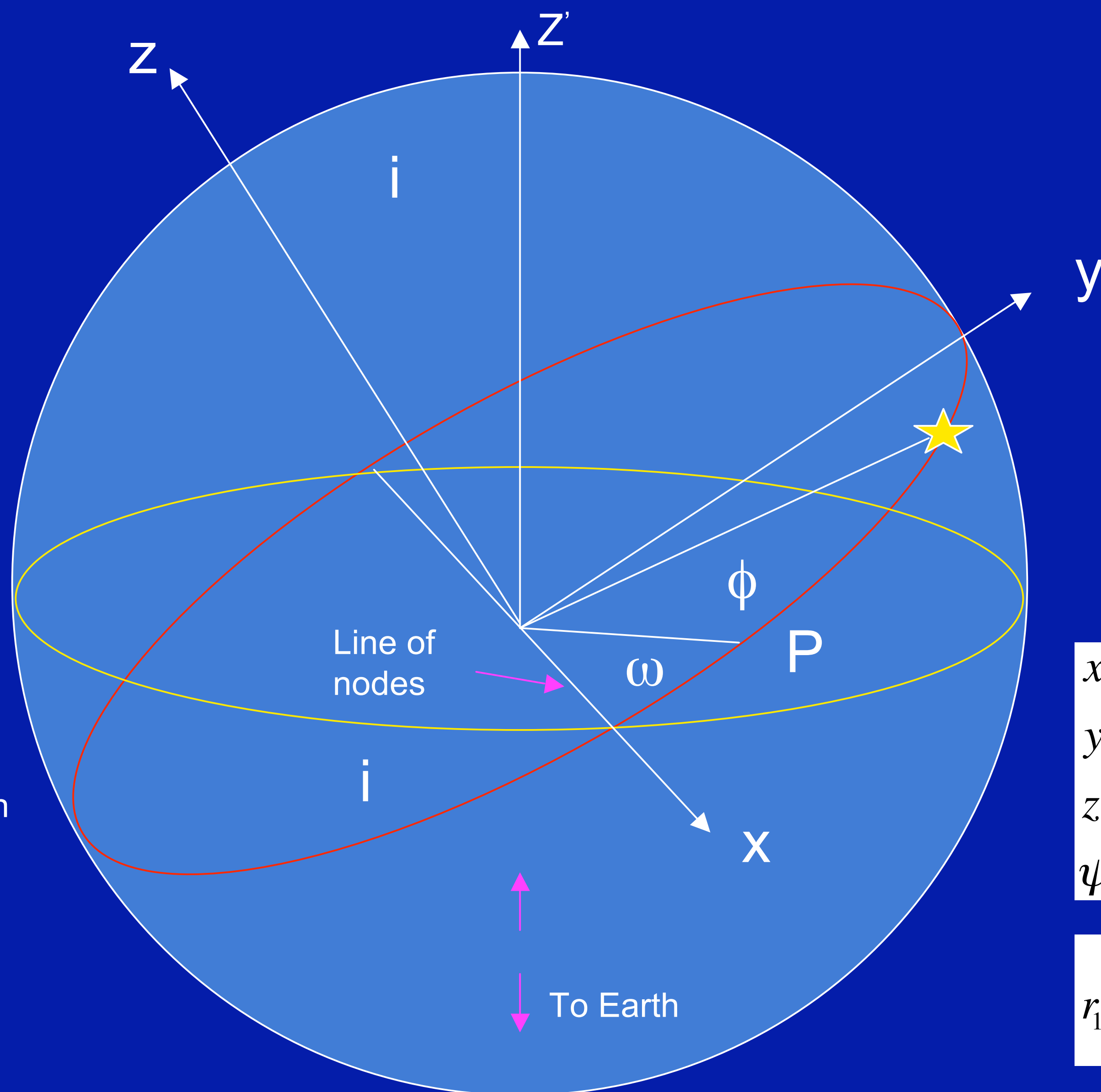
Doppler spectroscopy and astrometry

Theory and practice of planetary orbit measurements

Geometry of a binary orbit



i - orbital inclination
ω- longitude of periastron
ϕ- true anomaly
P- periastron

$$x = r_1 \cos\psi$$

$$y = r_1 \sin\psi$$

$$z = r_1 \sin\psi \sin i$$

$$\psi = \omega + \phi$$

$$r_1 = \frac{a_1\left(1 - e^2\right)}{1 + e\cos\phi}$$

Orbital elements

A binary orbit is defined by 7 elements:

- Size: $a = a_1 + a_2 \rightarrow$ semi-major axes of the orbits
- Shape: $e \rightarrow$ eccentricity
- Orientation in space: i, ω, Ω (longitude of periastron)
- "Location" in time: $T \rightarrow$ time of periastron passage, $P \rightarrow$ orbital period
- In Doppler spectroscopy, five orbital elements (a_1, e, ω, P, T) can be determined from radial velocity measurements of one binary companion

Determination of radial velocities

Radial velocity, V_r, is a time derivative of a component of the radius vector along the z-axis:

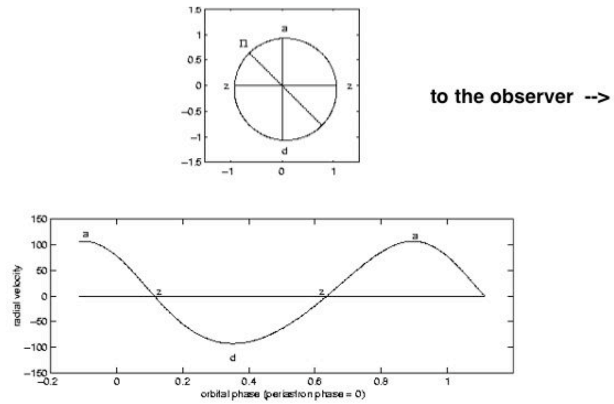
$$V_r = \dot{z} = \sin i[\dot{r}\sin(\theta+\omega) + r\dot{\theta}\cos(\theta+\omega)]$$

Time derivatives of r i Θ can be computed from the equation of elliptical motion and from the 2nd Kepler Law:

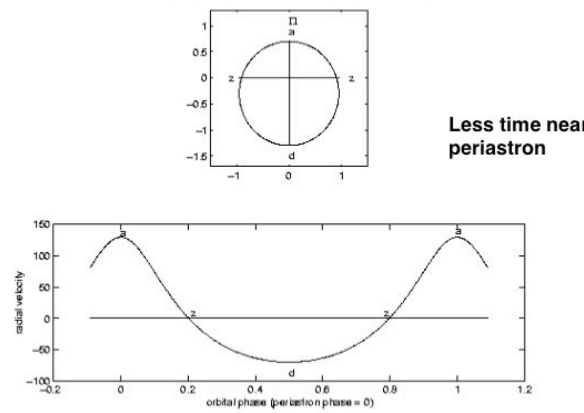
$$\dot{r} = \frac{e\sin\theta r\dot{\theta}}{1+e\cos\theta} \qquad r^2\dot{\theta} = \frac{2\pi a^2(1-e^2)^{1/2}}{P}$$

$$V_r = \frac{2\pi a\sin i}{P\sqrt{1-e^2}}\left[\cos(\theta+\omega) + e\cos(\omega)\right]$$

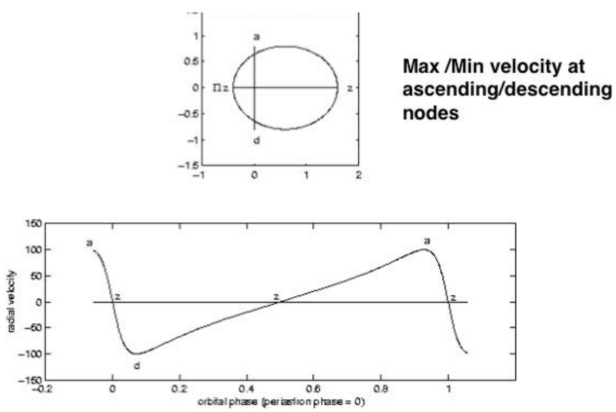
Examples of radial velocity curves



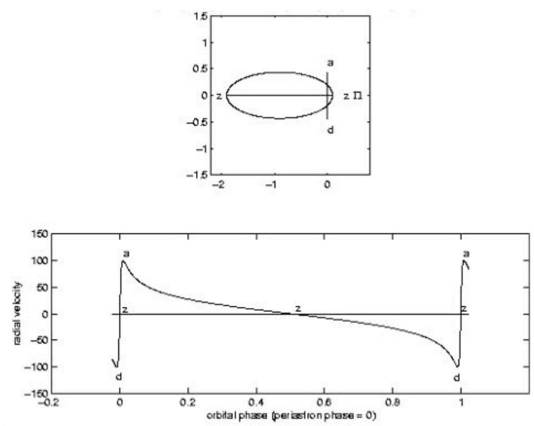
e = 0.1, w = 45°



e = 0.3, w = 0°



- ***e=0.6 , w= 90°***



- ***e=0.9 , w= 270°***

Models of orbits from V_r measurements

- Observations are given in the form of a time series, $V_r(i)$, at epochs t(i), i = 1,…,n
- A transition from t(i) to Θ(i) is accomplished in two steps:

$$E - e\sin E = \frac{2\pi}{P}(t - T)$$

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$$

$$V_r = K\left(\cos(\theta + \omega) + e\cos\omega\right)$$

← Equation for mean anomaly, *E*

$$K = \frac{2\pi a_1 \sin i}{P\sqrt{1-e^2}}$$

- From the fit (least squares, etc.), one determines parameters *K, e,* ω*, T, P*

Planetary mass determination

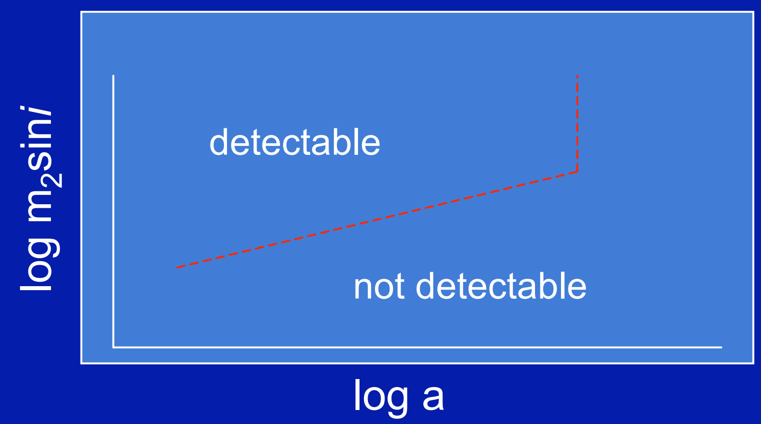
- From $K = (V_{max} - V_{min})/2$ we get:

$$a_1 \sin i = \frac{\sqrt{1-e^2}}{2\pi} KP$$

- A planetary mass (times sin *i*) is found by assuming that the mass of the star is known:

$$f(m_2) = \frac{m_2^{\,3} \sin^3 i}{M^2} = \frac{(1-e^2)^{3/2} K^3 P}{2\pi G}$$

For a fixed amplitude V*r*
$m_2 \sin i \sim a^{1/2}$

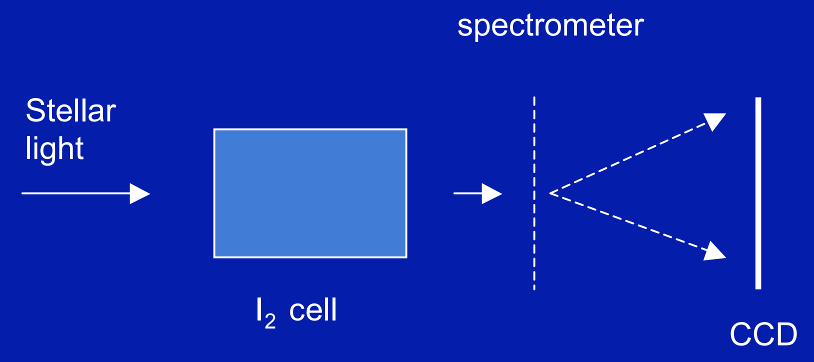


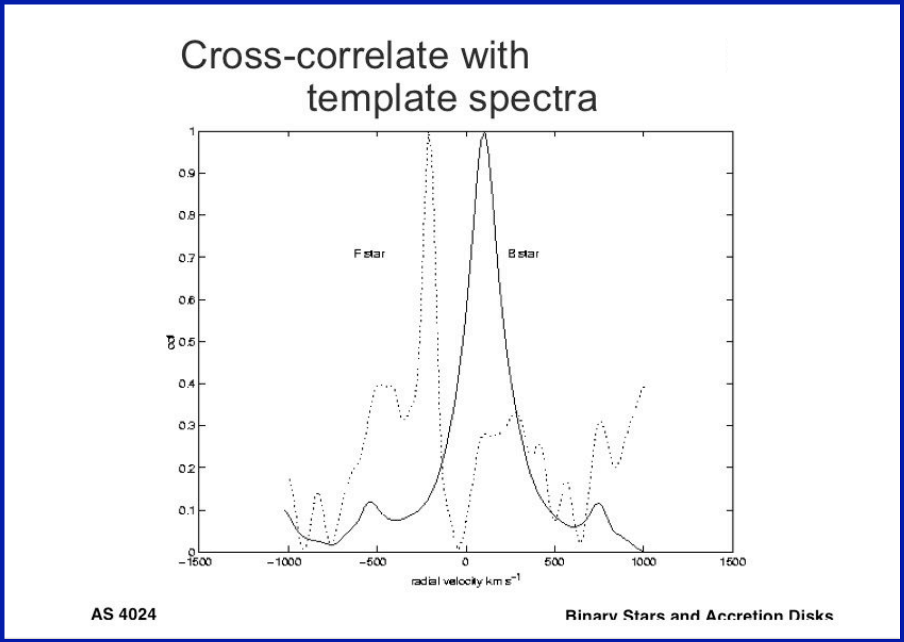
Applicability and limitations of Doppler spectroscopy

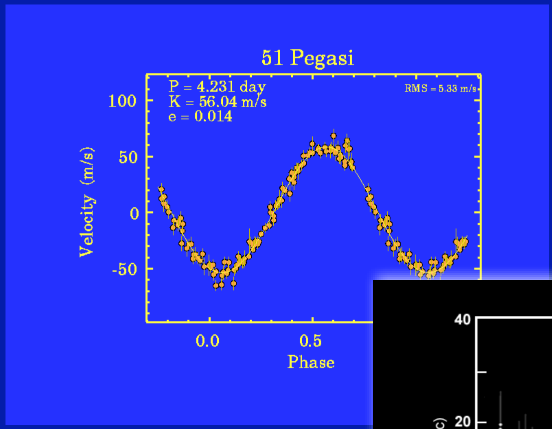
- The methods allows determination of 5 out of 7 parameters of the orbit projected onto the sky plane. Without an independent measurement of *i*, one gets only a lower limit to the mass of the planet
- Ability to measure very small changes of V_r are necessary (e.g. Jupiter - 12,5 m s^{-1}, Earth - 0,1 m s^{-1}, a spectrometer with the resolution of $R=10^5$ allows to measure V_r on the order of $10^{-5}c$ ~ a few km s^{-1})
- Photon noise (uncertainty of flux estimate $\sim N^{-1/2}$/pixel) provides an absolute limit of the precision of V_r measurement
- Measurements of large numbers of lines improves the signal-to-noise ratio, (S/N)~ (# of lines)$^{1/2}$, S/N depends on the spectral type of star
- For a G star with V=8, S/N ~ 200 can be achieved with a 3-m telescope. This gives a theoretical V_r precision ~ 1-3 m s^{-1}
- Another practical precision limitation results from stellar activity. This problem can be controlled to some extent by modeling. Currently attainable precision is ~3 m s^{-1} (Saturn mass for G-stars 10-20 masses of Neptune for K,M dwarfs)

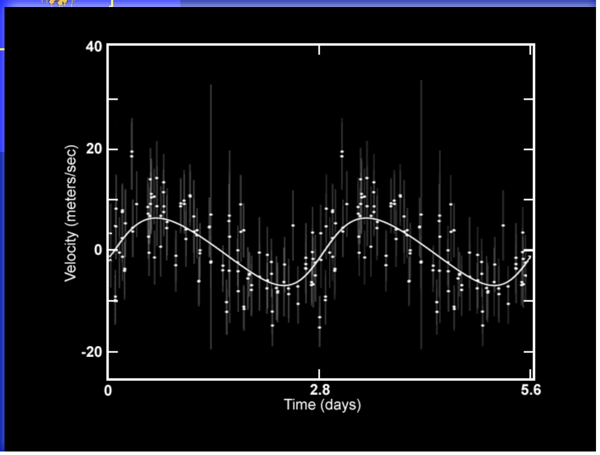
Calibration, analysis and examples of V_r curves

- Modern observing hardware and techniques of spectral analysis allow V_r measurements at a ~10^{-3} pixel precision
- Analysis is done by cross-correlating the spectra with high-S/N templates, the use of many spectral lines, and by accurate calibration with of I_2 cells installed in the optical path of the telescope

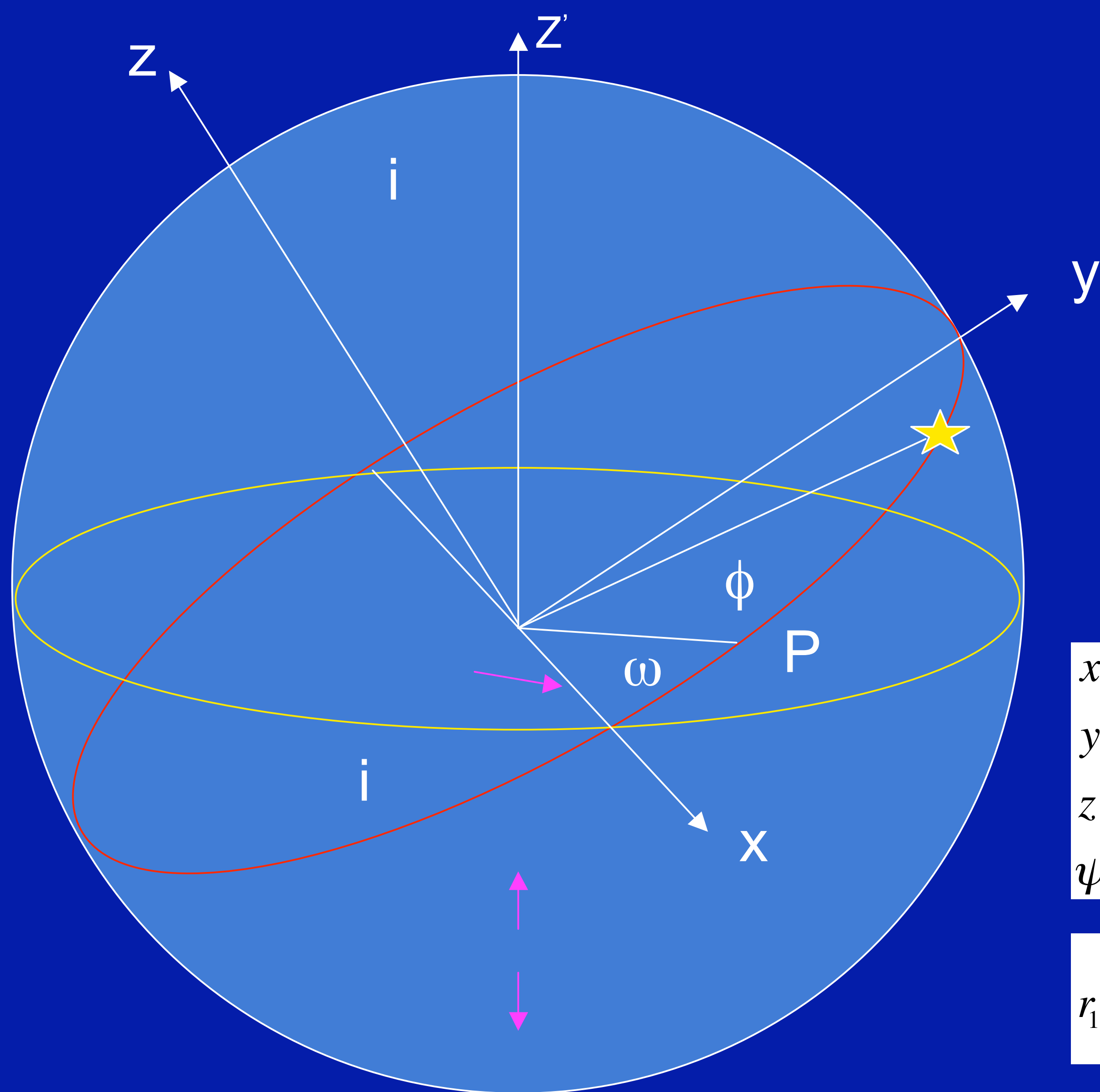








Astrometry



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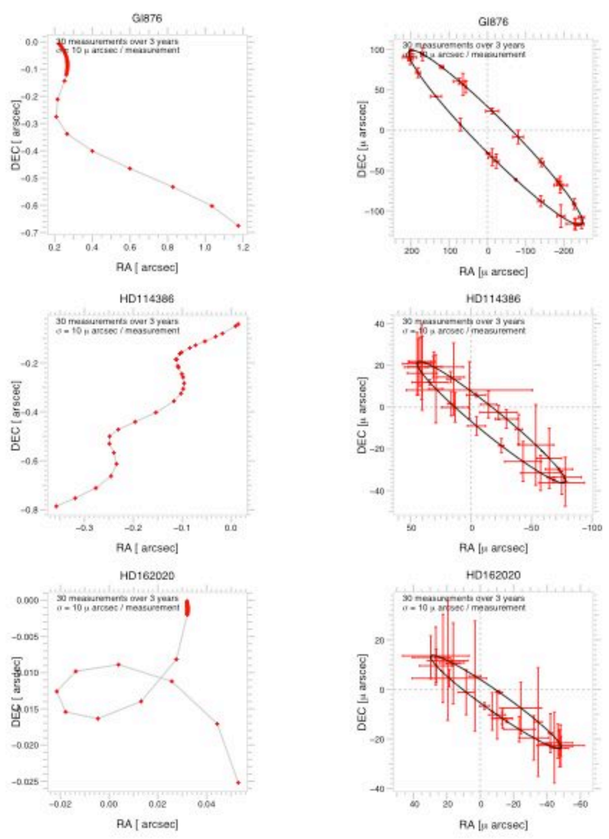
Astrometry: basic characteristics - I



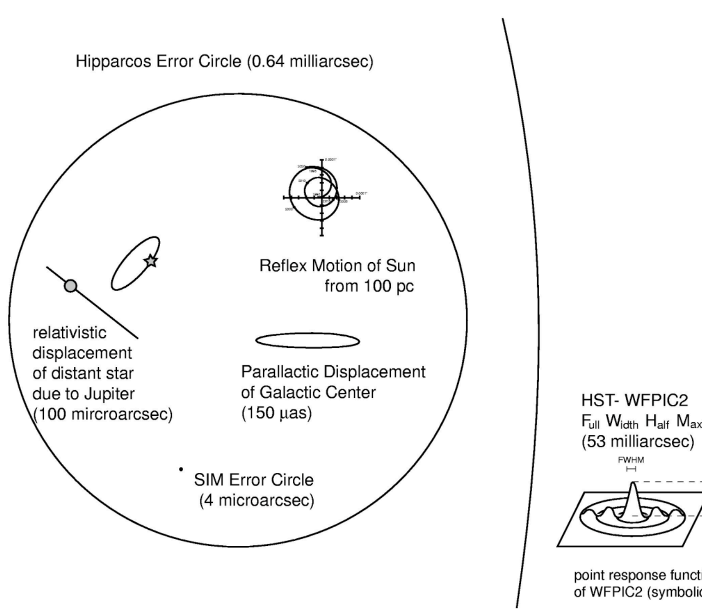
Examples of measurements and orbits

- Astrometry measures stellar positions and uses them to determine a binary orbit projected onto the plane of the sky
- Astrometry measures all 7 parameters of the orbit
- In analysis, one has to take the proper motion and the stellar parallax into account
- The measured amplitude of the orbital motion is simply $a_1 = (m_2/m_1)a$. Assuming $m_2 << m_1$ we have:

$$\Delta\theta = \left[\frac{m_2}{m_1}\right]\left[\frac{a}{d}\right]$$

Astrometry: basic characteristics - II

A comparison of some astrometry situations



Taking *q=m₂/m₁,we can calibrate the expression for Δθ:*

$$\Delta\theta = 0.5\left(\frac{q}{10^{-3}}\right)\left(\frac{a}{5AU}\right)\left(\frac{d}{10\,pc}\right)^{-1}$$

- A unit for $\Delta\theta$ is one millisecond of arc - very small effect
- Amplitude of the effect depends directly on *d*
- Dependence of m_2 on *a* is opposite to that in Doppler spectroscopy