MATH 249 Introductory Calculus

Examples on Limits of Functions: The Squeeze Theorem

I assume that you have worked through the basics of calculating limits and one-sided limits. Here, we want to look at another useful technique of finding limits. Let's with an example.

Example 1a. Find the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

Discussion. We know that one of the limit theorems says that the limit of a product is the product of the limits. That, of course, assumes that the limits of the factors exist in the first place. So, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)\,g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$. If we try and apply this theorem, we would say the limit of $x^2\sin\frac{1}{x}$ is the limit of x^2 times the limit of $\sin\frac{1}{x}$. The problem with this attempt is that the limit of $\sin\frac{1}{x}$ does not exist! It is not hard to see why the limit of $\sin\frac{1}{x}$ does not exist. Here, x is moving toward 0. This means that x is getting smaller and smaller. This implies that $\frac{1}{x}$ keeps growing in size indefinitely. But then $\sin\frac{1}{x}$ will represent the y-coordinate of a point on the unit circle that keeps moving round and round unceasingly. But this y-coordinate will be bouncing back and forth between the two (vertical) ends of the unit circle, namely -1 and 1. It just keeps oscillating like that and it never goes toward any place. So it has no limit.

Now, the fact that $\sin\frac{1}{x}$ does not have a limit as x goes to 0 means that the limit theorem for product cannot be applied! So what can we do? Well we have seen many situations where the limit theorems cannot be applied right away. Standard scenarios are when the denominator goes to zero, remember? There we found ways to rewrite the function in a form so that we can then apply the limit theorems, right? What about here in this example. The terms involved are x^2 and $\sin\frac{1}{x}$, and as far as we can recall and/or search there doesn't seem to be any identity that allows us to rewrite $\sin\frac{1}{x}$, so we are doomed!?

Now, let's sit back and ask, What's the problem here? Ah, it has to do with the indefinite oscillating behavior of $\sin\frac{1}{x}$. Well we can't say "Don't Wobble", but perhaps we can keep it under some kind of control? The answer is YES. Let's notice that even though $\sin\frac{1}{x}$ oscillates, it oscillates between fixed bounds, namely -1 and 1. That turns out to be exactly what we need to keep $\sin\frac{1}{x}$ under control and pursue the limit of $x\sin\frac{1}{x}$. Here are the details.

Solution. We know that for all x, as long as $x \neq 0$ so we can talk about $\frac{1}{x}$, we have

$$-1 \le \sin\frac{1}{x} \le 1.$$

What does it imply about the function $x^2\sin\frac{1}{x}$? Well, just multiply the inequality throughout by x^2. We get this:

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2, \; x \neq 0.$$

Notice that x^2 is always positive, and so when we multiply it to the inequality, we do not need to turning the inequality signs around. [Remember? If $a \leq b$ and $c > 0$, we have $ac \leq bc$, but if $c < 0$, we have $ac \geq bc$ instead.] Now, this says that the graph of the function $x^2 \sin \frac{1}{x}$ lies between the graphs of $-x^2$ and x^2 everywhere! Well... except at $x = 0$, because there the function $x \sin \frac{1}{x}$ is undefined. But if we look at the graphs of $-x^2$ and x^2 near 0, we see that both graphs tend to get near the origin. In fact, as $x \to 0$, both $-x^2 \to 0$ and $x^2 \to 0$. This does not leave much room for $x^2 \sin \frac{1}{x}$ to go! Being sandwiched between two graphs that merge together as $x \to 0$, the graph of $x^2 \sin \frac{1}{x}$ is forced to go there too! So, we can safely conclude that it also has limit 0. That is,

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

Reflection. The way we solved the above example suggests that we can write down a principle about a general "sandwiching" scenario. It is called the **Sandwich Theorem** or the **Squeeze Theorem**.

Theorem. Suppose that $f(x) \leq g(x) \leq h(x)$ for all x close to a, except possibly for $x = a$, and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

So, we interpret a significant use of the Sandwich Theorem as follows. We have a function g, and we want to find its limit at a. But somehow we find it impossible or hard to apply the normal limit theorems to find this limit. But "fortunately" we realize that abnormal it may be, g is bounded between two functions f and h which happen to be simple enough to handle. Furthermore, we notice that f and h tend to merge together as x approaches a. That leaves no room for g to go, and it has to go to the place where f and h converge. Simple as it may sound, a successful application of the theorem requires us to be "fortunate" enough to have f and h. And of course they are usually not given to us. After all, we are often just asked the find the limit of g. Period. No extra information. No extra hints. No f or h. Right? Yes, and that means we have to come up with f and h ourselves if we want to use the Sandwich theorem effectively. And that means we need to know basic features of standard classes of functions such as the power functions, exponential and logarithmic functions, and the trigonometric functions. The more we know about the standard functions, the better we can make use of the Sandwich theorem.

Let me show you another one. It is similar to the above example. It involves the oscillating function cosine. Note also that the Sandwich Theorem is clearly valid for one-sided limits as well as ordinary limits.

Example 1b. Find $\lim_{x \to 0^+} \sqrt{x} \cos \left(x + \frac{1}{x} \right)$.

Solution. We know that the cosine of any angle is between -1 and 1 since it represents the x-coordinate of a point on the unit circle. So,

$$-1 \leq \cos \left(x + \frac{1}{x} \right) \leq 1$$

for all $x > 0$. Now, multiply throughout by $\sqrt{x}$, we get

$$-\sqrt{x} \le \sqrt{x}\cos\left(x + \frac{1}{x}\right) \le \sqrt{x}.$$

We do not need to switch the inequality signs since $\sqrt{x} > 0$ always.

Now, since both $-\sqrt{x} \to 0$ and $\sqrt{x} \to 0$ as $x \to 0^+$, it follows from the Sandwich Theorem that

$$\lim_{x\to 0^+} \sqrt{x}\cos\left(x + \frac{1}{x}\right) = 0.$$

Want to try and write up the proof for the following?

Practice 1a. Use the Sandwich Theorem to argue that $\lim_{x\to 0^+} x^{1/3}\cos\left(2 + \frac{1}{x}\right) = 0.$

Solution. We start by observing that

$$-1 \le \cos\left(2 + \frac{1}{x}\right) \le 1$$

for all $x \neq 0$. Then if $x > 0$, then $x^{1/3} > 0$ and so

$$-x^{1/3} \le x^{1/3}\cos\left(2 + \frac{1}{x}\right) \le x^{1/3}.$$

Now, as $x \to 0^+$, we have both $-x^{1/3} \to 0$ and $x^{1/3} \to 0$. It follows from the Sandwich Theorem that $x^{1/3}\cos\left(2 + \frac{1}{x}\right) \to 0$ as well. That is,

$$\lim_{x\to 0^+} x^{1/3}\cos\left(2 + \frac{1}{x}\right) = 0.$$

Try one more?

Practice 1b. Use the Sandwich Theorem to prove that $\lim_{x\to 0^-} x\sin\frac{\sqrt{x+2}}{x}$.

Solution. We start by observing that

$$-1 \le \sin\frac{\sqrt{x+2}}{x} \le 1$$

for all x for which the expression $\sin\frac{\sqrt{x+2}}{x}$ is defined, i.e., for all $x \ge -2$ with $x \neq 0$. Now, if $-2 \le x < 0$, then multiplying the above inequality by x demands a switch in the inequality signs:

$$-x \ge x\sin\frac{\sqrt{x+2}}{x} \ge x.$$

Now, as $x \to 0^-$, we have both functions $x \to 0$ and $(-x) \to 0$. It follows from the Sandwich Theorem that $x\sin\frac{\sqrt{x+2}}{x} \to 0$ as well. That is,

$$\lim_{x\to 0^-} x\sin\frac{\sqrt{x+2}}{x} = 0.$$

You wouldn't think that the Sandwich Theorem always gives the answer 0? Let's look at the following example, which deals with limit at infinity. I suppose it is easy to see why the Sandwich Theorem is also valid in this case.

Example 2. Find $\lim_{x\to\infty} \frac{3x - \sin x}{4x+5}$.

Solution. We know that

$$-1 \le \sin x \le 1$$

for all values of x. Multiplying through by -1 and switching the inequality signs, we have

$$1 \ge -\sin x \ge -1.$$

Adding $3x$ to all the terms, we get

$$3x + 1 \ge 3x - \sin x \ge 3x - 1.$$

Now, dividing by $4x + 5$, we get

$$\frac{3x+1}{4x+5} \ge \frac{3x - \sin x}{4x+5} \ge \frac{3x-1}{4x+5}.$$

[Shouldn't we thought about if we have to switch the inequality signs?] Now, for limits at infinity, we have

$$\lim_{x\to\infty} \frac{3x+1}{4x+5} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{4+\frac{5}{x}} = \frac{3+0}{4+0} = \frac{3}{4},$$

and

$$\lim_{x\to\infty} \frac{3x-1}{4x+5} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{4+\frac{5}{x}} = \frac{3-0}{4+0} = \frac{3}{4}.$$

It follows from the Sandwich Theorem that

$$\lim_{x\to\infty} \frac{3x - \sin x}{4x+5} = \frac{3}{4}.$$

Practice 2a. Find $\lim_{x\to\infty} \frac{x + 7\sin x}{-2x+13}$.

Solution. We know that

$$-1 \le \sin x \le 1$$

for all x. So,

$$\begin{aligned} -7 &\le 7\sin x \le 7, \\ x - 7 &\le x + 7\sin x \le x + 7. \end{aligned}$$

Dividing through by $-2x + 13$, we get

$$\frac{x-7}{-2x+13} \ge \frac{x + 7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13}$$

for all x that are large. [Why did we switch the inequality signs?]

Now,

$$\lim_{x\to\infty} \frac{x-7}{-2x+13} = \lim_{x\to\infty} \frac{1-\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1-0}{-2+0} = -\frac{1}{2},$$

and

$$\lim_{x\to\infty}\frac{x+7}{-2x+13}=\lim_{x\to\infty}\frac{1+\frac{7}{x}}{-2+\frac{13}{x}}=\frac{1+0}{-2+0}=-\frac{1}{2}.$$

Practice 2b. Find $\lim\limits_{x\to-\infty}\dfrac{5x+2\cos x}{3x-14}$.

Solution. We know that

$$-1\le\cos x\le 1$$

for all x. So,

$$\begin{aligned}-2 &\le 2\cos x\le 2,\\ 5x-2 &\le 5x+2\cos x\le 5x+2.\end{aligned}$$

Dividing by $3x-14$, we get

$$\frac{5x-2}{3x-14}\ge\frac{5x+2\cos x}{3x-14}\ge\frac{5x+2}{3x-14}$$

for x large and negative. Now,

$$\lim_{x\to-\infty}\frac{5x-2}{3x-14}=\lim_{x\to-\infty}\frac{5x+2}{3x-14}=\frac{5}{3}.$$

It follows that

$$\lim_{x\to-\infty}\frac{5x+2\cos x}{3x-14}=\frac{5}{3}.$$

Example 3. Suppose that f is a function that $9x\le f(x)\le(x+1)(x+4)$ for all x that are near 2 but not equal to 2. Show that this fact contains enough information for us to find $\lim\limits_{x\to 2}f(x)$. Also, find this limit.

Solution. It is easy to see that

$$\lim_{x\to 2}9x=9(2)=18,$$

and

$$\lim_{x\to 2}(x+1)(x+4)=(2+1)(2+4)=18.$$

This has enough information for us to find $\lim\limits_{x\to 2}f(x)$. Indeed, it follows from the Sandwich Theorem that

$$\lim_{x\to 2}f(x)=18.$$

Practice 3a. Suppose that f is a function that $2x^2\le f(x)\le x(x^2+1)$ for all x that are near 1 but not equal to 1. Show that this fact contains enough information for us to find $\lim\limits_{x\to 1}f(x)$. Also, find this limit.

Solution. We see that

$$\lim_{x\to 1}2x^2=2(1)^2=2,$$

and

$$\lim_{x\to 1}x\left(x^2+1\right)=1\left(1^2+1\right)=2.$$

This is enough for us to find $\lim\limits_{x\to 1}f(x)$. Indeed, it follows from the Sandwich Theorem that

$$\lim_{x\to 1}f(x)=2.$$

Practice 3b. Suppose that f is a function that $2(3x-1) \leq f(x) \leq x(x^2+3)$ for all x that are near 1 but not equal to 1. Show that this fact contains enough information for us to find $\lim_{x\to 1} f(x)$. Also, find this limit.

Answer. 4.

Remarks.

1. In the statement of the Sandwich Theorem, we assume that $f(x) \leq g(x) \leq h(x)$ for all x near a, "except possibly at a". This means that it is not required that when $x = a$, we have the inequality for the functions. That is, it is not required that $f(a) \leq g(a) \leq h(a)$. The reason is that we are dealing with limits as x approaches a. So, we have x that is moving closer and closer to a. As long as $f(x) \leq g(x) \leq h(x)$ is true for all these x, we can be sure that the limit, i.e., the point where the function values are heading, must behave as the Sandwich Theorem indicates. In particular, unless we are given extra information about the functions and their values at a, the Sandwich Theorem does not allow us to make conclusions about function values at a. So, none of the following claims can be guaranteed by the assumptions in the Sandwich Theorem:

(a) $f(a) = g(a) = h(a)$. [Well, not even $f(a) \leq g(a) \leq h(a)$]

(b) $g(a) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.

(c) $\lim_{x\to a} g(x) = g(a)$.

etc.

2. The Sandwich Theorem allows us to draw a conclusion about $\lim_{x\to a} g(x)$ when $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. This is because the graph of g is being forced to head to that point where f and h converge. But what if $\lim_{x\to a} f(x) < \lim_{x\to a} h(x)$, i.e., the graphs of f and h do not converge as $x \to a$? Can we conclude that since the graph of g lies between f and h, the limit of g at a must lie between those of f and g? Well, yes and no. The catch here is that if we only know $\lim_{x\to a} f(x) < \lim_{x\to a} h(x)$, then snce g lies between them, its limit should lie between them *if g actually has a limit there*! The inequality $\lim_{x\to a} f(x) < \lim_{x\to a} h(x)$ allows a gap between the bounds so that there is no guarantee that g does not oscillate in such a fashion that it does not have a limit at a. An example we ahve already encountered is this: the funciton $g(x) = \sin\frac{1}{x}$ lies between -1 and 1. The limits of -1 and 1 as x approaches zero are of course -1 and 1, but g oscillates indefinitely within these bounds in such a way that it does not go toward a limit as x approaches 0. So, we can phrase the fact as follows:

Theorem. Suppose that $f(x) \leq g(x) \leq h(x)$ for all x close to a, except possibly for $x = a$, and if $\lim_{x\to a} f(x) \leq \lim_{x\to a} h(x)$ both exist, then *if* $\lim_{x\to a} g(x)$ *exists*, we have $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$.
