Background

Equations for tsunami gravity waves, based on linear theory (see the illustration at the end of this note):
We want to look at the energy flux along the long waves.

We assume that the velocity potential can be given by the Laplace equation:

$$\nabla^2\varphi = 0$$

With the following boundary conditions:

At z = 0:

$$\frac{\partial\varphi}{\partial t} + g\eta = 0$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t}$$

At z = -H:

$$\frac{\partial\varphi}{\partial z} = 0$$

It can be shown that the following are the solutions for the above differential equation with the given boundary conditions:

$$\eta = a\sin(kx - \omega t)$$

$$\varphi = -\frac{a\omega}{k\sinh(kH)}\cosh[k(z+H)]\cos(kx - \omega t)$$

Where ω is the angle velocity and k is the wave number defined by $\lambda = \frac{2\pi}{k}$ and λ is the wavelength.
Gravity ocean waves are dispersive waves which means that the phase velocity depends on the wavelength.
The dispersion relation is given by:

$$\omega^2 = gk\tanh(kH)$$

And g is the acceleration of gravity.

The phase velocity for the gravity waves is:

$$c = c_0\sqrt{\frac{\tanh(kH)}{kH}}\ ,\ c_0 = \sqrt{gH}$$

Long wave approximation, tsunamies

The phase velocity above can be written as follows, remember that: $\lambda = \frac{2\pi}{k}$

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh(2\pi\frac{H}{\lambda})} \qquad (1)$$

The gravity waves generated by earthquake in the ocean have a wavelength much larger than the depth. This means $\lambda >> H$, then $\tanh(2\pi\frac{H}{\lambda}) \approx 2\pi\frac{H}{\lambda}$

Then the equation (1) is reduced to:

$$c = \sqrt{gH} = c_g \quad (2)$$

And c_g is the group velocity. It can be shown that the wave energy propagates by the group velocity.

The energy density per area is given by:

$$E = \frac{1}{2}\rho g A^2 \quad (3)$$

and ρ is the density of the water in the ocean and A is the amplitude of the wave.

The energy flux is: $F = Ec_g \quad (4)$

Equations (2) and (3) gives this form in equation (4):
The energy flux:

$$\boxed{F = \frac{1}{2}\rho g^{\frac{3}{2}} A^2 \sqrt{H}}$$

A reasonable assumption is that we have a constant energy flux: $F = F_0$
So the amplitude relation is:

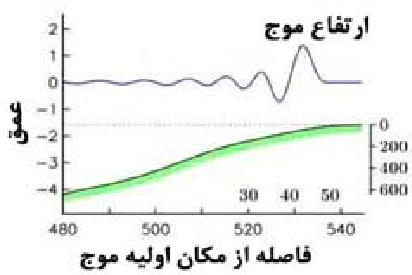
$$\boxed{A = A_0 (\frac{H_0}{H})^{\frac{1}{4}}}$$

Where A_0 and H_0 are the amplitude of the wave and the ocean depth at a reference point in the ocean.

Now if we arrange a table with a few assumed values for the wave amplitude and ocean depth, we can come up with something like this to compare the amplitudes:

Depth H [m]	**Amplitude A [m]**	**Wavelength λ [km]**	**Velocity c [km/t]**
4000	**1**	**100**	**713**
1000	1.4	50	356
100	2.5	16	113
10	**4.4**	**5**	**10**

An illustration for increasing amplitudes:



The values in this table can give us an idea about the amplitude variations as the depth decreases.

Non-linear phenomena

When the amplitude of the waves increases with decreasing depth, the waves can modify the medium and non-linear effects occur. We can then assume this form for the velocity:

$$c = \sqrt{g(H+\eta)} + u$$

Where u is the velocity of the stream in the medium generated by the wave. We can see that we have different velocities for the top side and down side of the waves ($\eta > 0$ and $\eta < 0$). So the front of the wave goes with a greater velocity. This non-linear effect breaks the wave and can generate a shock hydraulic waves and solitons.

For a short brief on solitons please look at this page:
http://oceanworld.tamu.edu/resources/ocng_textbook/chapter16/chapter16_02.htm

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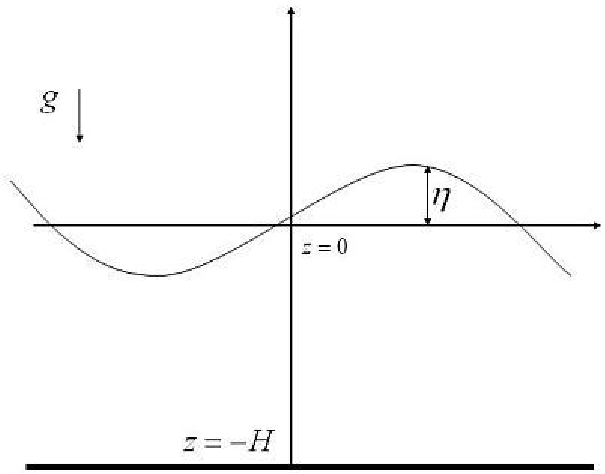
g
η
$z = 0$
$z = -H$