from the origin to the point: $R = \sqrt{X^2 + Y^2 + Z^2}$

W = constant: $W^2 = 1 - e^2 \sin^2 \varphi$

X, Y, Z = Cartesian, geocentric coordinates of a point

β = reduced or parametric latitude (sometimes referred to as eccentric latitude)

λ = geodetic longitude

φ = geodetic latitude

Ψ = geocentric latitude

Looking at the meridional ellipse, one can see that three different latitudes can be displayed as shown in figure 1 [Rapp, 1989]. The geodetic latitude, φ , is the angle formed in the meridional plane where the normal to the ellipsoid intersects the equatorial axis. Note that this point is generally not at the center of the ellipse. The reduced latitude, β , is the angle measured at the center of the equivalent circle from the equatorial plane to point P' on the circumference of the circle (line OP' in figure 1). The reduced latitude is frequently referred to as the parametric latitude. Finally, the geocentric latitude, ψ , is the angle measured at the center of the ellipse from the equatorial axis to the point on the ellipse (line OP in figure 1). From Rapp [1989] numerous mathematical relationships between these latitudes can be developed. For example, one can write

$$\tan \psi = \frac{Z}{X} = (1 - e^2)^{1/2} \tan \beta = (1 - e^2) \tan \varphi$$

$$\tan \beta = (1 - e^2)^{1/2} \tan \varphi$$

$$\tan \varphi = (1 + e^2)^{1/2} \tan \beta$$

CARTESIAN-GEODETC COORDINATE RELATIONSHIPS

The direct problem involves the computation of the X-, Y-, and Z-coordinates of a point given the latitude, longitude, and height. These relationships are well known and can be derived with the help of the relationships shown in figure 1. The direct problem is solved using:

$$X = (N + h) \cos \varphi \cos \lambda$$

$$Y = (N + h) \cos \varphi \sin \lambda$$

$$Z = [N(1 - e^2) + h] \sin \varphi$$

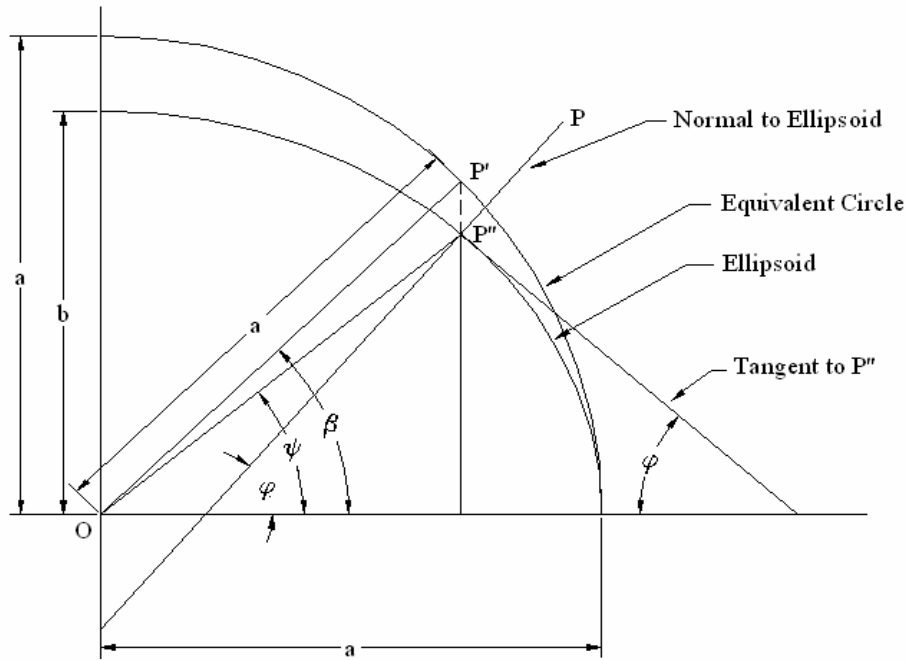


Figure 1. Meridional Ellipse.

The inverse association is not that easy, especially when solving for the latitude. The longitude can be easily computed by dividing Y by X. Hence,

$$\frac{Y}{X} = \frac{(N + h)\cos\varphi\cos\lambda}{(N + h)\cos\varphi\sin\lambda} = \tan\lambda$$

Therefore,

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right)$$

The complexity of the latitude computation lies in the fact that the radius of curvature in the prime vertical contains the latitude and one cannot easily isolate that variable without making some simplification of the formula. Because of this, two approaches have been devised. The first is an iterative approach while the second uses a direct solution.

ITERATIVE METHODS

Borkowski's Iterative Method

Borkowski [1989] presented an iterative approach where the successive approximations are performed on the reduced latitude (see also Gerdan and Deakin [1999] for a detailed description). Beginning with

$$p = a \cos \beta + h \cos \varphi$$

$$Z = b \sin \beta + h \sin \varphi$$

Rewrite these equations in terms of the height and geodetic latitude terms then divide the second with the first equation. Recognizing the relationship for the geodetic latitude in terms of the reduced latitude, we have

$$\frac{a}{b} \tan \beta = \frac{Z - b \sin \beta}{p - a \cos \beta}$$

Eliminating the denominator and multiplying the expression by 2,

$$2ap \sin \beta - 2bZ \cos \beta - 2(a^2 - b^2) \cos \beta \sin \beta = 0$$

Borkowski defines $\Omega = \tan^{-1}\left(\frac{bZ}{ap}\right)$ and $c = \frac{a^2 - b^2}{\sqrt{(ap)^2 + (bZ)^2}}$. Substitute

$\frac{\cos \Omega}{\sqrt{(ap)^2 + (bZ)^2}}$ for ap and $\frac{\sin \Omega}{\sqrt{(ap)^2 + (bZ)^2}}$ for bZ (Note the Gerdan and Deakin [1999] designate the denominator of both of these substitutions as q), we arrive at

$$f(\beta) = 2 \sin(\beta - \Omega) - c \sin(2\beta) = 0 \quad (1)$$

Solve this equation using the Newton-Raphson iterative technique. The first derivative of the function is

$$f'(\beta) = 2[\cos(\beta - \Omega) - c \cos(2\beta)] \quad (2)$$

For an initial estimate of the reduced latitude, Borkowski suggests using

$$\beta_o \cong \tan^{-1}\left(\frac{aZ}{bp}\right) \quad (3)$$

For points on the surface of the ellipsoid, (3) is exact. Evaluate the original function and the first derivative, equations (1) and (2), to arrive at the new estimate of the reduced latitude.

$$\beta_1 = \beta_o - \frac{f(\beta)}{f'(\beta)} \quad (4)$$

If the second term on the right hand side of (4) is sufficiently small $\left(\frac{f(\beta)}{f'(\beta)} \leq \text{criteria} \right)$ then the reduced latitude is β_1 as shown in (4). Otherwise, recalculate (1) and (2) again, this time using the updated value for β to arrive at a new estimate for β . Continue to iterate until the criteria is met. Once determined, the geodetic latitude is found using the formula

$$\varphi = \tan^{-1} \left[\left(\frac{a}{b} \right) \tan \beta \right] \quad (5)$$

Although the height can be determined using a number of different formulae, Borkowski suggests using the next formula because its accuracy over all ranges of latitudes.

$$h = (p - a \cos \beta) \cos \varphi + (Z - b \sin \beta) \sin \varphi \quad (6)$$

Borkowski [1989] claims that accuracies for the geodetic latitude at the 1×10^{-9} rad (0.0002") can be achieved in just two iterations for points that are over 1,000 km from the origin of the coordinate system. The results of his comparison with other methods are shown in figure 2. Note that the method identified at Heiskanen and Moritz [1967] is referred to as the Hirvonen and Moritz method in this paper and acknowledged as such by Heiskanen and Moritz (see also Rapp [1989]). Other methods mentioned here that will be discussed in this paper are those by Long [1975], Pick [1985], Heikkinen [Rapp, 1989], and Borkowski's exact method [1989].

Bowring's Iterative Method

Bowring [1976] presented an iterative procedure which has been described as a standard upon which other methods have been based. The principles are succinctly described in Rapp [1989]. Figure 3 shows the meridian ellipse. Point P is the location of the point and P'' is the corresponding point located at the intersection of the ellipsoid with the spheroidal normal. M is the radius of curvature in the meridian and C is the center of curvature of the meridian ellipse.

From figure 3,

$$x_C = x_{P''} - M \cos \varphi \quad (7)$$

From Rapp [1989], equation (3.42)

$$x_{P''} = \frac{a \cos \varphi}{W} = \frac{a \cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}$$

Substitute the value of $x_{P''}$ into (7) and recalling the equation for the radius of curvature in the meridian,

Position displacement errors (in mm) for ten algorithms. The numbers in brackets are for 3 iterations with procedures of Heiskanen and Moritz (1967), *Astronomical Almanac* (1987) and Bartelme and Meissl (1975), algorithms No. 1, 2 and 3, respectively. Other algorithms are based on approximate closed solutions: Izotov et al., (1974; No. 4), Long (1975; 5), Baranov et al., (1986; 6), Pick (1985; 7) and this paper (8), and exact solutions: Heikkinen (1982; 9) and this paper (10). The bottom line lists respective timing measurements in units of 0.1 ms.

φ deg	h km	Algorithm No.									
		1	2	3	4	5	6	7	8	9	10
89	100000	109(0)	0(0)	1(0)	721	190	1	925	.000000	.000015	.000000
89	10000	460(1)	0(0)	22(0)	721	158	0	126	.000004	.000002	.000000
89	1000	228(1)	0(0)	25(0)	721	857	0	0	.000000	.000000	.000000
89	0	0(0)	0(0)	-(-)	721	1219	0	0	.000000	.000000	.000000
89	-1000	431(3)	0(0)	64(0)	721	1836	0	1	.000000	.000001	.000001
70	100000	89(0)	0(0)	0(0)	550	112	248	13722	.000021	.000017	.000000
70	10000	377(1)	1(0)	10(0)	635	105	105	2094	.000000	.000000	.000000
70	1000	187(1)	6(0)	11(0)	660	370	5	6	.000000	.000001	.000000
70	0	0(0)	9(0)	-(-)	646	506	0	0	.000000	.000001	.000000
70	-1000	353(2)	12(0)	29(0)	611	734	10	34	.000000	.000000	.000001
45	100000	30(0)	1(0)	0(0)	167	24	452	10075	.000000	.000000	.000000
45	10000	128(0)	37(0)	0(0)	365	5	191	2683	.000000	.000000	.000000
45	1000	63(0)	181(1)	0(0)	587	215	9	14	.000000	.000001	.000001
45	0	0(0)	242(1)	-(-)	636	327	0	0	.000000	.000000	.000000
45	-1000	120(0)	340(1)	0(0)	695	519	18	81	.000000	.000000	.000000
20	100000	2(0)	2(0)	0(0)	13	22	91	13641	.000007	.000000	.000015
20	10000	7(0)	73(0)	10(0)	118	25	38	2090	.000000	.000000	.000002
20	1000	3(0)	358(2)	11(0)	424	72	2	7	.000000	.000001	.000001
20	0	0(0)	478(3)	-(-)	546	121	0	0	.000000	.000001	.000001
20	-1000	6(0)	672(5)	28(0)	736	205	4	35	.000000	.000001	.000000
1	100000	0(0)	0(0)	1(0)	0	1	0	922	.000015	.000015	.000009
1	10000	0(0)	5(0)	22(0)	5	1	0	126	.000000	.000000	.000000
1	1000	0(0)	25(0)	24(0)	25	10	0	0	.000000	.000001	.000000
1	0	0(0)	33(0)	-(-)	33	17	0	0	.000000	.000000	.000000
1	-1000	0(0)	47(0)	63(0)	47	27	0	1	.000000	.000001	.000000
Exec. time		56(78)	51(70)	50(63)	44	34	46	146	74	56	63

Figure 2. Results from Borkowski's evaluation of transformations from Cartesian to geodetic coordinates [Borkowski, 1989, p.53].

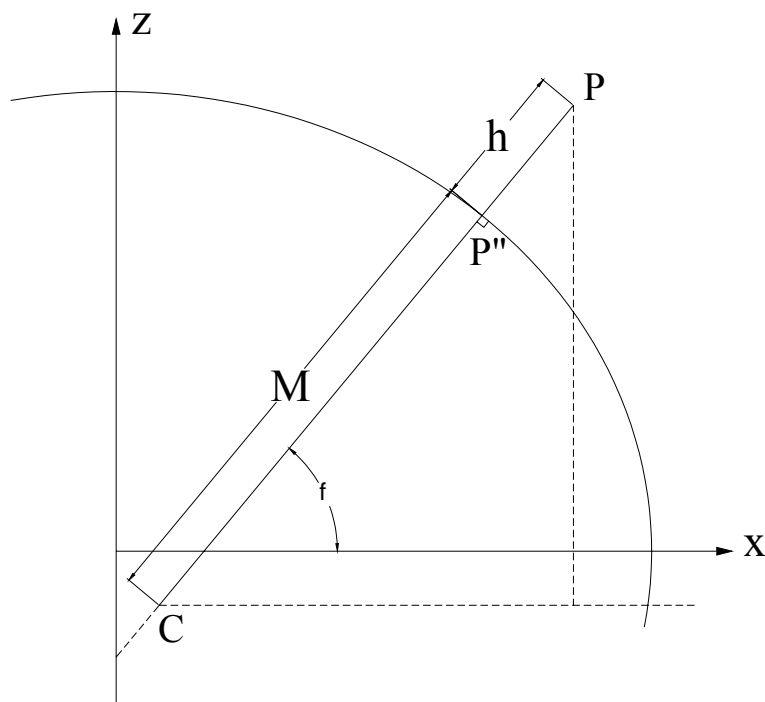


Figure 3. Meridian ellipse showing the relationship between the center of curvature in the meridian.

$$\begin{aligned}
x_C &= \frac{a \cos \varphi}{W} - \frac{a(1-e^2)\cos \varphi}{W^3} \\
&= \frac{a \cos \varphi}{W} \left[1 - \frac{1-e^2}{W^2} \right] = \frac{a \cos \varphi}{W} \left[\frac{e^2(1-\sin^2 \varphi)}{W^2} \right] \\
&= \frac{ae^2 \cos^3 \varphi}{W^3} = \frac{ae^2 \cos^3 \varphi}{(1-e^2 \sin^2 \varphi)^{3/2}}
\end{aligned} \tag{8}$$

Again from Rapp, equation (3.66)

$$\cos \varphi = W \cos \beta$$

Then, substitute this into (8) will give us the x-coordinate of the center of curvature of the meridian ellipse.

$$x_C = \frac{ae^2 W^3 \cos^3 \beta}{W^3} = ae^2 \cos^3 \beta \tag{9}$$

From figure 3, we can develop a similar formula for the z-coordinate of C. Write

$$z_C = z_{P''} - M \sin \varphi \tag{10}$$

From Rapp, equation (3.43)

$$z_{P''} = \frac{a(1-e^2)\sin \varphi}{W} = \frac{a(1-e^2)\sin \varphi}{\sqrt{1-e^2 \sin^2 \varphi}}$$

which, when substituted into (10) yields

$$\begin{aligned}
z_C &= \frac{a(1-e^2)\sin \varphi}{W} - \frac{a(1-e^2)\sin \varphi}{W^3} \\
&= \frac{a(1-e^2)\sin \varphi}{W} \left(1 - \frac{1}{W^2} \right)
\end{aligned} \tag{11}$$

In Rapp, equation (3.41), W^2 can be expressed as

$$W^2 = \frac{1}{1 + e^2 \sin^2 \beta}$$

Substitute into (11) yields

$$\begin{aligned}
z_C &= \frac{a(1-e^2)\sin\varphi}{W} [1 - (1 + e'^2 \sin^2 \beta)] \\
&= \frac{a(1-e^2)\sin\varphi}{W} (-e'^2 \sin^2 \beta)
\end{aligned}$$

But, Rapp shows that $\sin\varphi = V \sin\beta$ in (3.65), where $V = (1 + e'^2 \cos^2 \varphi)^{1/2}$ as shown in (3.40). This means

$$z_C = \frac{a(1-e^2)V \sin\beta}{W} (-e'^2 \sin^2 \beta) \quad (12)$$

Rapp shows in (3.49) that $\frac{V}{W} = \frac{a}{b}$. Substitute this into (12) yields the formula for the z-coordinate for the center of curvature of the meridian ellipse.

$$z_C = -be'^2 \sin^3 \beta \quad (13)$$

From figure 3, one can write the following relationship for the geodetic latitude.

$$\tan\varphi = \frac{z_P - z_C}{x_P - x_C} \quad (14)$$

Substitute the values of x_C and z_C in equations (9) and (13) respectively which yields

$$\tan\varphi = \frac{z_P + be'^2 \sin^3 \beta}{x_P - ae^2 \cos^3 \beta}$$

Express the values for x_P and z_P in terms of the Cartesian coordinates of the point. The solution for the latitude then can be expressed as [see Bowring equation (4)]:

$$\tan\varphi = \frac{Z + be'^2 \sin^3 \beta}{p - ae^2 \cos^3 \beta} \quad (15)$$

This formula is clearly iterative since both the geodetic and the reduced or parametric latitude are unknown. Bowring suggests using the next formula to determine the initial estimate of β .

$$\tan\beta = \left(\frac{Z}{p}\right)\left(\frac{a}{b}\right) \quad (16)$$

Thus, once (16) is computed, the value for β is substituted into (15) to arrive at the geodetic latitude. If a new value for the reduced latitude is needed then Rapp [1989] shows that it can be computed as

$$\tan \beta = (1 - f) \tan \varphi = \frac{b}{a} \tan \varphi \quad (17)$$

where φ is determined from (15). This value for β is then inserted into (15) to compute a new estimate of φ . Once the latitude is determined, the height can be computed from one of the following [see Bowring (8) and Rapp (6.98) and (6.99)].

$$h = \frac{p}{\cos \varphi} - N = \frac{Z}{\sin \varphi} - N + e^2 N \quad (18)$$

As both Bowring [1976] and Rapp [1989] point out, this method is very accurate for most terrestrial points. For example, Bowring shows that the error in latitude using his method can be defined as (Bowring's equation 6)

$$E_\varphi = \frac{3}{2} e^6 \left[\frac{ah^2}{(a+h)^3} \right] \sin^3 \varphi \cos^3 \varphi \quad (19)$$

Then, for earth-based points, the main error will be 0.000 000 030'' for a single iteration. The maximum latitude error is shown to be 0.0018'' when $h = 2a$. As Rapp points out, the error in height using the Bowring method is around 39 mm, at a height of 5,000 km. As Bowring [1985] states, the error in his method is not appropriate for outer-spaced scenarios. He identified the error in height to be

$$E_h = \left(\frac{p}{\cos^2 \varphi} - e' M \cos \varphi \right) \sin \varphi \Delta \varphi \quad (20)$$

Then, at the maximum latitude error of 0.0018'', the error in height is 17 cm. To alleviate this large of discrepancy, Bowring [1985] suggests using the following formula to calculate the height.

$$\left. \begin{aligned} h &= p \cos \varphi + Z \sin \varphi - \left(\frac{a^2}{N} \right) \\ &= R \cos(\varphi - \theta) - \left(\frac{a^2}{N} \right) \end{aligned} \right\} \quad (21)$$

where $\tan \theta = Z/p$.

Despite the fact that the latitude is a part of equation (21), Bowring shows that the error in height is insensitive to any error in latitude to the first order differential. In fact, he also shows that the worst case scenario for error is when $h = 4a$ and even then the error is -9.1×10^{-10} m. Bowring also addressed refining the algorithm for computing the latitude. In his 1976 paper, the initial estimate of the reduced latitude is as shown in (16). For a better approximation, Bowring [1985] suggests

$$\tan \beta = \frac{bZ}{ap} \left(1 + e' \frac{b}{R} \right) \quad (22)$$

Then using (15) solve for the geodetic latitude. This algorithm is estimated to be accurate to 0.000 000 1'' for any point, either terrestrial or outer space.

Rapp [1989] presents an alternative formula for the computation of height, which was developed by Bartelme and Meissl. The height is shown as

$$h^2 = (p - a \cos \beta)^2 + (Z - b \sin \beta)^2 \quad (23)$$

This formula is stable, although it does fail at the poles.

Fukushima's Iterative Method

As will be describe later in the section on closed form methods, Borkowski [1987, 1989] presents an alternative to the iterative method already discussed. He shows that the solution involves finding roots of a quartic equation shown as (see later section for what the variables represent):

$$t^4 + 2Et^3 + 2Ft - 1 = 0 \quad (24)$$

His solution was to employ Ferrari's formula. But, as Fukushima [1999] points out, the distance from the polar axis, p , which is a part of E and F , is ill-conditioned as p gets smaller. As the co-latitude tends towards zero, the error grows significantly. To solve the problem, Fukushima offers a different form for equation (24). We can write

$$f(t) \equiv pt^4 + ut^3 + vt - p = 0 \quad (25)$$

where: $t \equiv \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) = \tan \left[\frac{1}{2} \tan^{-1} \left(\frac{1}{e' \tan \varphi} \right) \right]$

$$u \equiv 2pE = 2(Z' - c)$$

$$v \equiv 2pF = 2(Z' + c)$$

$$c \equiv ae^2$$

$$Z' \equiv e'Z$$

$$E \equiv \frac{Z'-c}{p}$$

$$F \equiv \frac{Z'+c}{p}$$

The solution is referred to as an iterative trigonometric procedure performed using the Newton-Raphson method. Like any iterative approach, convergence can be accelerated by selecting a better estimate of the unknown parameters. Here, as in Borkowski, the evaluation is based on t which is the tangent of half the complement of the reduced latitude to 90° . Fukushima evaluates the first and second derivatives of (25), which yields (Fukushima equations 11 and 12)

$$f'(t) = 4pt^3 + 3ut^2 + v$$

$$f''(t) = 12pt^2 + 6ut$$

From this, it can be explicitly shown that the zeros for the second derivative are (Fukushima equation 12)

$$t = 0, \quad t = t_M \equiv \frac{-u}{2p} = \frac{c - Z'}{p}$$

Then, three different cases were identified depending upon T_M . In most cases, $t_M \leq 0$. In this case, the upper bound can be used as the initial estimate for t in the Newton-Raphson iteration. The initial estimate for t then becomes (Fukushima equation 13)

$$t = \frac{p - c + Z'}{p - c + 2Z'} \quad (26)$$

To look at the initial estimates for the other cases the reader is referred to Fukushima [1999], p.605.

The new estimate of t is

$$t \equiv t + \Delta t \quad (27)$$

where $\Delta t \equiv \frac{-f(t)}{f'(t)} = \frac{p - (pt^4 + ut^3 + vt)}{4pt^3 + 3ut^2 + v}$. Equation (27) is iterated until Δt is sufficiently small for the problem. Then, the latitude is found using:

$$\varphi = \tan^{-1} \left(\frac{2e't}{1-t^2} \right) \quad (28)$$

and the height is given as (Fukushima equation 9):

$$h = \frac{2pe't + Z(1-t^2) - ae'(1+t^2)}{\sqrt{(1+t^2)^2 - 4e^2t^2}} \quad (29)$$

or by using one of the forms of the height equation (21) given earlier.

Fukushima [1999] explains that the approach has several advantages to both the Borkowski and Bowring methods used in the comparison. First, the approach is faster than Bowring's despite the fact that it requires a few iterations to complete while Bowring's method needs, for all practical purposes, only a single iteration to find the latitude and height. A big part of the reason is that Fukushima requires no calls for transcendental functions in the iteration while Bowring requires 8-9 calls, including atan, tan, sin/cos, and sqrt. Secondly, Fukushima's method is reportedly accurate to 10^{-15} or 10 nm on the surface of the earth. Finally, this method is very stable for all types of inputs, from heights close to the geocenter to satellite altitudes, such as the Global Positioning Satellite constellation.

Hirvonen and Moritz Iterative Method²

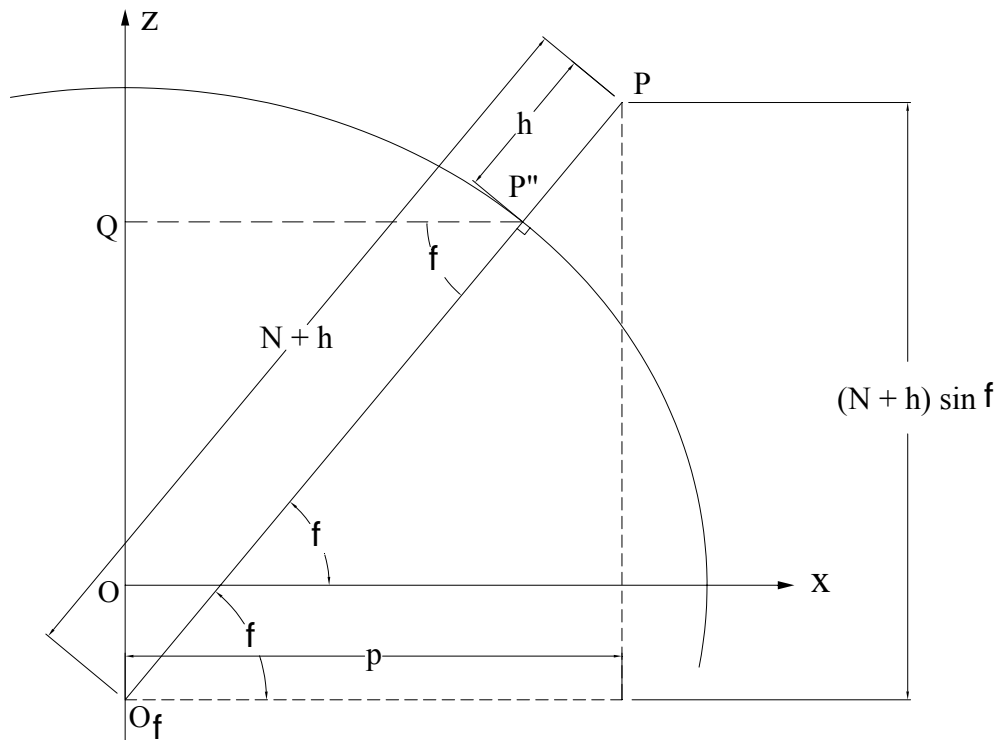


Figure 4. Meridian section showing the radius of curvature in the prime vertical (adapted from Rapp, 1989 and Pollard, 2002).

² Many authors refer to this as the Heiskanen and Moritz algorithm or method because it appears in their book. The authors themselves refer to the work by Hirvonen and Moritz within the textbook.

The iterative solution by Hirvonen and Moritz is described in Heiskanen and Moritz [1967] and Rapp [1989] and the geometry is presented in figure 4. From basic trigonometry we can write [Rapp, equation 6.94]:

$$\tan \varphi = \frac{(N + h) \sin \varphi}{p} \quad (30)$$

Recall earlier that it was shown that $Z = [N(1 - e^2) + h] \sin \varphi$, which, after rearranging, yields

$$(N + h) \sin \varphi = Z + e^2 N \sin \varphi$$

Substitute this into (30) gives us

$$\tan \varphi = \frac{Z + e^2 N \sin \varphi}{p} \quad (31)$$

This is the basic equation to iterate. Heiskanen and Moritz [1967] and Torge [2001] show the equation in a slightly different form:

$$\tan \varphi = \frac{Z}{p} \left(1 - e^2 \frac{N}{N + h} \right)^{-1} \quad (32)$$

In (32), both φ (which is imbedded in N) and h are unknowns on the right hand side of the equation. By setting $h = 0$, (32) becomes

$$\tan \varphi_0 = \left(\frac{Z}{p} \right) \left(\frac{1}{1 - e^2} \right) \quad (33)$$

Here, φ_0 indicates the initial estimate of the latitude. Use this value for the latitude to find the radius of curvature in the prime vertical. Then upgrade the estimate of the geodetic latitude using (31). If using the Heiskanen and Moritz form of the $\tan \varphi$, one needs to calculate the geodetic height using (18) [Heiskanen and Moritz give the first form of (18), namely $h = \frac{p}{\cos \varphi} - N$]. Evaluate $\Delta \varphi = |\varphi_0 - \varphi|$. If $\Delta \varphi$ is less than the desired criteria, the last value of the latitude is the correct value and a new value for the elevation is determined using (18). Otherwise, φ_0 is replaced by φ and the process continues.

From Rapp (equation 3.18) we can write $(1 - e^2)^{1/2} = (1 + e'^2)^{-1/2}$. Therefore, (33) can also be written as (see Gerdan and Deakin, 1999, equation 3.1):

$$\tan \varphi_0 = \frac{Z(1 + e'^2)}{p}$$

Lin and Wang Iterative Method

Like Fukushima and others, Lin and Wang [1995] recognized the computation burden that the Bowring method requires due to the use of trigonometric and inverse functions. Thus, they presented a more efficient method based employing the Newton-Raphson iterative procedure. The Lin and Wang approach also involves solving a quartic equation like Fukushima and Borkowski [Pollard, 2002]. To begin, the ellipsoid is modeled using

$$\frac{X_{P''}^2}{a^2} + \frac{Y_{P''}^2}{a^2} + \frac{Z_{P''}^2}{b^2} = 1 \quad (34)$$

where $X_{P''}$, $Y_{P''}$, $Z_{P''}$ are the geocentric coordinates of point P on the ellipsoid. The coordinates of any point on the normal to the ellipse passing through point P can be described from the parametric equation as:

$$\begin{aligned} x &= \frac{X}{1 + 2m/a^2} \\ y &= \frac{Y}{1 + 2m/a^2} \\ z &= \frac{Z}{1 + 2m/b^2} \end{aligned}$$

where m is a parameter that describes where along the normal the point (x, y, z) is located. For example, when $m = 0$, $x = X$, $y = Y$, and $z = Z$. Lin and Wang describe the problem as one of finding that value of m where $(x, y, z) = (X_{P''}, Y_{P''}, Z_{P''})$. Substitute the values for x, y, z into (34) gives us

$$\begin{aligned} f(m) &= \frac{X^2}{(a + 2m/a)^2} + \frac{Y^2}{(a + 2m/a)^2} + \frac{Z^2}{(b + 2m/b)^2} - 1 = 0 \\ &= \frac{p^2}{(a + 2m/a)^2} + \frac{Z^2}{(b + 2m/b)^2} - 1 = 0 \end{aligned} \quad (35)$$

Equation (35) is solved iteratively using the Newton-Raphson method. An initial estimate for m is given by Lin and Wang as

$$m_0 = \frac{ab(a^2 Z^2 + b^2 p^2)^{3/2} - a^2 b^2 (a^2 Z^2 + b^2 p^2)}{2(a^4 Z^2 + b^4 p^2)} \quad (36)$$

To begin the iteration process take the derivative of (35).

$$f'(m) = \frac{df(m)}{dm} = -4 \left[\frac{p^2}{a(a + 2m/a)^3} + \frac{Z^2}{b(b + 2m/b)^3} \right] \quad (37)$$

Compute a new estimate of m .

$$m_i = m_{i-1} \frac{f(m_{i-1})}{f'(m_{i-1})} \quad (38)$$

Equations (35), (37) and (38) are solved for iteratively until $f(m)$ converges towards zero (the criteria established by the user). Then, using the most current value for m , compute

$$p_{p''} = \left| \frac{p}{\left(1 + \frac{2m}{a^2}\right)} \right| \quad (39)$$

$$Z_{p''} = b \sqrt{1 - \frac{p_{p''}^2}{a^2}} \quad (40)$$

The latitude is found using

$$\varphi = \tan^{-1} \left(\frac{a^2 Z_{p''}}{b^2 p_{p''}} \right) \quad (41)$$

and the height is

$$h = \sqrt{(p - p_{p''})^2 + (Z - Z_{p''})^2} \quad (42)$$

According to Lin and Wang, the error in their algorithm has the same magnitude as the error found in Bowring's approach. Both methods also converge at about the same rate, but the algorithm here is about 50% faster than Bowring's approach. Gerdan and Deakin

[1999] conclude that from the methods they tested, the Lin and Wang approach was the most efficient method in terms of execution time.

Pollard's Height First Iterative Method

Pollard [2002] presented two vector methods from converting Cartesian coordinates to geodetic coordinates. As he points out that while the vector method is simple to follow because it does not require the solution of a quartic equation, it does require the separation of the height from the latitude determination thus making this approach iterative. The first method presented is based on the computation of the height first through an iterative process and, once determined, to compute the latitude directly.

The theory of Pollard's method begins with the general equation of an ellipse of revolution for the earth, written in general form as [note that this is a slightly different form of equation (34)]

$$x^2 + y^2 + z^2 \left(\frac{a}{b} \right)^2 = a^2$$

Designate $k_{P''} = \left(\frac{a}{b} \right)^2$. Then differentiate the general ellipsoid formula and substitute into the results the coordinates of point P'', which lies on the surface of the ellipsoid. This yields the direction numbers of the normal at P'' [Pollard, 2002]. Then, from figure 4, the distance $QO_{\varphi} = k_{P''} Z_{P''}$ and

$$N = \sqrt{X_{P''}^2 + Y_{P''}^2 + k_{P''}^2 Z_{P''}^2}$$

The unit normal rectangular coordinates, designated by Pollard as l, m, and n, are found using the following relationships

$$l = \frac{X_{P''}}{N} \quad ; \quad m = \frac{Y_{P''}}{N} \quad ; \quad n = \frac{k_{P''} Z_{P''}}{N}$$

The coordinates of P'' can then be shown in terms of the height, h, to be

$$X_{P''} = X - lh$$

$$Y_{P''} = Y - mh$$

$$Z_{P''} = Z - nh$$

Write the equation of the ellipse in terms of the coordinates of P''.

$$X_{P''}^2 + Y_{P''}^2 + k_{P''}^2 Z_{P''}^2 = a^2$$

Substitute the values of $X_{P''}$, $Y_{P''}$, and $Z_{P''}$ yields

$$(X - lh)^2 + (Y - mh)^2 + k_{P''}(Z - nh)^2 = a^2$$

This can be shown in a quadratic form as

$$(l^2 + m^2 + k_{P''}n^2)h^2 - 2(lX + mY + k_{P''}nZ)h + (X^2 + Y^2 + k_{P''}Z^2 - a^2) = 0$$

Pollard [2002] defines r , s , and t as

$$\begin{aligned} r &= l^2 + m^2 + k_{P''}n^2 = 1 + (k_{P''} - 1)n^2 = 1 + e'^2 n^2 \\ s &= lX + mY + k_{P''}nZ \\ t &= X^2 + Y^2 + k_{P''}Z^2 - a^2 \end{aligned}$$

Thus, the height can be found as

$$h = \frac{s \pm \sqrt{s^2 - rt}}{r}$$

As Pollard point out, the root with the minus sign yields the right answer.

The algorithm employed by Pollard can be outlined in the following steps. The problem is to find a solution to $Z_{P''}$ for the point on the ellipsoid where the normal to the ellipsoid passing through point P intersects the ellipsoid. For most points on the earth's surface, the $Z_{P''}$ coordinate can be approximated by the Z -coordinate of the point in question. For other points located farther away from the ellipsoid, Pollard recommends scaling the Z -coordinate and using an initial value of

$$Z_{P''_{ini}} = \frac{bZ}{R} \quad (43)$$

Make the initial value of Z on the ellipse, $Z_{P''_{ini}} = Z_{P''}$.

$$k_{P''} = \left(\frac{a}{b}\right)^2$$

The estimated distance from P'' to O_φ is found using

$$|P''O_\varphi| = \sqrt{X^2 + Y^2 + (Z + e'^2 Z_{P''})^2} \quad (44)$$

$$l = \frac{X}{|P''O_\varphi|} \quad m = \frac{Y}{|P''O_\varphi|} \quad n = \frac{Z + e'^2 Z_{P''}}{|P''O_\varphi|} \quad (45)$$

$$\left. \begin{aligned} r &= l^2 + m^2 + k_{p''} n^2 \\ s &= lX + mY + k_{p''} Z \\ t &= X^2 + Y^2 + k_{p''} Z^2 - a^2 \end{aligned} \right\} \quad (46)$$

$$h = \frac{s - \sqrt{s^2 - rt}}{r} \quad (47)$$

$$Z_{p''} = Z - nh \quad (48)$$

Compare this new estimate for the Z -coordinate for point P'' with the initial estimate. If the difference is within the desired criteria, then the geodetic height is as determined in (46). Otherwise, using the new estimate for $Z_{p''}$ and recalculate the variables in equations (43) – (47). This iterative process is continued until the solution converges. Then, once the geodetic height has been finalized, compute the geodetic latitude using the algorithm as follows. Calculate the coordinates of the point of intersection of the normal to the ellipse passing through the point P with the ellipse ($X_{p''}$ and $Y_{p''}$).

$$\begin{aligned} X_{p''} &= X - lh \\ Y_{p''} &= Y - mh \end{aligned}$$

Next, compute the length of the normal, N , using

$$N = |P''O_{\varphi}| - h$$

Then, the latitude is found using

$$\varphi = \tan^{-1} \left(\frac{Z + e'^2 Z_{p''}}{p} \right) \quad (49)$$

Pollard's Latitude First Iterative Method

Pollard [2002] presented an alternative iterative vector approach to finding the geocentric to geodetic coordinate transformation where the latitude is created first. The algorithm presented by Pollard is as follows. Begin by computing the initial value of the Z -coordinate along the normal at the ellipsoid (43). Then compute the tangent of the parametric latitude, t .

$$t = \tan \beta = \frac{b}{a} \cdot \frac{Z + e'^2 Z_{p''}}{p} \quad (50)$$

Update the estimate of $Z_{p''}$.

$$Z_{p''} = b \frac{t}{\sqrt{1+t^2}} \quad (51)$$

Iterate on $Z_{p''}$ using either a simple iteration or, as recommended by Pollard, a Newton-Raphson method where the new estimate of $Z_{p''}$ is found by

$$Z_{p''_{New}} = \frac{\frac{bt}{\sqrt{1+t^2}} - \frac{bZ_{p''}t'}{\left(\sqrt{1+t^2}\right)^3}}{1 - \frac{bt'}{\left(\sqrt{1+t^2}\right)^3}} \quad (52)$$

where, $t' = \frac{dt}{dZ_{p''}} = \frac{be'^2}{ap}$.

Iterate until the discrepancy between the estimate on the right hand side of (52) for $Z_{p''}$ and the new value for $Z_{p''}$ on the left hand side are sufficiently small. Once done, compute the latitude and height using:

$$\varphi = \tan^{-1} \left(\frac{Z + e'^2 Z_{p''}}{p} \right) \quad (53)$$

$$h = \frac{p}{\cos \varphi} - N \quad (54)$$

Pollard [2002] compared his methods with Bowring's approach in terms of errors and speed. For terrestrial points, the error in latitudes are comparable. In terms of height errors, Pollard found the errors in his approach are lower than those found in Bowring's formulas. In terms of execution time, all algorithms had similar levels of performance. Pollard's vector methods were slightly better when the programs were optimized.

Barteleme and Meissl's Iterative Method

Rapp [1976], in his documentation for the Fortran program XYZ2PLH, gives a concise overview of the algorithm attributed to Barteleme and Meissl³. The basic equation to iterate is

³ Although not identified specifically, it is believed that the reference is (see for example the references in Sjöberg [1999]): Bartelme, N., and P. Meissl, 1975. "Ein einfaches, rasches und numerisch stabiles Verfahren zur Bestimmung des kürzesten Abstandes eines Punktes von einem sphäroidischen Rotationsellipsoid", Allg. Verm. Nachr., 12: 436-439.

$$1 + \frac{\mu^{k+1}}{R^2} = \sqrt{\left(\frac{p}{a}\right)^2 \left(\frac{1 + \frac{\mu^k}{R^2}}{1 + \frac{\mu^k}{a^2}}\right)^2 + \left(\frac{Z}{b}\right)^2 \left(\frac{1 + \frac{\mu^k}{R^2}}{a + \frac{\mu^k}{b^2}}\right)^2} \quad (55)$$

Iterations begins with $\mu^k = 0$. Rapp states that the speed of the convergence depends on the selection of R. The method for calculating R from Bartelme and Meissl is

$$R^2 = \frac{a^2}{1 + \frac{e'^2 Z^2}{Z^2 + p^2}} \quad (56)$$

or using the approximation

$$R^2 = ab \quad (57)$$

Rapp states the Vincenty offers an alternative form that speeds up the convergence. This is

$$R^2 = a^2 \left\{ 1 - \frac{\left[1 - \left(\frac{b}{a} \right)^2 \right] Z^2}{p^2 \left(\frac{b}{a} \right)^4 + Z^2} \right\} \quad (58)$$

Once the algorithm converges, then the geodetic height is found using

$$h = \sqrt{(p - p_E)^2 + (Z - Z_E)^2} \quad (59)$$

where $p_E = p \left(1 + \frac{\mu}{a^2} \right)^{-1}$

$$Z_E = Z \left(1 + \frac{\mu}{b^2} \right)^{-1}$$

Finally, the geodetic latitude is found using

$$\varphi = \tan^{-1} \left(\frac{Z_E a^2}{p_E b^2} \right) \quad (60)$$

Sjöberg's Iterative Method

The iterative method developed by Sjöberg [1999] begins in a similar fashion as that by Hirvonen and Moritz. He begins by expressing equation (31) in the following form

$$p \tan \varphi - Z = \frac{e^2 a \sin \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} \quad (61)$$

Squaring both sides of (61) and making the following substitutions

$$\alpha = \tan \varphi \quad \alpha_{oo} = Z/p \quad \delta = \frac{e^2 a}{p}$$

We can, after some manipulation, arrive at Sjöberg's equation (7):

$$(\alpha - \alpha_{oo})^2 [1 + (1 - e^2) \alpha^2] - \delta^2 \alpha^2 = 0 \quad (62)$$

Sjöberg acknowledges that equation (62) does become unstable in polar regions. In those cases, he suggests making the following substitutions:

$$\beta = \alpha^{-1} = \cot \varphi \quad \beta_{oo} = p/Z \quad \gamma = \frac{e^2 a}{Z}$$

Then, (62) can be shown as

$$(\beta - \beta_{oo})^2 [\beta^2 + (1 - e^2)] - \gamma^2 \beta^2 = 0 \quad (63)$$

Sjöberg's method is iterative in finding an acceptable value for either α or β . For non-polar regions, an initial estimate for α is found using equation (33) noting that Sjöberg's notation denotes $\alpha = \tan \varphi$. Then, the iteration is shown as

$$\alpha_{k+1} = \alpha_{oo} + \delta \frac{\alpha_k}{\sqrt{1 + A \alpha_k^2}} \quad (64)$$

where α_{oo} and δ are given above and $A = 1 - e^2$. The main difference between (63) and (32) is that Sjöberg's method does not require iterations involving transcendental functions, like the arctangent. Instead, each iteration uses the tangent value for the latitude and only at the end does one find the actual angular value for the latitude. This speeds up the algorithm processing and Sjöberg [1999] indicates that this new method performed faster than the Hirvonen and Moritz method.

For the polar regions, Sjöberg suggests using

$$\beta_0 = \frac{p}{Z}(1 - e^2) \quad (65)$$

which is the cotangent of φ_0 as identified earlier in equation (33). Then the equation to iterate is

$$\beta_{k+1} = \beta_0 - \gamma \frac{\beta_k}{\sqrt{A + \beta_k^2}} \quad (66)$$

where the variables have been defined above. Sjöberg [1999] tested this method with those iterative methods identified in Borkowski [1989] and found that his method performed better. It works for all latitudes and heights close to the surface of the earth. He also compared his iterative method to the two closed methods developed by Fotiou [1998], which will be presented later. He found the height method performed much better than the distance method and recommends the height method as a possible alternative approach to the method he presented.

Seemkooei's Iterative Method

Seemkooei [2002] presented an iterative method that is very similar to that of Hirvonen and Moritz. Recall from (31) that

$$\tan \varphi = \frac{Z + e^2 N \sin \varphi}{p} \quad (31)$$

This can be rewritten as (Seemkooei equation 11)

$$p \sin \varphi = Z \cos \varphi + Ne^2 \sin \varphi \cos \varphi (\sin^2 \varphi + \cos^2 \varphi)$$

or as

$$\tan \varphi = \frac{Z + Ne^2 \sin^3 \varphi}{p - Ne^2 \cos^3 \varphi} \quad (67)$$

Equation (67) is solved iteratively. An initial estimate of the latitude is given in equation (33). Note that this equation can be easily transformed into the form depicted by Hirvonen and Moritz.

Seemkooei [2002] compared his method with those of Bowring, Borkowski (iterative), Lin and Wang, and Hirvonen and Moritz, along with a closed form where he solves the quartic equation. The accuracy, as measured in terms of the latitude, were slightly better using Seemkooei's method, but not significantly. He did find in his experiments that the closed form was considerably less precise than the iterative methods tested. In terms of

speed in calculations⁴, Seemkoeei found that Bowring's method only required one iteration and was the fastest algorithm employed, about 35% faster than his own method (using average times). The next fastest was Seemkoeei, which required 2.3 iterations on average, followed by Borkowski, which averages 2 iterations, followed by the closed form solution, then Hirvonen and Moritz, which required about 3.2 iterations as an average, with Lin and Wang's algorithm being the slowest, requiring only about 1.4 iterations on average.

CLOSED FORM METHODS

Borkowski's Direct Method

In addition to the iterative solution, Borkowski [1987, 1989] developed a closed or direct solution to the conversion of Cartesian to geodetic coordinates. It was shown in Borkowski's Iterative Method section in equation (1) that

$$2 \sin(\beta - \Omega) - c \sin(2\beta) = 0$$

Expressing this equation in terms of $\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right) \equiv t$ ⁵, a fourth-degree quartic or biquadratic equation can be developed. This is shown as

$$t^4 + 2Et^3 + 2Ft - 1 = 0 \quad (68)$$

where Borkowski defines

$$E = \frac{bZ - (a^2 - b^2)}{ap} \quad (69a)$$

$$F = \frac{bZ + (a^2 - b^2)}{ap} \quad (69b)$$

Using Ferrari's formula for the solution, we have

$$t = \pm \sqrt{G^2 + \frac{F - vG}{2G - E}} - G \quad (70)$$

where

$$v = \sqrt[3]{\sqrt{D} - Q} - \sqrt{\sqrt{D} + G} \quad (71a)$$

$$G = \pm \frac{\sqrt{E^2 + v} + E}{2} \quad (71b)$$

⁴ See Seemkoeei [2002] for specifics on experimentation he performed.

⁵ This is a tangent function of half the complement of the reduced latitude to 90°.

$$D = P^3 + Q^2 \quad (71c)$$

$$P = \left(\frac{4}{3}\right)(E^2 - F^2) \quad (71d)$$

$$Q = 2(E^2 - F^2) \quad (71e)$$

It is clear from the \pm in (70) and (71b) that the quartic solution contains four solutions. While not significant, if $D < 0$, which occurs when the point is less than about 45 km from the center, Borkowski does suggest using the following equation for v :

$$v = 2\sqrt{(-P)} \cos \left\{ \frac{1}{3} \cos^{-1} \left[\frac{Q}{P} \sqrt{(-P)} \right] \right\}$$

The final solution for the latitude and height can be found using:

$$\varphi = \tan^{-1} \left[\frac{a(1-t^2)}{2bt} \right] \quad (72)$$

$$h = (p - at) \cos \varphi + (Z - b) \sin \varphi \quad (73)$$

Borkowski states that using the positive square roots in (70) and (71b) will give the desired answer for most applications provided that $a > b$ and $\varphi > 0^\circ$. The algorithm does not work when $\varphi = 90^\circ$. For latitudes in the southern hemisphere the user must apply a negative value to the semi-minor axis (i.e., $-b$). Another weakness in the algorithm is the cube root required in (53a). The approach by Borkowski found round-off problems close to the Z-axis. He points out that Heikkinen's approach is free from this problem, even though the cube root is used in Heikkinen's method, but the trade-off is that Heikkinen's algorithm cannot be used when $D < 0$. To solve Borkowski's problem, he suggests using the cubic resolvent, v' (one of the three roots) to compute v as

$$v = -\frac{(v'^3 + 2Q)}{3P}$$

CONCLUSION

This paper presents a general overview of some methods used in the transformation from Cartesian coordinates to geodetic coordinates. Two general approaches are employed in these types of transformations: iterative or direct. The latter approaches employ some form of approximation to arrive at the desired solution.

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