Econ 201C: General Equilibrium and Welfare Economics

First Welfare Theorem

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1 The First Welfare Theorem

The idea that markets provide efficient allocations goes back several hundred years (probably long before Smith), but the mathematical justification of this concept in an idealized setting came much later and is now what we refer to as the "First Theorem of Welfare Economics." In these notes, I hope to prove this theorem in several different contexts.

1.1 Preliminaries

In order to define exactly what is meant by price-taking equilibrium and efficiency, we will require several definitions. Let I be an index set for consumers and J be an index set for producers. (It need not be that I or J are finite (or non-empty), but it may be easier to think about $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$.)

Definition 1 *An **economy** is a vector* $E \equiv \left[(X_i, \succeq_i)_{i \in I}, (Y_j)_{j \in J}, \omega \right]$ *of **consumption sets** (feasible consumption bundles for the consumer), **preference orderings, production sets** (feasible input-output bundles for the producer), and the **aggregate endowment**.*

Definition 2 *A **state** for E is a vector $[(x_i), (y_j)]$ of consumption bundles for the consumers and input-output bundles for the producers.*

Definition 3 *We say that a state $[(x_i), (y_j)]$ is **attainable** for E if*

1. $x_i \in X_i$ *for all* i
2. $y_j \in Y_j$ *for all* j
3. $\sum_{i \in I} x_i - \sum_{j \in J} y_j = \omega$

Definition 4 *A state $\left[(x_i^*), (y_j^*) \right]$ is **(strongly) Pareto optimal** if*

1. $\left[(x_i^*), (y_j^*) \right]$ *is attainable*
2. *There is no allocation $\left[(x_i'), (y_j') \right]$ which are feasible and satisfies $x_i' \succeq_i x_i^*$ for all i and for at least one $i' \in I$, $x_{i'}' \succ_{i'} x_{i'}^*$. (i.e. no one can be made better off without making someone else worse off.)*

Definition 5 *A **private ownership economy** for E is a vector $\mathcal{E} = [(X_i, \succeq_i, \omega_i), (\theta_{ij}), (Y_j)]$ of consumption sets, preference orderings, **individual endowments**, **profit shares**, and production sets where $\sum_{i \in I} \omega_i = \omega$, $\theta_{ij} \geq 0$ and $\sum_{i \in I} \theta_{ij} = 1$ for all j.*

Definition 6 *In a private ownership economy $\mathcal{E}$, we define i's **wealth function** as a function of **prices** as $w_i(p) = p \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi_j(p)$ where $\pi_j(p)$ is the **maximized profit** for firm j. (i.e. $\pi_j(p) = \max pY_j$)*

Definition 7 *A **price-taking equilibrium** for the private ownership economy $\mathcal{E}$ is a vector $\left[(x_i^*), (y_j^*), p^* \right]$ satisfying*

1. $\left[(x_i^*), (y_j^*) \right]$ *is attainable for* E
2. $x_i^* \succeq_i x_i'$ *for all* x_i' *satisfying* $p^* \cdot x_i' \leq w_i(p^*)$ *(Or equivalently* $x_i^* \in \xi_i(p^*, w_i(p^*))$*)*
3. $y_j^* = \max p^* Y_j$ *(or equivalently* $y_j^* \in \eta_j(p^*)$*)*

1.2 Standard Exchange Economy

Suppose there is no production in the economy. That is, either $J = \emptyset$ or $Y_j = 0$ for all j. Then we have

Theorem 8 (First Welfare Theorem) *Let $[(x_i^*), p^*]$ be a price-taking equilibrium for $\mathcal{E}$. Then $[(x_i^*)]$ is Pareto optimal for E.*

Proof. In order to get a contradiction, suppose $[(x_i^*)]$ is not Pareto optimal for E. Then there exists some attainable state $[(x_i')]$ for E satisfying $x_i' \succeq_i x_i^*$ for all i and $x_{i'}' \succ_{i'} x_{i'}^*$ for at least one $i' \in I$. Since $[(x_i^*), p^*]$ is a price-taking equilibrium for $\mathcal{E}$, we must have $p^* \cdot x_i' \geq w_i(p^*)$ for all i (or else i would have chosen x_i') and $p^* \cdot x_{i'}' > w_{i'}(p^*)$. But this is a contradiction to $[(x_i')]$ being feasible. To see why, note that since $[(x_i^*)]$ and $[(x_i')]$ are feasible, we have

$$\sum_{i \in I} x_i^* = \sum_{i \in I} \omega_i = \sum_{i \in I} x_i'$$

But we also have

$$\begin{aligned} \sum_{i \in I} p^* \cdot x_i' &> \sum_{i \in I} w_i(p^*) \\ &= \sum_{i \in I} p^* \cdot \omega_i \end{aligned}$$

Which is not possible. ■

1.3 Standard Production Economy

The proof of the first welfare theorem is easily extended to the case where we have production. That is, suppose $J \neq \emptyset$ and $Y_j \neq 0$ for at least one $j \in J$. Then we have

Theorem 9 (First Welfare Theorem) *Let $\left[(x_i^*), (y_j^*), p^*\right]$ be a price-taking equilibrium for $\mathcal{E}$. Then $\left[(x_i^*), (y_j^*)\right]$ is Pareto optimal for E.*

Proof. In order to get a contradiction, suppose $\left[(x_i^*), (y_j^*)\right]$ is not Pareto optimal for E. Then there exists some attainable state $\left[(x_i'), (y_j')\right]$ for E satisfying $x_i' \succeq_i x_i^*$ for all i and $x_{i'}' \succ_{i'} x_{i'}^*$ for at least one $i' \in I$. Since $\left[(x_i^*), (y_j^*), p^*\right]$ is a price-taking equilibrium for $\mathcal{E}$, we must have $p^* \cdot x_i' \geq w_i(p^*)$ for all i (or else i would have chosen x_i') and $p^* \cdot x_{i'}' > w_{i'}(p^*)$. Also, since y_j^* was profit maximizing at prices p^*, it must be that $p^* \cdot y_j^* \geq p^* \cdot y_j'$ for all j. But this is a contradiction to $\left[(x_i'), (y_j')\right]$ being feasible. To see why, note that since $\left[(x_i^*), (y_j^*)\right]$ and $\left[(x_i'), (y_j')\right]$ are feasible, we have

$$\sum_{i \in I} x_i^* - \sum_{j \in J} y_j^* = \sum_{i \in I} \omega_i = \sum_{i \in I} x_i' - \sum_{j \in J} y_j'$$

But we also have

$$\begin{aligned} \sum_{i \in I} p^* \cdot x_i' - \sum_{j \in J} p^* \cdot y_j' &> \sum_{i \in I} w_i(p^*) - \sum_{j \in J} p^* \cdot y_j^* \\ &= \sum_{i \in I} p^* \cdot \omega_i + \sum_{j \in J} p^* \cdot y_j^* - \sum_{j \in J} p^* \cdot y_j^* \\ &= \sum_{i \in I} p^* \cdot \omega_i \end{aligned}$$

That is,

$$\sum_{i \in I} p^* \cdot x_i' - \sum_{j \in J} p^* \cdot x_j' > \sum_{i \in I} p^* \cdot \omega_i = \sum_{i \in I} p^* \cdot x_i^* - \sum_{j \in J} p^* \cdot x_j^*$$

Which is not possible. ■

1.4 Quasilinear General Equilibrium Economy

For the quasilinear model, we need to alter some of the definitions slightly. In the quasilinear model, we have that preferences are given by

$$\begin{aligned} U_i(x_i, m_i) &= u_i(x_i) + m_i \\ &= v_i(z_i) + m_i \end{aligned}$$

Where $v_i(z_i) \equiv u_i(z_i + \omega_i)$, $z_i \in Z_i$ and $v_i(z_i) = -\infty$ if $z_i \notin Z_i$, where $Z_i \equiv X_i - \omega_i$.

Definition 10 *An **economy** is a vector* $\mathbf{v} = (v_i)_{i \in I}$.

Definition 11 *A **state** of* $\mathbf{v}$ *is a vector* $\left[(z_i, m_i)_{i \in I}\right]$. *We say that a state is **attainable** if it satisfies* $\sum_{i \in I} z_i = 0$ *and* $\sum_{i \in I} m_i = 0$.

Definition 12 *A state* $[(z_i^*, m_i^*)]$ *is **Pareto optimal** if*

$$\sum_{i \in I} v_i(z_i) = \max_{(z_i)} \left\{ \sum_{i \in I} v_i(z_i) : \sum_{i \in I} z_i = 0 \right\} \equiv v_I(0).$$

Definition 13 *A **price-taking equilibrium** for* $\mathbf{v}$ *is a vector* $[(z_i^*, m_i^*), p^*]$ *satisfying*

1. $\sum_{i \in I} z_i^* = 0$, $\sum_{i \in I} m_i^* = 0$
2. $p^* \cdot z_i^* + m_i^* = 0$ *for all* i
3. $v_i^*(p^*) \equiv \max_{z_i} \{v_i(z_i) + m_i : p^* \cdot z_i + m_i = 0\} = v_i(z_i^*) + m_i^*$.

The first welfare theorem then states that

Theorem 14 (First Welfare Theorem) *Let* $[(z_i^*, m_i^*), p^*]$ *be a price-taking equilibrium for* $\mathbf{v}$. *Then* $\sum_{i \in I} v_i(z_i^*) = v_I(0)$. *(i.e.* $[(z_i^*, m_i^*)]$ *is Pareto optimal.)*

Proof. Let p be an arbitrary price vector and let $[(z_i, m_i)]$ be an arbitrary attainable state for $\mathbf{v}$. Then

$$\begin{aligned} \sum_{i \in I} v_i^*(p) &= \sum_{i \in I} \max_{(z_i)} \{v_i(z_i) + m_i : p \cdot z_i + m_i = 0\} \\ &\geq \sum_{i \in I} v_i(z_i) + \sum_{i \in I} m_i \\ &= \sum_{i \in I} v_i(z_i) \end{aligned}$$

That is, for all (z_i) and for all p, $\sum_{i \in I} v_i^*(p)$ is an upper bound for $\sum_{i \in I} v_i(z_i)$. Next, note that since $[(z_i^*, m_i^*), p^*]$ is a price-taking equilibrium, we have that

$$\begin{aligned} \sum_{i \in I} v_i^*(p) &= \sum_{i \in I} [v_i(z_i^*) + m_i^*] \\ &= \sum_{i \in I} v_i(z_i^*) \end{aligned}$$

That is, $\sum_{i \in I} v_i(z_i^*)$ attains its upper bound, or

$$\sum_{i \in I} v_i(z_i^*) = \max_{(z_i)} \left\{ \sum_{i \in I} v_i(z_i) : \sum_{i \in I} z_i = 0 \right\} \equiv v_I(0)$$

That is, $[(z_i^*, m_i^*)]$ is Pareto optimal. ■

Note that in the quasilinear model, v_i can represent either a consumer or a producer. This proof did not rely on either specification for any v_i. Therefore, this proof is quite general in the sense that it applies equally well to an exchange economy as it does to an economy with production.

1.5 Lindahl Equilibrium in QLGE Economy

We may also consider economies in which either the consumption of one individual affects the utility of another (externalities) or each consumer must consume the same quantity of a particular good (public goods). It turns out that, with the appropriate notion of pricing (Lindahl pricing or idealized pricing), we can still define a price-taking equilibrium for such an economy and demonstrate its efficiency.

Definition 15 *A* ***Lindahl Equilibrium*** *for the economy* $(v_0, v_1, \ldots, v_n)$ *is a vector of individualized prices and quantities* $\left[\left(p^i\right), p^0, \left(z_i\right), z_0\right]$ *satisfying*

1. $z_i - z_0 \leq 0$ *for all* i *(quantity clearing)*
2. $\sum_{i=1}^n p^i - p^0 = 0$ *(price clearing)*
3. $v_i^*\left(p^i\right) = v_i\left(z_i\right) - p^i z_i$ *for all* i *(consumer optimality)*
4. $v_0^*\left(p^0\right) = p^0 z_0 - c_0\left(z_0\right)$*, where* $v_0\left(z_0\right) = -c_0\left(z_0\right)$*. (producer optimality)*

Definition 16 *A allocation of public goods* z *is* ***efficient*** *if*

$$z = \arg\max\left\{\left[\sum_{i=1}^n v_i\right](z) - c_0(z)\right\}$$

Proposition 17 *Let* $\left[\left(p^i\right), p^0, \left(z_i\right), z_0\right]$ *be a Lindahl Equilibrium. Then* z_0 $(= z_i\ \forall i)$ *is efficient.*

Proof. Let z be an arbitrary public goods allocation and let $\left[\left(p^i\right), p^0\right]$ be an arbitrary attainable price vector. (i.e. $\sum_{i=1}^n p^i - p^0 = 0$.) Then

$$\begin{aligned}
\sum_{i=1}^n v_i^*\left(p^i\right) + v_0^*\left(p^0\right) &= \sum_{i=1}^n \max\left\{v_i\left(z_i\right) - p^i z_i\right\} + \max\left\{p^0 z_0 - c_0\left(z_0\right)\right\} \\
&\geq \sum_{i=1}^n \left[v_i(z) - p^i z\right] + p^0 z - c_0(z) \\
&= \left[\sum_{i=1}^n v_i\right](z) - z\left(\sum_{i=1}^n p^i - p^0\right) - c_0(z) \\
&= \left[\sum_{i=1}^n v_i\right](z) - c_0(z)
\end{aligned}$$

That is, for all $\left[\left(p^i\right), p^0\right]$ and for all z, $\sum_{i=1}^n v_i^*\left(p^i\right) + v_0^*\left(p^0\right)$ is an upper bound for $\left[\sum_{i=1}^n v_i\right](z) - c_0(z)$. Let $\left[\left(p^i\right), p^0, \left(z_i\right), z_0\right]$ be a Lindahl Equilibrium. By the quantity clearing condition, $z_0 = z_i$ for all i. By consumer and producer optimality,

$$\begin{aligned}
\sum_{i=1}^n v_i^*\left(p^i\right) + v_0^*\left(p^0\right) &= \sum_{i=1}^n \left[v_i\left(z_i\right) - p^i z_i\right] + p^0 z_0 - c_0\left(z_0\right) \\
&= \left[\sum_{i=1}^n v_i\right]\left(z_0\right) - z_0\left(\sum_{i=1}^n p^i - p^0\right) - c_0\left(z_0\right) \\
&= \left[\sum_{i=1}^n v_i\right]\left(z_0\right) - c_0\left(z_0\right)
\end{aligned}$$

That is, $\left[\sum_{i=1}^n v_i\right]\left(z_0\right) - c_0\left(z_0\right)$ attains its upper bound or

$$\left[\sum_{i=1}^n v_i\right]\left(z_0\right) - c_0\left(z_0\right) = \max_z\left\{\left[\sum_{i=1}^n v_i\right](z) - c_0(z)\right\}$$

That is, z_0 is efficient. ■