The Absolute Arithmetic Continuum and its Peircean Counterpart[1]

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> a continuum is a collection so vast…that in the whole universe of possibility there is not room for them to obtain their distinct identities; but they become welded into one another. Thus the continuum is all that is possible, in whatever dimension it be continuous.
>
> Charles Sanders Peirce
> *The Logic of Relatives*
> 1898a, p. 160

> the possibility of determining more than any given multitude of points…at every part of the line, makes it *continuous*….
>
> Charles Sanders Peirce
> *Infinitesimals*
> 1900, p. 364

Introduction

In the decades bracketing the turn of the twentieth century the real number system was dubbed the *arithmetic continuum* because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the *geometric linear continuum* is assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to ensure this would be the case. In honor of Georg Cantor [1872] and Richard Dedekind

[1872], who first proposed this mathematico-philosophical thesis, the presumed isomorphism of the two structures is sometimes called the *Cantor-Dedekind axiom.* Given the Archimedean nature of the real number system, once this axiom is adopted we have the classic result of standard mathematical philosophy that infinitesimals are superfluous to the analysis of the structure of a continuous straight line. [2]

More than twenty years ago, however, we began to suspect that while the Cantor-Dedekind theory succeeds in bridging the gap between the domains of arithmetic and of classical Euclidean geometry, it only reveals a glimpse of a far richer theory of continua that not only allows for infinitesimals but leads to a vast generalization of portions of Cantor's theory of the infinite, a generalization that also provides a setting for Abraham Robinson's infinitesimal approach to analysis [1961; 1966] as well as for the profound and all too often overlooked non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese [1891; 1894], Tullio Levi-Civita [1892; 1898], David

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[2] In the case of an ordered field $\langle A,+,\cdot,<,0,1\rangle$ (such as the ordered field of real numbers), the *Archimedean axiom* asserts: for all $a,b \in A$, if $0 < a < b$, there is a positive integer n such that $na > b$. Since every ordered field A contains a unique isomorphic copy $\mathbb{Q}_A$ of the ordered field of rational numbers, an element a of A may be said to be *infinitesimal* if $|a|$ is less than every positive member of $\mathbb{Q}_A$, and it may be said to be *infinite* if $|a|$ is greater than every positive member of $\mathbb{Q}_A$. An ordered field is Archimedean if and only if it contains neither infinite nor infinitesimal elements. Non-Archimedean ordered fields, by contrast, contain infinite as well as non-zero infinitesimal elements, the latter being the multiplicative inverses of the former.

  Following Abraham Robinson [1961, p. 434; 1966, p. 51], it has become commonplace to say an element of an ordered field A is *finite*, if it lies between two members of $\mathbb{Q}_A$. In accordance with this convention, the infinitesimals of A are finite. The geometer, Robin Hartshorne [2000, p. 159], on the other hand, calls the elements that lie between two members of $\mathbb{Q}_A$ *finitely bounded* and defines the *finite* elements of A to be the finitely bounded elements that are not infinitesimal. While there is historical precedence for each of these conventions, in this paper we follow the latter.

  For a thorough discussion of the properties of ordered fields, the reader may consult [Lam 1980]. For the sake of convenience, however, the definition of an ordered field as well as the definitions of some of the other basic conceptions employed in the paper are collected together in the Appendix.

Hilbert [1899] and Hans Hahn [1907] in connection with their work on non-Archimedean ordered algebraic and geometric systems and by Paul du Bois-Reymond (cf. [1870-71;1875; 1877; 1882]), Otto Stolz [1883; 1885], Felix Hausdorff [1907; 1909] and G. H. Hardy [1910; 1912] in connection with their work on the rate of growth of real functions.[3] Central to the theory is J.H. Conway's ordered field of *surreal numbers* [1976; 2001], a system of numbers containing the reals and the ordinals as well as a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega - \pi$ to name only a few. Indeed, this particular number system, which Conway calls *No*, is so remarkably inclusive that, subject to the proviso that numbers--construed here as members of ordered number fields--be individually definable in terms of sets of *von Neumann-Bernays-Gödel set theory with global choice*, henceforth NBG,[4] it may be said to contain *all numbers great and small*![5]

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[3] For historical overviews of some of these pioneering contributions to non-Archimedean mathematics, see [Ehrlich 1994 (General Introduction); 1995; 2006] and the papers by Veronese and Poincaré in [Ehrlich 1994].

[4] NBG is a conservative extension of ZFC (Zermelo-Fraenkel set theory with the *axiom of choice*), that is, the same statements of the language of ZFC are provable in ZFC and in NBG. This being the case, ZFC is consistent if and only if NBG is consistent. Unlike ZFC, however, NBG admits sets as well as *proper classes*. Throughout the text, we follow the standard practice of referring to sets and proper classes collectively as *classes*.

Unless otherwise specified, throughout the text it is assumed that the underlying set theory is NBG. In virtue of the *axiom of global choice*--which is equivalent to the assertion that all classes can be well ordered--and the *axiom of foundation* for classes, all proper classes have the same 'cardinality' in NBG. We will denote their 'cardinality' by $\aleph_{On}$, *On* being the class of all ordinals. While the axiom of global choice plays no role in either the introduction of the surreal numbers or in the development of most aspects of the theory, it plays a crucial role in proving embedding theorems for structures that are proper classes including establishing (up to isomorphism) various uniqueness properties for the system of surreal numbers considered as a whole.

For a detailed discussion of NBG and its relation to ZFC, see [Fraenkel, Bar-Hillel and Lévy, 1973; Lévy, 1976], where NBG is referred to as $VNBC_{\sigma}$.

[5] By "ordered number field", we simply mean an ordered field whose elements are customarily called "numbers". When we say "numbers--construed here as members of

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In a number of works [Ehrlich 1987; 1989; 1992; 1994; 2005; Forthcoming 1; Forthcoming 2], we have suggested that whereas the real number system should be regarded as constituting an arithmetic continuum modulo the Archimedean axiom, *No* may be regarded as a sort of *absolute arithmetic continuum* (modulo NBG). And, in [Ehrlich 2002; 2004; 2005; and, especially, Forthcoming 2] we have drawn attention to the unifying framework this absolute arithmetic continuum provides for the reals and the ordinals as well as the various other sorts of systems of numbers great and small alluded to above.[6] In the present paper, we introduce a formal replacement for the intriguing extension of the classical linear continuum sketched by Charles Sanders Peirce [*circa* 1897; 1898a; 1900] at the turn of the twentieth century, and point out that by limiting *No* to its substructure No_P consisting of its finite and infinitesimal members (see Note 2), one obtains a model of this *Peircean linear continuum*, as we call it, whose properties mimic the remarkable properties of *No* established by the present author in [Ehrlich 1992].[7.8] In the course of so doing, we will also clarify some of the senses in which a

ordered number fields", we are not of course proposing that the term "number" should be limited to such numbers, but rather we are simply reporting on how we are employing the term in the phrase "*all numbers great and small*".

[6] Some readers might find it puzzling that we have included the ordinals as members of ordered number fields. After all, being noncommutative, aren't the sums and products of ordinals incompatible with the arithmetic of ordered fields? However, the sums and products of ordinals in ordered fields are *not* the familiar noncommutative sums and products of Cantor, but rather the *natural sums* and *natural products* of ordinals due to Hessenberg [1906] and Hausdorff [1927], respectively. For a discussion of these operations and further references to the literature, see [Ehrlich 2006, pp. 24-25].

[7] There is also an exact parallel between the properties of No_P and the properties of *No* established in Theorem 5 of [Ehrlich 2001: p. 1239]. To prove the analog for No_P, however, would require the development of No_P's structure as an *s-hierarchical finitely bounded ordered (integral) domain*, something space will not permit.

[8] Following the author's talk at the Carlsberg Academy referred to in Note 1, John Bell informed the author that while he had never developed the idea, he likewise realized that

Peircean linear continuum, so defined, realizes some of the key characteristics envisioned by Peirce for his purported linear continuum as well as draw attention to some of the differences that exist between Perice's intuitive conception and our proposed formal replacement thereof. Some of the differences point to limitations in Peirce's intuitive conception while others are byproducts of the underlying set-theoretic framework which, in accordance with standard geometrical practice and in marked contrast with Peirce, treats the collection of points on a line as an *actual* as opposed to a potential collection.[9]

Moreover, unlike Peirce, we will go beyond a largely order-theoretic exploration of Peircean linear continua by shedding light on the ordered algebraic structure No_P inherits from No. Since a relational structure M is a model of *absolutely continuous elementary Euclidean geometry* if and only if M is isomorphic to a Cartesian space defined over No [Ehrlich 1987; Forthcoming 1: Section 14], this will reveal just how rich a system of

No mirrored to some extent Peirce's purported continuum. See [Bell 2005, p. 209: Note 1] for a reiteration of this point.

[9] According to Peirce, whereas a continuum should be construed as a collection or an aggregate, it is a "potential aggregate." Moreover,

> being a potential aggregate only, it does not contain any individuals at all. It only contains general conditions which *permit* the determination of individuals. [Peirce 1898b, p. 247]

In the case of points, in particular, writes Peirce:

> We must…conceive that there are only so many points on the line as have been marked, or otherwise determined, upon it. Those do form a collection; but ever a greater collection remains determinable upon the line. *All* the determinable points cannot form a collection…the points on a line not yet actually determined are mere potentialities. [Peirce 1900, p. 363]

However, as our remark in the main body of the text alludes to, since the latter part of the nineteenth century, when geometers embraced the ideas of Cantor, geometers have treated the collection of points on a line as an actual infinite collection (cf. [Hilbert 1899; 1971; Tarski 1959; Greenberg 1993; Hartshorne 2000]).

numbers must be to provide length measures for the full range of segments of a straight line of absolutely continuous elementary Euclidean space that are finite or infinitesimal relative to a given unit segment.[10] We will find that, whereas *No* may be said to exhibit all possible types of algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, No_P may be said to exhibit all possible types of algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as a finitely bounded ordered (integral) domain (see Section 10). Finally, to help clarify the import of affixing the qualifying phrase "set-theoretically defined" to the expression "order-theoretic gradations", in the Postscript we will draw attention to the fact that if one is willing to allow for types of order-theoretic gradations definable in terms of proper classes as well as sets—a possibility neither countenanced by Peirce nor permitted in NBG--there *are* distinguished extensions of *No* and No_P that can be constructed in *Ackermann's set theory*[11] whose types of order-theoretic gradations are a

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[10] The axioms for absolutely continuous elementary Euclidean geometry consist of a set of axioms that are collectively equivalent to Hilbert's axioms for classical Euclidean geometry less the continuity axioms [Schwabhäuser, Szmielew and Tarski 1983; Ehrlich 1997, p. 61], together with Tarski's *elementary continuity schema* [Tarski 1959; Ehrlich 1997, p. 67] and an axiom that ensures that a straight line is an absolute linear continuum in the sense defined above in Section 1 [Ehrlich 1987, p. 243; Forthcoming 1]. For the definition of a Cartesian space appropriate for the just-cited system of axioms, see [Ehrlich 1997, p. 62].

[11] Ackermann's set theory is a conservative extension of Zermelo-Fraenkel set theory (see Note 4) having classes as well as sets (cf. [Fraenkel, Bar-Hillel and Lévy l973, pp. 148-153; Lévy 1976, pp. 207-212]). Following Fraenkel, Bar-Hillel and Lévy, here we include the *axiom of foundation* (for sets) among the axioms of Ackermann's set theory. Since Ackermann [1956] did not do so, some authors distinguish between *A* and A^*, where by *A* they mean Ackermann's original axioms (or some equivalent set thereof) and by A^* they mean what we have called Ackermann's set theory (cf. [Lévy and Vaught 1961; Reinhardt 1970]).

  Since A^* is a conservative extension of ZF (Zermelo-Freankel set theory without the axiom of choice), it is equiconsistent with ZF; accordingly, if A^* is consistent, it is consistent with both the axiom of choice and the axiom of global choice.

good deal more refined than those of *No* and *No$_P$*. This will also make clear why in our characterization of *No* as an absolute arithmetic continuum (modulo NBG) the qualifying phrase "modulo NBG" is not superfluous.

All of the new results are contained in the latter portions of the paper concerned with Peircean continua. Readers who are primarily interested in the expository aspects of the text may skip the proofs without serious disruption to the exposition. Since the theory of surreal numbers is still not as well known among philosophers and logicians as it deserves to be, to keep the paper relatively self-contained we will begin with a detailed overview of those aspects of the theory that will be appealed to in the subsequent discussion of Peircean linear continua and our proposed arithmetization thereof. Readers desiring even more detailed expositions of the relevant aspects of the theory may consult [Ehrlich Forthcoming 1] as well as [Ehrlich 1992; 1994 and 2001].

Part I: Absolute Continua

1. The Absolute Arithmetic Continuum

An ordered number field is a collection of numbers, linearly ordered, whose elements can be added, subtracted, multiplied and divided in accordance with the algebraic and algebraico-order-theoretic properties familiar from the systems of rational and real numbers. Like the ordered field of real numbers, the ordered field of surreal numbers is *real-closed*. Following Emil Artin and Otto Schreier [1926], an ordered field *A* may be said to be real-closed if it admits no extension to a more inclusive ordered field that results from supplementing *A* with solutions to polynomial equations with

coefficients in *A*.[12] *Intuitively speaking, real-closed ordered fields are precisely those ordered fields having no holes that can be filled by algebraic means alone*. Tarski [1948] shed important model-theoretic light on Artin and Schreier's celebrated algebraic conception by showing that real-closed ordered fields are precisely the ordered fields that are first-order indistinguishable from the ordered field of reals. For this reason, real-closed ordered fields are sometimes called *elementary arithmetic continua*.

Unlike the classical arithmetic continuum, the ordered field of rational numbers is not real-closed--it can be extended to a richer ordered field by supplementing the rationals with solutions to polynomial equations with rational coefficients (including, for example, the real-valued solution to the equation $x^2 - 2 = 0$). The richest ordered field that can be obtained from the rationals in this fashion is (to within isomorphism) the ordered field of real algebraic numbers. The system of real algebraic numbers is in fact (up to isomorphism) the smallest real-closed ordered field containing the ordered field of rationals as a subfield. Among the important discoveries to emerge from the theory of real-closed ordered fields is that the relation the ordered field of rational numbers bears to the ordered field of real algebraic numbers is a special case of a far more general relation. In particular, by a celebrated result of Artin and Schreier [1926], every ordered field *A* has (up to isomorphism) a unique *real-closure*, i.e. a smallest real-closed ordered field containing *A*. If *A* is itself real-closed, then *A* is its own real-closure, otherwise the real-closure of *A* is a (cardinality preserving) proper extension of *A*.

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[12] For overviews of some of the seminal results in the theory of real-closed ordered fields, see [Chang and Keisler 1990, pp. 345-348] and [Lam 1980]; for the historical development and philosophical significance of these important structures, see [Sinaceur 1994].

Since there is a multitude of real-closed ordered fields, it is natural to inquire if, like $\mathbb{R}$, it is possible to distinguish *No* (to within isomorphism) from the remaining real-closed ordered fields by appealing solely to its order. As Theorem 1 below shows, the following definition enables one to do just that.

DEFINITION 1 [Ehrlich 1987]. An ordered class $\langle A, < \rangle$ will be said to be an *absolute linear continuum* if for all sub*sets* L and R of A where $L < R$ there is a $y \in A$ such that $L < \{y\} < R$ (where the notation "$L < R$" indicates that every member of L precedes every member R).[13]

The reader will notice that an absolute linear continuum cannot be a set. Indeed, for any ordered set $\langle A, < \rangle$ one can always find a pair of subsets L and R of A where every member of L precedes every member R but for which there is no member of A that lies strictly between the members of L and the members of R by simply selecting L and R so that $L < R$ and $L \cup R = A$. Accordingly, if there are absolute linear continua in NBG, they must be proper classes.

Following tradition, a totally ordered class $\langle A, < \rangle$ is said to be *dense* if for each pair of members x and y of A where $x < y$, there is a $z \in A$ such that $x < z < y$. Extending this idea in NBG to its set-theoretic extreme, a totally ordered class $\langle A, < \rangle$ will be said to be *absolutely dense* (modulo NBG) if for each pair of nonempty sub*sets* X and Y of A where $X < Y$, there is a $z \in A$ such that $X < \{z\} < Y$. An absolute linear

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[13] In [Ehrlich 1988; 1992; 1994] and a number of other works, we refer to absolute linear continua as "η_{On} – orderings" since they extend to proper classes Hausdorff's [1907; 1914] idea of an η_α –*ordering of power* $\aleph_\alpha$, that is, an ordered set A of power $\aleph_\alpha$ such that for all subsets L and R of A where $L < R$ and $|L|, |R| < \aleph_\alpha$ there is a member of A lying strictly between those of L and those of R. The appellation "absolute linear continuum" was introduced in [Ehrlich Forthcoming 1].

continuum $\langle A,< \rangle$ is both absolutely dense in this sense and *absolutely extensive* in the sense that given any sub*set X* of *A* there are members *a* and *b* of *A* that are respectively smaller than and greater than every member of *X*. In fact, since in the definition of an absolute linear continuum *L* and/or *R* may be empty, one can readily show that an ordered class is an absolute linear continuum if and only if it has both of the just-stated properties.[14] Accordingly, since every element of an ordered class must either lie between two of its nonempty subclasses or be greater than or less than every member of some (possibly empty) subclass, *absolute linear continua are ordered classes having no order-theoretic limitations that are definable in terms of sets of standard set theory.*

In his *Contributions to the Founding of the Theory of Transfinite Numbers* [1895, §11], Cantor provided a non-metrical characterization of a closed interval of the *classical linear continuum* and showed that a closed interval of $\mathbb{R}$ is (up to isomorphism) the unique such structure.[15] From the latter one can readily obtain a categorical

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[14] In [Ehrlich 1987, p. 243; 1992, p. 172; 1994a, p. xxvii], we described the property employed in the definition of an absolute linear continuum as a condition of absolute density. While we continue to believe this description is apt, we now believe that, at least for some purposes, it is more revealing to split the condition into absolute density and absolute extensivity, as we have done here.

[15] In his [1892; 1889], Peirce criticized Cantor's earlier metrical characterization of a linear continuum [Cantor 1883 in Ewald 1996, p. 906] and offers a non-metrical alternative. Some recent writers on Peirce's theory of continua, perhaps influenced by the just-cited papers of Peirce, write as if Cantor only introduced a metrical characterization (cf. [Myrvold 1995, pp. 519-524; Herron 1997, pp. 606-608]). Potter and Schields [1977, pp. 25, 33: Note 13], on the other hand, are aware of Cantor's non-metrical definition and, based on the Peirce-Cantor correspondence, suggest that Peirce read Cantor's memoir containing it in December 1900. However, whether or not Peirce actually read or appreciated Cantor's non-metrical definition is not clear. By contrast, Cantor's non-metrical conception was made a *cause célèbre* by Bertrand Russell in his *Principles of Mathematics* [1903, Ch. XXXVI] and, soon thereafter, was widely discussed by E.V. Huntington [1917: Ch. V] and others.

It is perhaps also worth noting that Cantor was already in possession of a non-metrical characterization of a continuous series in 1887 but he did not specify the means of

characterization of the ordered set $\mathbb{R}$ itself. The following is the analog of the latter result for absolute linear continua.

THEOREM 1. [Ehrlich 1988: Lemma 1]. $\langle No,< \rangle$ is (up to isomorphism) the unique absolute linear continuum.

We hasten to add that, unlike the ordered field of real numbers, the ordered field of surreal numbers is not characterized (up to isomorphism) as an ordered field by its structure as an ordered class. Indeed, there are infinitely many pairwise non-isomorphic ordered fields that are absolute linear continua.[16] Happily, however, as we alluded to above, what one can prove in NBG is

THEOREM 2. [Ehrlich 1988: Lemma 2]. $\langle No,+,\cdot,<,0,1 \rangle$ is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum.

In virtue of Theorem 2, *No* (considered as an ordered field) is not only devoid of set-theoretically defined order-theoretic limitations, it is devoid of algebraic limitations as well; moreover, to within isomorphism, it is the unique ordered field that is devoid of both types of limitations or "*holes*", as they might more colloquially be called. That is, *No* not only *exhibits all possible types of algebraic and **set**-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, it is to within isomorphism the unique such structure that does.* It is ultimately this together with a number of closely related results, some of which will be discussed in Section 3 below,

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distinguishing segments of the classical linear continuum from segments of more general continuous series at that time. For a discussion of these matters, see [Ehrlich 2006, pp. 50-51].

[16] Since every α-saturated ordered field is an η_α-ordering [see Note 13], to obtain such an ordered field one need only form the union of a continuous chain A_α $(\alpha < On)$ of ordered fields whose universes are sets where A_0 is not real-closed and $A_{\alpha+1}$ is an $\alpha+1$-saturated elementary extension of A_α for each α. See, for example, [Chang and Keisler 1990, Ch. 5] for details.

that underlies our contention that *No* may be naturally regarded as an absolute arithmetic continuum (modulo NBG).

2. Absolute Linear Continua

Before continuing our exploration of *No*'s structure as an absolute arithmetic continuum, however, we will focus our attention more narrowly on its structure as an ordered class. In particular, in this section we will shed further light on two other senses in which *No*, considered as an ordered class, may be said to exhibit all possible types of order-theoretic gradations that are individually definable in terms of sets of NBG. The source of additional illumination is a pair of mutually reinforcing categorical characterizations of absolute linear continua provided in Theorem 3 below. Unlike the purely order-theoretic characterizations offered in Section 1, the characterizations of absolute linear continua stated in Theorem 3 are of a model-theoretic nature and are based on the author's extensions to proper classes of conceptions developed by Bjarni Jónsson [1956; 1960] and Michael Morley and Robert Vaught [1962]. The first is based upon

DEFINITION 2 [Ehrlich 1987; 1988; 1989; 1989a; 1992]. An ordered class A is said to be *universally extending* if for each ordered subset B of A and each ordered class A' extending B, there is an isomorphism $f : A' \rightarrow A$ that is an extension of the identity map on B.

Intuitively speaking, A is a universally extending ordered class if every possible way of enriching the order-theoretic gradations of an ordered subset B of A that is consistent with NBG is already (isomorphically) realized as an extension of B in A (see Figure 1).
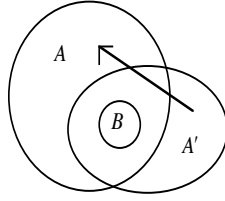
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Figure 1

The reader will notice that insofar as every ordered class is an extension of the empty ordered class, every universally extending ordered class contains an isomorphic copy of every ordered class. There is, however, a plethora of ordered classes that are inclusive in this sense that are not universally extending. However, as Theorem 3 below makes clear, the following definition provides the means to distinguish between the universally extending ordered classes and those that are "merely" universal.

DEFINITION 3 [Ehrlich 1987; 1988; 1989; 1989a; 1992]. An ordered class *A* is said to be *homogeneous universal* if it is *universal*—there is an isomorphic copy of every ordered class in *A*--and it is *homogeneous*--every isomorphism between ordered subsets *B* and *B'* of *A* can be extended to an automorphism of *A*, i.e. to an isomorphism of *A* onto itself (see Figure 2).[17]
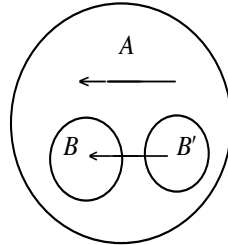
Figure 2

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[17] Since in the context of the theory of *homogeneous universal* relational structures, model theorists use the terms *universal*, *homogeneous*, and *homogeneous universal* in more general senses than they are employed here (cf. [Jónsson 1956; 1960; Morley and Vaught 1962]), in the model-theoretic settings of [Ehrlich 1987; 1989; 1989a; 1992] the terms *absolutely universal, absolutely homogeneous* and *absolutely homogeneous universal* were respectively employed in their steads.

As we alluded to above, the relation between homogeneous universal ordered classes, universally extending ordered classes, and absolute linear continua is encapsulated by the following result.

THEOREM 3 [Ehrlich 1988; 1989; 1992]. $\langle A,< \rangle$ is an absolute linear continuum if and only if $\langle A,< \rangle$ is a universally extending ordered class if and only if $\langle A,< \rangle$ is a homogeneous universal ordered class.[18]

The reader will notice that insofar as an automorphism of A is an isomorphism from A onto itself, the homogeneity condition essentially ensures that any pair of structurally indistinguishable ordered subsets of A have structurally indistinguishable surroundings as well. Consequently, if one cannot distinguish structurally between a pair of ordered subsets considered in isolation, they remain indistinguishable when the structures of their surroundings are taken into account as well.

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[18] It is interesting to note that an alternative version of Theorem 3 can be established using conceptions of homogeneous universality and universal extensibility that make no references to proper classes in their definiens. Call the latter conceptions *s-homogeneous universal* and *s-universally extending*, respectively. If A is an ordered class, then A is s-homogeneous universal, if A is s-*universal*—there is an isomorphic copy of every ordered set in A--and A is s-*homogeneous*--whenever $f : B \rightarrow B'$ is isomorphism between ordered subsets of A and C is an ordered subset of A that extends B, f can be extended to an isomorphism $g : C \rightarrow C'$ where C' is an ordered subset of A that extends B'. Similarly, A is s-universally extending, if for each ordered subset B of A and each ordered set A' extending B, there is an isomorphism $f : A' \rightarrow A$ that is an extension of the identity map on B. The proof of the equivalence of the s-notion and the corresponding class-notion employed in the main body of the text uses the axiom of global choice to show that an ordered proper class is the union of a chain (indexed over the class of all ordinals) of ordered sets.

Using the axiom of global choice in analogous fashions, completely analogous s-formulations also can be obtained for Theorem 4, PC 2 and PC 5 contained in Sections 3, 8 and 10, respectively.

It is perhaps also worth emphasizing that the homogeneity condition is extremely strong insofar as there is a veritable ubiquitous array of isomorphic copies of each ordered class in an absolute linear continuum. Indeed, the pervasiveness of such isomorphic copies only begins to become apparent when one considers the universality of absolute linear continua in conjunction with the fact that *every open interval of an absolute linear continuum is itself an absolute linear continuum*.[19] Accordingly, given the ubiquity of isomorphic copies of each ordered set in an absolute linear continuum, its homogeneity ensures that it is as amorphous as possible with respect to isomorphic ordered subsets. A classical linear continuum, by contrast, is merely as amorphous as possible with respect to finite isomorphic ordered subsets!

3. The Arithmetic and Absolute Arithmetic Continua:
The Reals and the Surreals

As we mentioned above, central to our contention that *No* may be naturally regarded as an absolute arithmetic continuum (modulo NBG), is the fact that in significant senses, that can be made precise, *No* may be said to exhibit all possible algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as an ordered field. In this regard, *No* bears much the same relation to ordered fields that $\mathbb{R}$ bears to Archimedean ordered fields. In this section, we will shed further light on this relation making use of variants of the conceptions employed in Sections 1 and 2 above. As we will later see, this relation foreshadows, to a great extent, the

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[19] As was noted in Note 13, absolute linear continua are straightforward extensions for proper classes of the η_α−orderings of power $\aleph_\alpha$ introduced by Felix Hausdorff [1907; 1914]. Among the properties of η_α−orderings established by Hausdorff is that every open interval (a,b) of an η_α−ordering A is isomorphic to A. His and subsequent such proofs naturally extend to η_{On}−orderings and, hence, to absolute linear continua.

corresponding relation that exists between our proposed model of Peirce's continuum considered as an ordered algebraic system and its classical counterpart.

Mimicking the corresponding definitions for ordered classes, an ordered field (Archimedean ordered field) A is said to be *homogeneous universal* if there is an isomorphic copy of every ordered field (Archimedean ordered field) in A and every isomorphism between ordered subfields of A whose universes are sets can be extended to an automorphism of A; and an ordered field (Archimedean ordered field) A is said to be *universally extending* if for each ordered subfield B of A whose universe is a set and each ordered field (Archimedean ordered field) A' extending B, there is an isomorphism $f : A' \to A$ that is an extension of the identity map on B.[20]

The following theorem further brings to the fore the intimate relation that exists between the classical and absolute arithmetic continua considered as inclusive ordered fields.

THEOREM 4 [Ehrlich 1987; 1989a; 1992]. (I) The following sets of axioms constitute (categorical) axiomatizations of $\langle \mathbb{R},+,\cdot,<,0,1 \rangle$; (II) by deleting the Archimedean axiom from the following axiomatizations one obtains categorical axiomatizations of $\langle No,+,\cdot,<,0,1 \rangle$.

Axioms for ordered fields

Archimedean axiom

Axiom of Homogeneous Universality

or, alternatively,   } (*Continuity Axioms*)

Axiom of Universal Extensibility,

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[20] The reader will notice that in the above two definitions implicit use is made of the fact that every subfield of an Archimedean ordered field is itself an Archimedean ordered field whose universe is a set.

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where by the Axiom of Homogeneous Universality (Axiom of Universal Extensibility) we mean the assertion: The collection of numbers together with the corresponding relations defined on it constitutes a Homogeneous Universal (Universally Extending) model of the above stated axioms.

Intuitively speaking, Theorem 4 asserts that whereas $\mathbb{R}$ is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small modulo the Archimedean axiom, *No* is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small that are individually definable in terms of sets of NBG.[21] We believe we are justified in referring to the Axioms of Universal Extensibility and of Homogeneous Universality as "continuity axioms" since in the context of the above axiomatizations they are equivalent to any of the more familiar continuity conditions including those due to Cantor, Dedekind and Hilbert.

Despite the revealing model-theoretic nature of the above comparative axiomatizations of the classical and absolute arithmetic continua, it is natural to inquire if it is possible to provide a comparative axiomatization that builds upon the classical intuition that, unlike discrete entities, continua are highly divisible. It is to such a comparative axiomatization that we now turn.

By a *cut* of an ordered class $\langle A,<\rangle$ we mean an ordered pair (L,R) of subclasses of A such that every member of L precedes every member of R and $L \cup R = A$. Unlike a *Dedekind cut*, where L and R are always nonempty, L and/or R may be empty. We will

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[21] Another important way of lending precision to the idea that $\langle No,+,\cdot,<,0,1\rangle$ is (up to isomorphism) the unique ordered number field containing all possible types of numbers great and small modulo NBG is to say that $\langle No,+,\cdot,<,0,1\rangle$ is (up to isomorphism) the unique *absolutely saturated model* for the theory of real-closed ordered fields [Ehrlich 1989]. The definition of an absolutely saturated model--a saturated model of power *On*--makes use of the classical model-theoretic conception of a *type* of an element.

say that a cut (L,R) is *definable in terms of sets*, or is a *set-cut*, if L contains a *cofinal* subset and R contains a *coinitial* subset, where a subclass Y of an ordered class X is said to be cofinal (coinitial) with X if for each $x \in X$ there is a $y \in Y$ such that $y \geq x$ ($y \leq x$). Moreover, if B is an ordered class that extends A and $x \in B$, x will be said to *fill* the cut (L,R) of A, if $L < \{x\} < R$.

DEFINITION 4. Let T be a theory containing the theory of dense linear orderings. A model A of T will be said to be a *maximally s-dense* model of T if it is impossible to extend A to a model of T (containing an element) that fills a cut in A that is definable in terms of sets.

The following theorem shows that when it comes to being maximally s-dense elementary arithmetic continua the only difference between the reals and the surreals is the satisfaction of the Archimedean axiom.

THEOREM 5 [Ehrlich 1992]. (I) The following set of axioms constitutes a categorical axiomatization of $\langle \mathbb{R},+,\cdot,<,0,1 \rangle$; (II) by deleting the Archimedean axiom from the following axiomatization one obtains a categorical axiomatization of $\langle No,+,\cdot,<,0,1 \rangle$.

Axioms for real-closed ordered fields (*Elementary Continuity Axioms*)
Archimedean axiom
Axiom of maximal s-density (*Continuity Axiom*),
where by the axiom of maximal s-density we mean the assertion: The collection of numbers together with the corresponding relations defined on it constitutes a maximally s-dense model of the above stated axioms.

4. The Surreal Numbers: A Prelude

In addition to its inclusive structure as an ordered field, *No* has a rich hierarchical structure that emerges from the recursive clauses in terms of which it is defined. From the standpoint of Conway's construction, this algebraico-tree-theoretic structure, or *simplicity hierarchy*, as we have called it [Ehrlich 1994a], depends upon *No*'s *implicit* structure as a *lexicographically ordered binary tree* and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with *No*'s structure as an ordered field, it being understood that *x* is *simpler than y* just in case *x* is a predecessor of *y* in the tree. In [Ehrlich 1994a] the just-described simplicity hierarchy was brought to the fore and made part of an algebraico-tree-theoretic definition of *No*. Following a review of some of the prerequisite definitions from the theory of lexicographically ordered binary trees, we will introduce the surreal numbers via the latter approach, which structurally mirrors and substantially amplifies a binary tree-theoretic process envisioned by Peirce whereby the points of a line could be marked off, thereby transforming them from potential to actual entities. It was in this process based on lexicographically ordered binary sequences of arbitrarily large finite length where Peirce claimed to find "symptoms of incipient cohesiveness…a premonition of continuity" [Peirce *circa* 1897, p. 87].

5. Lexicographically Ordered Binary Trees: Preliminary Definitions

A *tree* $\langle A, <_A \rangle$ is a partially ordered class (see Appendix) such that for each $x \in A$, the class $\{y \in A : y <_A x\}$ of *predecessors* of *x*, written $pr_A(x)$, is a set well ordered by $<_A$. A maximal subclass of *A* well ordered by $<_A$ is called a *branch* of the tree. Given any two distinct elements *x* and *y* of *A*, precisely one of the following is the

case: either x is a predecessor of y (i.e. $x <_A y$), y is a predecessor of x (i.e. $y <_A x$) or neither $x <_A y$ nor $y <_A x$. In the latter case, x is said to be *incomparable* with y. Thus x is incomparable with y if and only if x and y lie on different branches of the tree. An *initial subtree* of $\langle A, <_A \rangle$ is a subclass A' of A with the order inherited from $<_A$ such that for each $x \in A'$, the set of predecessors of x in A' coincides with the set of predecessors of x in A. The *tree-rank* of $x \in A$, written $\rho_A(x)$, is the ordinal corresponding to the well ordered set of predecessors of x; the αth *level* of A is the set of all members of the tree having tree-rank α; and a *root* of A is a member of the zeroth level. If $x, y \in A$, then y is said to be an *immediate successor* of x if $x <_A y$ and $\rho_A(y) = \rho_A(x) + 1$; and if $(x_\alpha)_{\alpha < \beta}$ is a chain in A (i.e., a subclass of A well ordered by $<_A$), then y is said to be an *immediate successor of the chain* if $x_\alpha <_A y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. The *length* of a chain $(x_\alpha)_{\alpha < \beta}$ in A is the ordinal β.

A tree $\langle A, <_A \rangle$ is said to be *binary* if each member of A has at most two immediate successors and every chain in A of limit length has at most one immediate successor. If every member of A has two immediate successors and every chain in A of limit length (including the empty chain) has an immediate successor, then the binary tree $\langle A, <_A \rangle$ is said to be *full*. Since a full binary tree has a level for each ordinal, the universe A of a full binary tree $\langle A, <_A \rangle$ is a proper class.

Using the axiom of global choice (or simply the axiom of choice, if A is a set) a tree may be shown to be binary if and only if it is isomorphic to an initial subtree of the *canonical full binary tree* $\langle B, <_B \rangle$, where B is the class of all sequences of -s and +s

indexed over some ordinal and $x <_B y$ signifies that x is a proper initial subsequence of y (cf. [Drake 1974, p. 216]).[22]

By an ordered tree $\langle A, <, <_A \rangle$, we mean a tree $\langle A, <_A \rangle$ whose universe is totally ordered by $<$. Thus, an ordered tree $\langle A, <, <_A \rangle$ has two orderings: a partial ordering $<_A$ in virtue of which $\langle A, <_A \rangle$ is a tree and a total ordering $<$ in virtue of which $\langle A, < \rangle$ is an ordered class.

As is well known, $\langle B, <_B \rangle$ can be totally ordered (lexicographically) in accordance with the definition:

$$(x_\alpha)_{\alpha < \mu} <_{lex(B)} (y_\alpha)_{\alpha < \sigma} \text{ if and only if } x_\beta = y_\beta \text{ for all } \beta < \text{ some } \delta, \text{ but } x_\delta < y_\delta,$$

it being understood that $- < \textit{undefined} < +$.

The resulting structure $\langle B, <_{lex(B)}, <_B \rangle$ is called the *lexicographically ordered canonical full binary tree*.

Figures 3-5 (below) exhibit levels 0-3 of $\langle B, <_{lex(B)}, <_B \rangle$, the level of the tree being indicated in Figure 3 on the right. In Figure 4 the members of the tree that are connected
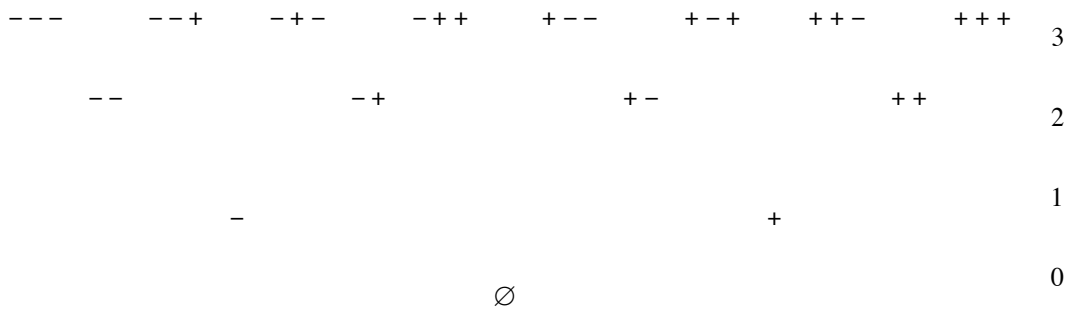

Figure 3

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[22] Many authors, including Peirce [*circa* 1897, p. 87], employ sequences of 0s and 1s rather than -s and +s to represent binary trees.

to a given member by a strictly descending sequences of dots are the predecessors of the given member. And in Figure 5 the ordering of the positions of the projected sequences

$$- - - \qquad - - + \qquad - + - \qquad - + + \qquad + - - \qquad + - + \qquad + + - \qquad + + +$$

$$- - \qquad\qquad - + \qquad\qquad + - \qquad\qquad + +$$

$$- \qquad\qquad\qquad\qquad\qquad + $$

$$\varnothing$$
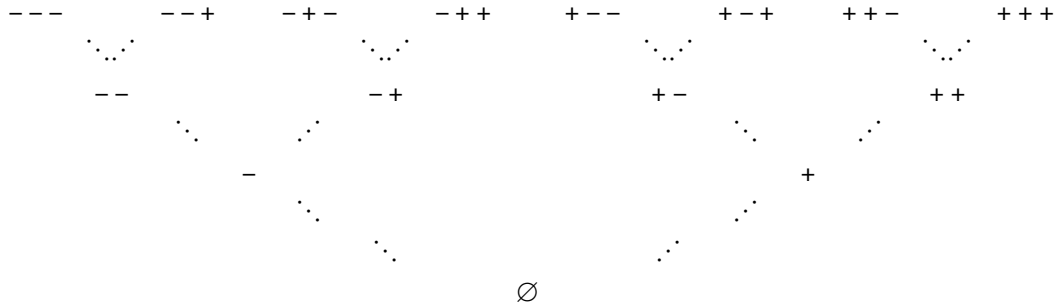
Figure 4

on the line indicates the total ordering of the members of levels 0-3 in accordance with the lexicographical ordering.

$$- - - \qquad - - + \qquad - + - \qquad - + + \qquad + - - \qquad + - + \qquad + + - \qquad + + +$$

$$- - \qquad\qquad - + \qquad\qquad + - \qquad\qquad + +$$

$$- \qquad\qquad\qquad\qquad\qquad + $$

$$\varnothing$$

Figure 5

In the following section, after introducing the surreal numbers, we will explore the relation between the surreal number tree and $\left\langle B, <_{lex(B)}, <_B \right\rangle$.

6. The Surreal Number Tree

Von Neumann defines an ordinal as a transitive set that is well ordered by the membership relation. As a result, for von Neumann, an ordinal emerges as the set of all of its predecesso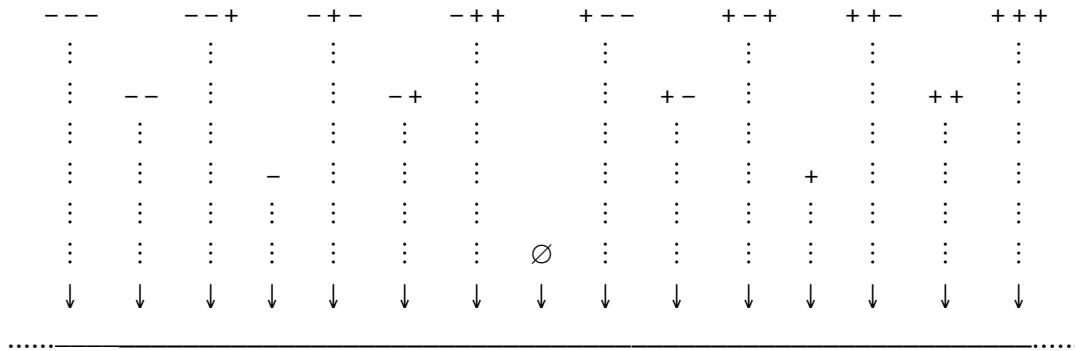rs in the long though rather trivial binary tree $\langle On, \in \rangle$ of all ordinals. So, so example, 0 is identified with $\varnothing$, 1 is identified with $\{\varnothing\} = \{0\}$, 2 is identified with $\{\varnothing, \{\varnothing\}\} = \{0,1\}$ and so on. In our construction of surreal numbers, which generalizes von Neumann's ordinal construction, each surreal number x emerges as an ordered pair (L_x, R_x) of sets of surreal numbers that are predecessors of x. Although L_x and R_x are defined independently of the total ordering that will be imposed on the number tree, they ultimately coincide with the sets of all predecessors of x that are *less than x* and *greater than x*, respectively.

$\langle On, \in \rangle$ is not usually described as a binary tree since its familiar structure as a well ordered class is indistinguishable from its structure as a tree. In the theory of surreal numbers, however, one must distinguish between its binary tree structure, on the one hand, and its structure as a totally ordered class, on the other hand.

Inspired by von Neumann's aforementioned definition of an ordinal, in [Ehrlich 1994a, p. 242] we introduced an analogous explicit definition of a surreal number, and in [Ehrlich 1994a, Appendix III, p. 265; 2002a; Forthcoming 1] we introduced an inductive version thereof. These constructions have the virtue that they permit the development of the theory of ordinals within the theory of surreal numbers [Forthcoming 1]. Here, however, for the sake of space, we presuppose that the ordinals are already at hand and introduce the surreal numbers inductively using the contents of a theorem that emerges from the former approaches [Ehrlich Forthcoming 1; 1994a: Theorem 1.1].

DEFINITION 5. $(\varnothing,\varnothing)$ is a surreal number; if $x = (L_x, R_x)$ is a surreal number, then $(L_x, \{x\} \cup R_x)$ and $(L_x \cup \{x\}, R_x)$ are surreal numbers; moreover, if $\{x_\alpha\}_{\alpha < \beta}$ is a sequence of surreal numbers of infinite limit length β for which $x_\mu \in L_{x_\nu} \cup R_{x_\nu}$ whenever $\mu < \nu < \beta$, then $\left(\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha} \right)$ is a surreal number. Nothing is a surreal number except in virtue of the above. By *No* we mean the class of surreal numbers so defined.

Intuitively speaking, a surreal number x is simpler than a surreal number y, if y cannot be constructed until x has already been constructed. This intuition is mirrored set-theoretically by

DEFINITION 6: A surreal number x is said to be *simpler than* a surreal number $y = (L_y, R_y)$, written $x <_s y$, if $x \in L_y$ or $x \in R_y$.

At this point it is not difficult to show that $\langle No, <_s \rangle$ is a full binary tree. In fact, the reader will note that $(L_x, \{x\} \cup R_x)$ and $(L_x \cup \{x\}, R_x)$ are the immediate successors in $\langle No, <_s \rangle$ of the surreal number x, and that $\left(\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha} \right)$ is the immediate successor in $\langle No, <_s \rangle$ of the chain x_α of surreal numbers indexed over the infinite limit ordinal β. In fact, as it turns out, $(L_x, \{x\} \cup R_x)$ is the immediate successor of x less than x, $(L_x \cup \{x\}, R_x)$ is the immediate successor of x greater than x, and $\left(\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha} \right)$ is always greater than the members of $\bigcup_{\alpha < \beta} L_{x_\alpha}$ and less than the members of $\bigcup_{\alpha < \beta} R_{x_\alpha}$. To lend precision to these ideas and to ideas regarding *No* considered as a totally ordered class more generally, however, we require

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DEFINITION 7 (The Rule of Order). For all surreal numbers $x = (L_x, R_x)$ and $y = (L_y, R_y)$,

$x < y$ if and only if $x \in L_y$ or $y \in R_x$ or $R_x \cap L_y \neq \varnothing$.

The reader will note that the three cases that comprise the rule of order correspond to the three mutually exclusive and collectively exhaustive relations that two surreal numbers x and y may bear to one another when x is less than y; namely, x is simpler than y, y is simpler than x, and x is incomparable with y, respectively. Using this in conjunction with Definitions 6 and 7, one may prove

COROLLARY 1 [Ehrlich 1994a]. For each surreal number x,

$$x = \left(L_{s(x)}, R_{s(x)} \right)$$

where $L_{s(x)} = \{ a \in No : a <_s x \text{ and } a < x \}$ and $R_{s(x)} = \{ a \in No : a <_s x \text{ and } x < a \}$.

For each surreal number x there is a unique enumeration $(x_\alpha)_{\alpha < \rho_{No}(x)}$ of the predecessors of x well ordered by the simpler than relation. Henceforth, we refer to this enumeration as the *genealogy* of x. By the *sign-expansion* of x we mean the sequence $(g_\alpha(x))_{\alpha < \rho_{No}(x)}$ defined by the condition: for all $\alpha < \rho_{No}(x)$

$$g_\alpha(x) = \begin{cases} +, \text{if } x_\alpha \in L_{s(x)} \\ -, \text{if } x_\alpha \in R_{s(x)} \end{cases}$$

where x_α is the predecessor of x having tree-rank α.[23]

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[23] Our definition and treatment of sign-expansions differs considerably from Conway's [1976, pp. 29-30]. In Conway's treatment, sign-expansions are defined using the ordered additive structure of *No*.

It is not difficult to show that every surreal number has a sign-expansion, distinct surreal numbers have distinct sign-expansions, and every sequence of elements of $\{+,-\}$ indexed over an ordinal is the sign-expansion of some surreal number [Ehrlich 1994a: Theorem 1.4]. Moreover, and more importantly, however, we have

THEOREM 5. [Ehrlich 1994a]. $\langle No, <, <_s \rangle$ and $\langle B, <_{lex(B)}, <_B \rangle$ are isomorphic as ordered trees, the map that sends each surreal number to its sign-expansion being the unique such isomorphism.

In virtue of Theorem 5, the theory of surreal numbers can be based just as well on $\langle B, <_{lex(B)}, <_B \rangle$ as on $\langle No, <, <_s \rangle$, and in [Ehrlich 2001] and [Gonshor 1986] it is developed in just that way. Readers inspired by Peirce's writings on continua might lean towards the treatment based on $\langle B, <_{lex(B)}, <_B \rangle$ since, as we noted earlier, it was essentially in the finite levels of $\langle B, <_{lex(B)}, <_B \rangle$ where Peirce claimed to find "a premonition of continuity". Accordingly, those so inspired are free to equate $\langle No, <, <_s \rangle$ with $\langle B, <_{lex(B)}, <_B \rangle$ for the remainder of the discussion.[24, 25]

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[24] In his discussion of the use made of binary sequences (indexed over ordinals) in Peirce's treatment of infinitesimals, Timothy Herron remarks: "It is not clear, however, that one can embed a theory of infinitesimals inside ordinally-indexed binary expansions if one wants to keep all of the usual theorems of the real line true for infinitesimal and infinite real numbers as well. At the very least, one would have to devise a clever set of restrictions specifying which ordinally-indexed binary expansions are valid and which are not valid as numbers" [1997, p. 615]. However, as our previous remarks make clear, Herron is mistaken.

[25] Still another approach to the development of the theory of surreal numbers is to base it on the generalization of the Dedekind cut due to Cuesta Dutari [1954]--what we have referred to as a "cut" in the main body of the text (see Section 3). This approach, which uses Cuesta Dutari's method of successively filling or adjoining cuts (see Note 32), was introduced by the author in 1982 (in a paper that appeared as [Ehrlich 1988]) and was later incorporated into [Alling and Ehrlich 1986; 1987]. For additional remarks on this

7. The Ordered Field of Surreal Numbers

Central to the algebraico-tree-theoretic development of the theory of surreal numbers is the following simple consequence of *No*'s structure as an ordered tree: if L and R are two sub*sets* of *No* for which $L < R$, there is a *simplest* member of *No* lying between the members of L and the members of R [Ehrlich 2001, p. 1235]. Co-opting notation introduced by Conway, the simplest member of *No* lying between the members of L and the members of R is denoted by the expression

$$\{L \mid R\}.$$

It is not difficult to show that each surreal number x is the simplest member of *No* lying between its predecessors on the left and its predecessors on the right, i.e.

$$x = \left\{ L_{s(x)} \mid R_{s(x)} \right\}.$$

Using this representation, the algebraico-tree-theoretic formulation of the central theorem in the theory of surreal numbers may be stated as follows.

THEOREM 6 [Conway 1976; Ehrlich 2001]. $\langle No, +, \cdot, <, <_s \rangle$ is an ordered field when $+, -$ and $\cdot$ are defined by recursion as follows where x^L, x^R, y^L and y^R are understood to range over the members of $L_{s(x)}, R_{s(x)}, L_{s(y)}$ and $R_{s(y)}$, respectively.

DEFINITION of $x + y$

$$x + y = \left\{ x^L + y, x + y^L \mid x^R + y, x + y^R \right\}.$$

DEFINITION of $-x$

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approach including the definition that transforms the structure into a lexicographically ordered binary tree, see [Ehrlich 1994a, p. 257].

$$-x = \left\{ -x^R \mid -x^L \right\}.$$

DEFINITION of xy

[26]

$$xy = \left\{ x^L y + xy^L - x^L y^L , x^R y + xy^R - x^R y^R \mid x^L y + xy^R - x^L y^R , x^R y + xy^L - x^R y^L \right\}.$$

Although the algebraico-tree-theoretic definitions of sums, products and additive inverses of surreal numbers are apt to appear rather cryptic to readers unfamiliar with the theory of surreal numbers, they have simple interpretations. To begin with, in virtue of the nature of the representations of x and y, we have

$$x^L < x < x^R \text{ and } y^L < y < y^R$$

for all x^L, x^R, y^L and y^R. By combining this with the routine (high school) algebra of ordered fields, one may readily show that in an ordered field $x + y$ must lie between the sums on the left of $x + y$ and the sums on the right of $x + y$ in the above definition of $x + y$ and, similarly, xy must lie between the arithmetical expressions on the left of xy and the arithmetical expressions on the right of xy in the above definition of xy [Ehrlich, 1994a, pp. 252-253; 2001, p. 1236; Forthcoming 1]. Accordingly, since $x + y$ and xy must lie between the arithmetic expressions on the left and the arithmetic expressions on the right in their respective recursive definitions, their definitions respectively require that they be the simplest member of the surreal number tree so situated. Finally, the constraint on additive inverses, which is a consequense of the definition of addition [Ehrlich 2001,

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[26] Following convention, to enhance the readability of the definitions of $+, -$ and $\cdot$, the set-theoretic brackets that enclose the sets of "right-sided members" and the sets of "left-sided members" have been omitted.

p. 1237], ensures that the portion of the surreal number tree less than 0 is an exact mirror image of the portion of the surreal number tree greater than 0, 0 being the simplest element of, as well as the unique root in, the surreal number tree.

Recursively defining the sums and products of members of *No* to be the simplest possible elements of *No* consistent with *No*'s intended structure as an ordered field ensures that *the sums and products of elements of No get defined just as soon as there is sufficient previously defined ordered algebraico-tree theoretic information to do so.* Moreover, as was shown in [Ehrlich 2001], this has profound implications for the structure of *No* that goes well beyond its already impressive structure as an ordered field. For example, it was established that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining *No*'s arithmetic in turn provides an unfolding of the entire spectrum of ordered number fields in such a way that an isomorphic copy of each such system either emerges as an initial subtree of *No* or is contained in a theoretically distinguished instance of such a system that is. In particular, we showed that *every real-closed ordered field is isomorphic to an initial subfield of No* [Ehrlich 2001: Theorem 19, p. 1253]. In fact, outside of *No*'s ordered field of real algebraic numbers (which has no proper real-closed subfield), every real-closed ordered field, including *No* itself, is the union of a chain of increasingly more inclusive real-closed ordered number fields each of which is a proper initial subfield of *No* whose universe is a set, and every real-closed initial subfield of *No*, whose universe is a set, is a component of just such a chain.

Although *No* contains an entire proper class of isomorphic copies of the ordered field of real numbers, only one of them is an initial subtree of *No*. In writings on the theory of surreal numbers, the latter subfield is identified as *No*'s subfield of real numbers, the set of whose members may be defined as follows

DEFINITION 8. Let $\mathbb{D}$ be the set of all surreal numbers having finite tree-rank and further let $\mathbb{R} = \mathbb{D} \cup \left\{ \{L \mid R\} : (L,R) \text{ is a Dedekind gap in } \mathbb{D} \right\}$.[27]

Except for inessential changes, the following result regarding the structure of $\mathbb{R}$ is due to Conway [1976, pp. 23-25].

THEOREM 7. $\mathbb{R}$ (with $+, -, \cdot$ and $<$ defined *à la No*) is isomorphic to the ordered field of real numbers defined in any of the more familiar ways, $\mathbb{D}$ being *No*'s ring of dyadic rationals (i.e., rationals of the form $m/2^n$ where m and n are integers); $n = \{0,\ldots,n-1 \mid \varnothing\}$ for each positive integer n, $-n = \{\varnothing \mid -(n-1),\ldots,0\}$ for each positive integer n, $0 = \{\varnothing \mid \varnothing\}$, and the remainder of the dyadics are the arithmetic means of their left and right predecessors of greatest tree-rank; e.g., $\frac{1}{2} = \{0 \mid 1\}$.

In virtue of Theorem 7, the first few levels of the surreal number tree may be depicted as follows where, as in Figure 4, the strictly descending sequences of dots connect a member of the tree to its predecessors.

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[27] A Dedekind cut (L,R) of an ordered class *A* is said to be a *Dedekind gap*, or simply a *gap*, if *L* has no greatest member and *R* has no least member.

$$-3 \qquad -1\tfrac{1}{2} \qquad -\tfrac{3}{4} \qquad -\tfrac{1}{4} \qquad \tfrac{1}{4} \qquad \tfrac{3}{4} \qquad 1\tfrac{1}{2} \qquad 3$$

$$-2 \qquad\qquad -\tfrac{1}{2} \qquad\qquad \tfrac{1}{2} \qquad\qquad 2$$
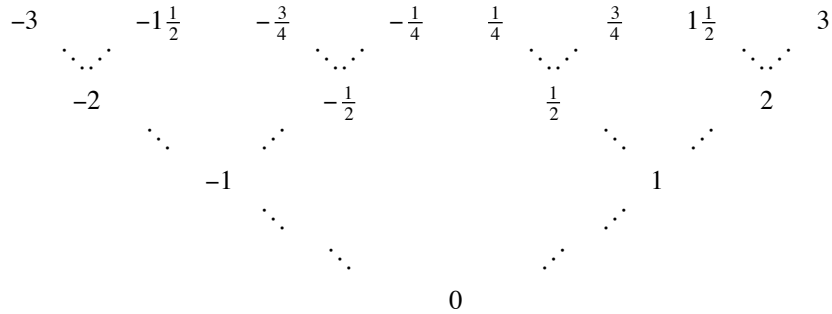
$$-1 \qquad\qquad\qquad 1$$

$$0$$

Figure 6

A striking feature of the system of surreal numbers is that each surreal number can be assigned a canonical proper name that is a reflection of its characteristic individual properties [Conway 1976; Ehrlich 2001, pp. 1244-1248; Forthcoming 1: Section 11]. To provide a sketch of how these *Conway names*, as we call them, are introduced we require a number of definitions beginning with the following classical ones applied to *No*.

Two elements *a* and *b* of *No* are said to be *Archimedean equivalent,* written $a \approx b$, if there are positive integers *m* and *n* such that $m|a| > |b|$ and $n|b| > |a|$; if $a \not\approx b$ and $|a| < |b|$, then we write $|a| << |b|$ and *a* is said to be *infinitesimal (in absolute value) relative to b* and *b* is said to be *infinite (in absolute value) relative to a*; the class of all members of *No* that are Archimedean equivalent to some member of *No* is said to constitute an *Archimedean class* of *No*. 0, which is infinitesimal (in absolute value) relative to every other surreal number, is the sole surreal number that is not a member of an Archimedean class.

Following Conway, an element of *No* is said to be a *leader* if it is the simplest member of the positive elements of an Archimedean class of *No*. Every Archimedean class of *No* has a leader. One of the salient features of the class of *No*'s leaders is given by

THEOREM 8 [Ehrlich 2001]. The ordered subtree of *No* consisting of the leaders of *No* is a lexicographically ordered full binary tree and, as such, there is a unique isomorphism of ordered trees from the leaders of *No* onto *No*.

Relying on the just-said theorem, one can recursively define [Conway 1976, p. 31; Ehrlich 2001, p. 1246: Definition 12] a unique appellation of the form ω^y for each leader in *No* in such a manner that $\omega^y <_s \omega^x$ if and only if $y <_s x$ for all $x, y \in No$. In effect, the definition assigns the appellation "ω^0" to 1, which is the simplest leader in *No*, the appellation "ω^{-1}" to the simplest leader in *No* less than ω^0, the appellation "ω^1" to the simplest leader in *No* greater than ω^0, and so on (see Figure 6). Having done so, one may show that if $x \in No - \{0\}$, there is a unique $r \in \mathbb{R} - \{0\}$, a unique leader ω^y, and a unique $a \in No$ such that $x = r\omega^y + a$ where a is infinitesimal (in absolute value) relative to ω^y. Moreover and more importantly, by combining repeated uses of this observation with properties of *No*'s structure as an ordered tree, one may prove

THEOREM 9 [Conway 1976; Ehrlich 2001]. For each surreal number x we can define a unique expression

$$\sum_{\alpha < \beta} \omega^{y_\alpha} . r_\alpha$$

(the *Conway name* or *normal form* of x) where $\{y_\alpha : \alpha < \beta \in On\}$ is a (possibly empty) descending sequence of members of *No* and $\{r_\alpha : \alpha < \beta\}$ is a sequence of members of $\mathbb{R} - \{0\}$. Distinct surreal numbers have distinct Conway names, and every expression of the above form is the Conway name of some surreal number (the Conway name of 0 being the empty series, i.e. the unique such expession where $\beta = 0$).

Conway names provide representations of surreal numbers that are both perspicuous and perspicacious. $\omega - \pi$ is indeed the surreal number obtained by subtracting π from ω, $\frac{\omega}{2}$ is ω divided by 2, $\frac{1}{\omega}$ is the multiplicative inverse of ω as well as the cube root of $\frac{1}{\omega^3}$, $\omega^\omega + \omega.3 + 4$ and $\sqrt{2}$ are the ordinal and real number respectively so named, and so on (where, in accordance with standard practice, π is written in place of $\omega^0.\pi$, $\frac{\omega}{2}$ is written in place of $\omega^1.\frac{1}{2}$, $\frac{1}{\omega}$ is written in place of ω^{-1}, and so on).

Making use of Conway names of surreal numbers, Figure 7 provides a slightly more expanded picture of the early stages of the recursive unfolding of the surreal number tree than that given in Figure 6 above.
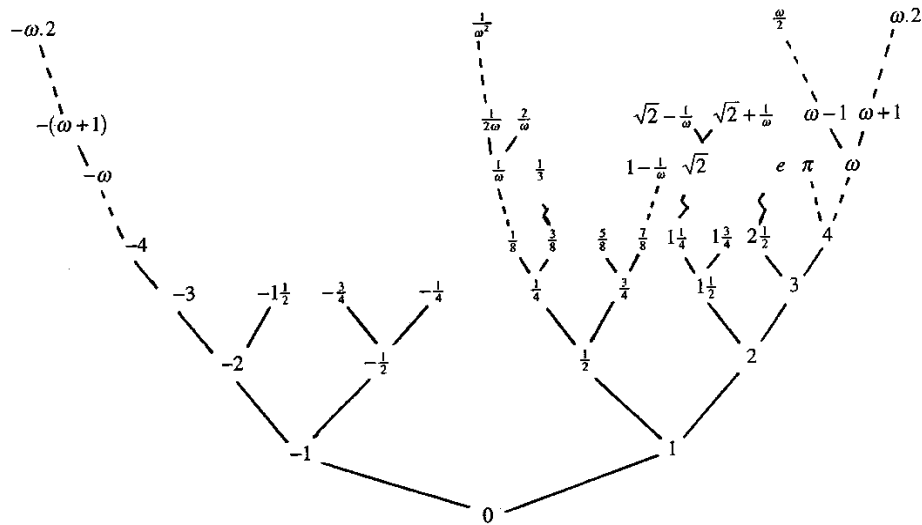

Figure 7

Besides providing perspicuous and perspicacious representations of surreal numbers, Conway names make surreal numbers more tractable from an algebraic point of view. Indeed, when surreal numbers are denoted by their respective Conway names, the basic field operations can be performed on them much like the familiar operations on

polynomials, as can the familiar ordering of elements by first-differences. In particular, one may prove the following result where the operations $+_H$ and $\cdot_H$ are defined on, and the relation $<_H$ is defined between, surreal numbers denoted by their respective Conway names (supplemented with "dummy" terms with zeros for coefficients to permit a uniform representation of all surreal numbers).

THEOREM 10 [Conway 1976; Ehrlich 2001]. $\langle No, +_H, \cdot_H <_H, <_S \rangle = \langle No, +, \cdot, <, <_S \rangle$ when $+_H$, $\cdot_H$ and $<_H$ are defined by the following conditions where terms with zeros for coefficients are understood to be inserted and deleted as needed:

$$\sum_{y \in No} \omega^y . a_y +_H \sum_{y \in No} \omega^y . b_y = \sum_{y \in No} \omega^y . (a_y + b_y),$$

$$\sum_{y \in No} \omega^y . a_y \cdot_H \sum_{y \in No} \omega^y . b_y = \sum_{y \in No} \omega^y . \left[\sum_{\substack{(\mu, \nu) \in No \times No \\ \mu +_H \nu = y}} a_\mu b_\nu \right],$$

$$\sum_{y \in No} \omega^y . a_y <_H \sum_{y \in No} \omega^y . b_y \text{ , if } a_y = b_y \text{ for all } y > \text{some } x \in No \text{ and } a_x < b_x.$$

As we mentioned in the Introduction, by No_P we mean the ordered subclass of No consisting of all finite and infinitesimal members of No. It is not difficult to see that $No_P - \{0\}$ consists of all the members of No whose Conway names solely have exponents that are less than or equal to 0, those having only negative exponents being the non-zero infinitesimals of No.[28] It is to an analysis of No_P and Peircean linear continua, more generally, to which we now turn.

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[28] On occasion, it might be advantageous to have a definition of No_P at hand that makes no reference to No. One way of doing this is to define the members of No_P inductively

Part II: Peircean Continua

8. Peircean Linear Continua: A Proposed Model

As the reader will recall, an ordered class is an absolute linear continuum if and only if it is both absolutely dense and absolutely extensive. Moreover, being absolutely extensive, an absolute linear continuum contains neither a cofinal nor a coinitial subset, where a subclass Y of an ordered class X is said to be cofinal (coinitial) with X if for each $x \in X$ there is a $y \in Y$ such that $y \geq x$ ($y \leq x$).

By contrast, we will say that a totally ordered class $\langle A, < \rangle$ is a *Peircean linear continuum* if it is absolutely dense and it contains an isomorphic copy of the ordered set of real numbers that is both cofinal and coinitial with $\langle A, < \rangle$.[29] Accordingly, an ordered

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as follows: $(\varnothing, \varnothing)$ is a *p*-surreal number; if $x = (L_x, R_x)$ is a *p*-surreal number, then $(L_x, \{x\} \cup R_x)$ and $(L_x \cup \{x\}, R_x)$ are *p*-surreal numbers; moreover, if $\{x_\alpha\}_{\alpha < \beta}$ is a sequence of *p*-surreal numbers of infinite limit length β for which $x_\mu \in L_{x_\nu} \cup R_{x_\nu}$ whenever $\mu < \nu < \beta$, and $L_{x_\alpha} = \varnothing$ for at most finitely many α and $R_{x_\alpha} = \varnothing$ for at most finitely many α, then $\left(\bigcup_{\alpha < \beta} L_{x_\alpha}, \bigcup_{\alpha < \beta} R_{x_\alpha} \right)$ is a *p*-surreal number. No_P is the class of all *p*-surreal numbers so defined.

Alternatively, if one wants to base No_P on sequences of +s and −s (in harmony with the alternative approach to *No* mentioned at the end of Section 6), one can define No_P as the class of all sequence of +s and −s (indexed over an ordinal) that neither begin with an infinite sequence of +s nor an infinite sequence of −s.

[29] Equivalently, one could define a totally ordered class $\langle A, < \rangle$ to be a *Peircean linear continuum* if it is absolutely dense and it contains an isomorphic copy of the ordered set of integers that is both cofinal and coinitial with $\langle A, < \rangle$. Plainly, the latter definition is implied by the definition in the main body of the text. Conversely, by two applications of

class $\langle A, < \rangle$ is a Peircean linear continuum if and only if it is absolutely dense and it contains an isomorphic copy of the ordered set of real numbers, say, $A_{\mathbb{R}}$, such that every member of A lies between two members of $A_{\mathbb{R}}$.

Insofar as Peirce appears to have envisioned his purported linear continuum to be an extension of a Cantor-Dedekind linear continuum, the former of whose "nonstandard" points lie between pairs of members of the Cantor-Dedekind linear continuum in question, a Peircean linear continuum, as defined above, is compatible with this aspect of Peirce's vision.

Let $\langle No_P, < \rangle$ be the class No_P of finite and infinitesimal elements of No together with the order it inherits from $\langle No, < \rangle$. The relation between $\langle No_P, < \rangle$ and Peircean linear continua is given by

PC 1. $\langle No_P, < \rangle$ is (up to isomorphism) the unique Peircean linear continuum.

Proof. Plainly, since $\langle No, < \rangle$ is an absolute linear continuum, $\langle No_P, < \rangle$ is a Peircean linear continuum. Accordingly, to complete the proof it suffices to show that in NBG any two Peircean linear continua are isomorphic. For this purpose, let $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ be Peircean linear continua, A_0 be an isomorphic copy of the ordered set of real numbers that is a cofinal and coinitial subset of $\langle A, <_A \rangle$, B_0 be an isomorphic copy of the ordered

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absolute density one may show that the definition in the main body of the text is implied by the just-stated condition. Indeed, let $A_{\mathbb{Z}}$ be an isomorphic copy of the ordered set of integers that is both cofinal and coinitial with A. In virtue of the absolute density of A, there is an isomorphic copy of the ordered set of rationals, say $A_{\mathbb{Q}}$, such that $A_{\mathbb{Z}} \subset A_{\mathbb{Q}} \subset A$; moreover, by a second application of the absolute density of A, there is an isomorphic copy of the ordered set of reals, say $A_{\mathbb{R}}$, such that $A_{\mathbb{Z}} \subset A_{\mathbb{Q}} \subset A_{\mathbb{R}} \subset A$ where the members of $A_{\mathbb{R}} - A_{\mathbb{Q}}$ are arbitrarily selected members of A that fill the various Dedekind gaps in $A_{\mathbb{Q}}$, each such gap being filled by precisely one member of $A_{\mathbb{R}} - A_{\mathbb{Q}}$. Finally, since $A_{\mathbb{Z}}$ is both cofinal and coinitial with A, so are $A_{\mathbb{Q}}$ and $A_{\mathbb{R}}$.

set of real numbers that is a cofinal and coinitial subset of $\langle B, <_B \rangle$, and $\bar{a}$ and $\bar{b}$ be well

orderings of $A - A_0$ and $B - B_0$, respectively. The existence of $\bar{a}$ and $\bar{b}$ are guaranteed

by the axiom of global choice. We obtain the desired surjection $F : A \rightarrow B$ by defining a

chain $f_\alpha (\alpha < On)$ of isomorphisms where $F = \bigcup_{\alpha < On} f_\alpha$ as follows. If $\alpha = 0$, we let f_0

be an order preserving isomorphism from A_0 onto B_0; if $\alpha = 2\beta + 1$, we take the first

unused element of $\bar{a}$, call it $a_{2\beta+1}$, and let $f_{2\beta+1} : A_{2\beta+1} = A_{2\beta} \cup \{a_{2\beta+1}\} \rightarrow B$ be the unique

order injection extending $f_{2\beta}$ that sends $a_{2\beta+1}$ to the first unused element of $\bar{b}$ that fills

the corresponding cut in $B_{2\beta} = f_{2\beta}(A_{2\beta})$ that $a_{2\beta+1}$ fills in $A_{2\beta}$;[30] if $\alpha = 2\beta + 2$, we take

the first unused element of $\bar{b}$, call it $b_{2\beta+2}$, and let $f_{2\beta+2} = g_{2\beta+2}^{-1}$ where

$g_{2\beta+2} : B_{2\beta+2} = \left(f_{2\beta+1}(A_{2\beta+1}) \cup \{b_{2\beta+2}\} \right) \rightarrow A$ is the unique order injection extending $f_{2\beta+1}^{-1}$

that sends $b_{2\beta+2}$ to the first unused element of $\bar{a}$, call it $a_{2\beta+2}$, that fills the corresponding

cut in $A_{2\beta+1}$ that $b_{2\beta+2}$ fills in $B_{2\beta+1} = f_{2\beta+1}(A_{2\beta+1})$; and if α is an infinite limit ordinal, we

let $f_\alpha = \bigcup_{\tau < \alpha} f_\tau$. The existence of f_0 is evident and the existence of the $f_{\alpha+1}$s for all α is

a simple consequence of the absolute density of Peircean linear continua.

As the epigraph dated 1898 that dons the title page of our text indicates, Peirce

held that a continuum "is all that is possible, in whatever dimension it be continuous" or,

to put this another way, it exhibits all possible gradations "in whatever dimension it be

continuous". Interestingly, however, while Peirce *may* have encountered Veronese's

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[30] If A is an ordered class, B is an ordered subclass of A and $a \in A - B$, then there is a
unique cut (X,Y) of B such that $X < \{a\} < Y$. In this case, as was noted in Section 3, a is
said to *fill* the cut (X,Y). Moreover, if $f : B \rightarrow C$ is an order-preserving isomorphism
from B onto C, then $(f(X), f(Y))$ is said to be the cut in C (modulo f) that corresponds
to (X,Y) in B.

work on non-Archimedean geometry prior to writing the above words,[31] he does not appear to have considered lines in his theory of continua having the property that given an arbitrary unit of measure there are segments of the line that are infinitely large and others that are infinitely small, relative to the unit of measure, i.e., the kind of lines investigated by Veronese [1891; 1894] and the other pioneering non-Archimedean geometers of Peirce's day. Rather, he envisions his linear continuum as supplementing the points that lie "at a finite distance from one another" on the Cantor-Dedekind linear continuum with "points [that] can be at infinitesimal distances" [Peirce 1900, p. 363]. Indeed, as we noted earlier, Peirce appears to presuppose that an isomorphic copy of the Cantor-Dedekind linear continuum is both cofinal and coinitial with the linear continuum he has in mind. Accordingly, when Peirce tells us that a continuum "is all that is possible, in whatever dimension it be continuous" his words must be understood with this very real proviso in mind.

The concept of absolute density provides one important sense in which Peircean Linear Continua are exceptionally rich in order-theoretic possibilities that are definable in terms of sets of NBG. Our next result sheds further light on their richness in this regard

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[31] In an incomplete letter to William E. Story dated March 22, 1896, Peirce says of his then unpublished work *New Elements of Mathematics*: "It is something like Veronese's Geometry, but is (I think) far deeper logically, and certainly far simpler. Nor do I directly go in to non-Euclidean geometry" [Peirce March 22, 1896, p. v]. Since Veronese only published three books on geometry prior to the date of Peirce's letter, his *Fondamenti di Geometria* [1891], a treatise in which he develops non-Euclidean geometry and non-Archimedean geometry, the German translation thereof [1894], and his *Elementi di Geometria* [1895], a textbook devoted to Euclidean geometry, presumably Peirce is referring to Veronese's *Fondamenti* or its German translation. Assuming this to be the case, by 1896 Peirce *may* have been familiar with some aspects of Veronese's contributions to non-Archimedean geometry.

by establishing the equivalence between absolute density and two revealing alternative conceptions.

Let $\langle A,< \rangle$ be an ordered class. A subclass X of A will be said to *bounded* if there are $a,b \in A$ such that $\{a\} < X < \{b\}$. Moreover, if whenever X is a bounded sub*set* of A and B is an ordered class that extends X, there is an isomorphism from B into A that is an extension of the identity map on X, then $\langle A,< \rangle$ will be said to be *universally extending with respect to bounded subsets*. Furthermore, $\langle A,< \rangle$ will be said to be *homogeneous universal with respect to bounded subsets*, if it is *universal with respect to bounded subsets*--every ordered class is isomorphic to a bounded subset of $\langle A,< \rangle$--and it is *homogeneous with respect to bounded subsets*--every isomorphism between bounded sub*sets* of $\langle A,< \rangle$ can be extended to an automorphism of $\langle A,< \rangle$.

The reader will notice that insofar as every absolute linear continuum is absolutely extensive, every subclass X of an absolute linear continuum is bounded. This, of course, is not true of Peircean linear continua. It is for this reason that for their characterization we have to employ the "bounded" variants of the definitions of universally extending ordered classes and homogeneous universal ordered classes employed in Section 2 in our characterizations of absolute linear continua.

The following two simple lemmas are emplyed in the proof of the aforementioned equivalence theorem. The second lemma is a consequence of the first and the fact that every open interval (a,b) of an absolute linear continuum is itself an absolute linear continuum [see Note 19]; and the first follows from the fact that $\langle No_P,< \rangle$, which is (up to isomorphism) the unique Peircean linear continuum, is a *convex* subclass of $\langle No,< \rangle$,

where a subclass *I* of an ordered class $\langle A, < \rangle$ is said to be convex (or a *segment*) if every member of *A* that lies between two members of *I* is likewise a member of *I*.

LEMMA A. A Peircean Linear Continuum is a convex subclass of an absolute linear continuum.

LEMMA B. Every open interval (a,b) of a Peircean linear continuum is an absolute linear continuum.

PC 2. The following are equivalent for a nontrivial ordered class $\langle A, < \rangle$:

i. $\langle A, < \rangle$ is absolutely dense;

ii. $\langle A, < \rangle$ is universally extending with respect to bounded subsets;

iii. $\langle A, < \rangle$ homogeneous universal with respect to bounded subsets.

Proof. Suppose $\langle A, < \rangle$ is absolutely dense. Further suppose *X* is a subset of *A* for which there are $a,b \in A$ such that $\{a\} < X < \{b\}$, and let *B* be an ordered class that extends *X*. By Lemma B, the open interval (a,b) of $\langle A, < \rangle$ is an absolute linear continuum, and so, by Theorem 3, (a,b) is a universally extending ordered class, from whence it follows that $\langle A, < \rangle$ is universally extending with respect to bounded subsets. Now suppose $\langle A, < \rangle$ is universally extending with respect to bounded subsets. Since $\langle A, < \rangle$ is nontrivial, there are $a,b \in A$ for which $a < b$. Furthermore, insofar as $\{a\} < \varnothing < \{b\}$, the empty subset $\varnothing$ of *A* is bounded by *a* and *b*. Accordingly, since every nonempty ordered class is isomorphic to an extension of the empty ordered set, every ordered class is isomorphic to a bounded subset of $\langle A, < \rangle$ that extends $\varnothing$. Now suppose *B* and *B'* are bounded subsets of *A* and f_0 is an isomorphism from *B* onto *B'*. Since *B* and *B'* are bounded subsets of

A, there are $a,b \in A$ such that $\{a\} < B \cup B' < \{b\}$. Let $\overline{B}$ be a well ordering of $(a,b) - B$

and $\overline{B}'$ be a well ordering of $(a,b) - B'$. To show that f_0 can be extended to an

automorphism of *A* it suffices to show that f_0 can be extended to an automorphism *F* of

(a,b) since such an automorphism can always be extended to an automorphism f of *A*

by setting $f(x) = F(x)$ for $x \in (a,b)$ and $f(x) = x$, otherwise. The proof of the existence

of *F*, the details of which are left to the reader, is essentially the same as the back and

forth argument used above to prove that any two Peircean linear continua are isomorphic

except that one now appeals to universal extensibility with respect to bounded subsets

rather than to absolute density to establish the existence of the requisite members of $\overline{B}$

and $\overline{B}'$ needed to fill the corresponding cuts in the increasingly richer subsets of (a,b)

that are constructed in the course of the proof. Finally, suppose $\langle A, < \rangle$ homogeneous

universal with respect to bounded subsets, and *X* and *Y* are bounded subsets of *A* for

which $X < Y$. Since *X* and *Y* are bounded, there are $a,b \in A$ such that $\{a\} < X \cup Y < \{b\}$.

Let *Z* be an ordered set having universe $X \cup \{z\} \cup Y$ where $X < \{z\} < Y$. Using the

technique of *successively filling or adjoining cuts,*[32] one can show that such an ordered

set exists. Moreover, since every ordered class is isomorphic to a bounded subset of

$\langle A, < \rangle$, there is a bounded isomorphic copy of *Z* in $\langle A, < \rangle$ whose universe is given by

$X' \cup \{z'\} \cup Y'$ where $X' < \{z'\} < Y'$ and for which there is an order preserving

isomorphism $g : X' \cup Y' \to X \cup Y$. But since $X' \cup Y'$ and $X \cup Y$ are bounded, it follows

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[32] The technique of successively filling or adjoining cuts was introduced by Norberto
Cuesta Dutari [1954; 1958-59] and rediscovered by Egbert Harzheim [1964]. For a good
discussion of the method, see [Harzheim 2005, pp. 110-114].

from the hypothesis that g can be extended to an automorphism g' of $\langle A, < \rangle$. But then $X < \{g'(z')\} < Y$, which establishes the absolute density of $\langle A, < \rangle$.

9. Peircean Linear Continua and Peirce's Continuity Condition

As we have already noted, Peirce held that a continuum "is all that is possible, in whatever dimension it be continuous." Moreover, as the second quotation from the title page of our text indicates, according to Peirce: "the possibility of determining more than any given multitude of points…at every part of the line, makes it *continuous*." Peirce appears to take the notion of a "part of a line" to be sufficiently well understood, which seems to suggest that, following philosophical tradition, he takes the parts of a line to be its rays and its segments determined by pairs of distinct points. Accordingly, insofar as a ray contains a proper class of points if each of its nontrivial segments does, we will say that an ordered class satisfies *Peirce's continuity condition* if every nontrivial closed interval of the ordered class contains a proper class of elements.[33]

Every Peircean linear continuum in our sense satisfies Peirce's continuity condition. On the other hand, there is vast array of ordered classes that satisfy Peirce's continuity condition that are neither Peircean linear continua in our sense nor even convex subclasses of absolute linear continua more generally. Indeed, despite what Peirce apparently believed, his continuity condition is far too weak to ensure the presence of all possible gradations that are individually definable in terms of sets (even when one factors in the aforementioned Peircean constraint regarding the cofinal and coinitial containment of the classical continuum). This point could be illustrated by simply pointing to the

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[33] This formulation of Peirce's continuity condition appears to be in harmony with the related remarks of Myrvold [1995, p. 535] and Herron [1997, pp. 608-609].

existence of ordered classes that satisfy Peirce's continuity condition that are not absolutely dense. However, we believe it would be more revealing to first decompose absolute density into two independent conditions and provide illustrations that show that Peirce's continuity condition implies neither.

For this purpose we require the following generalizations for ordered classes of classical concepts for ordered sets due to Hausdorff [1906; 1914, Chapter VI; also see Harzheim 2005, pp. 77-79]. An ordered class A will be said to have *cofinal character* α (*coinitial character* α^*) if α is the least ordinal $\leq On$ such that there is a cofinal (coinitial) subclass of A that is isomorphic with α (α^*, where α^* is the inverse of α).[34] Moreover, a gap (X,Y) in A (see Note 27) will be said to have *character* (α,β^*) if α is the cofinal character of X and β^* is the coinitial character of Y; and, in harmony with the terminology introduced in Section 3, the gap of will be said to be *definable in terms of sets*, or be a *set-gap*, if α and β^* are both sets. In addition, if x is a member of an ordered class A, then x will be said to have *character* (α,β^*) if $\{y \in A : y < x\}$ has cofinal character α and $\{y \in A : x < y\}$ has coinitial character β^*. Finally, by an *extremal element* of an ordered class we mean the least element or the greatest element, i.e., an element whose character has the form $(0,\beta^*)$ or $(\alpha,0)$, respectively.

If an element of an ordered class has an immediate successor or an immediate predecessor, it has character $(\alpha,1)$ or $(1,\alpha)$, respectively, for some ordinal α. Using this

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[34] The inverse of an ordered class $\langle A,< \rangle$ is the ordered class $\langle A,<^* \rangle$ characterized by the property: $a <^* b$ whenever $b < a$, for all $a,b \in A$.

together with the fact that every Dedekind cut of a nontrivial densely ordered class is either a gap or a continuous cut (see Appendix), it is straightforward matter to prove

PC 3. For a nontrivial ordered class A the following are equivalent:

(i) A is absolutely dense;

(ii) A has no set-gaps and every element of A has character (On, On^*), $(0, On^*)$ or $(On, 0)$ depending on whether it is a nonextremal element, a least element or a greatest element, respectively.

Since a Peircean linear continuum is a nontrivial densely ordered class having neither a least nor a greatest element, it follows from PC 3 that absolute density for Peircean linear continua is equivalent to the absence of set-gaps and the ubiquitous presence of elements of character (On, On^*). The following examples show that Peirce's continuity condition is independent of each.

First consider the ordered subclass of No consisting of $No_P \cup No_P{}'$ where $No_P{}'$ is the set of all $x \in No$ such that $\omega - n < x < \omega + n$ for some positive integer n. Since $No_P{}'$ is a Peircean linear continuum, $No_P \cup No_P{}'$ consists of No_P followed by an isomorphic copy of itself, and therefore satisfies Peirce's continuity condition. On the other hand, $No_P \cup No_P{}'$ is not absolutely dense since it contains a set-gap. In particular, since the ordered set of non-negative integers is a cofinal subset of No_P and the ordered set of

surreal numbers of the form $\ldots, \omega - n, \ldots, \omega - 1, \omega - 0 = \omega$ (where n is a non-negative

integer) is a coinitial subset of ${No_P}'$, $\left({No_P}, {No_P}'\right)$ is gap of character $\left(\omega, \omega^*\right)$.[35]

Next, consider the ordered subclass of *No* that results from supplementing

$No_P \cup {No_P}'$ with $\omega/2$. Plainly, $No_P \cup \{\omega/2\} \cup {No_P}'$ satisfies Peirce's continuity

condition. Nevertheless, $No_P \cup \{\omega/2\} \cup {No_P}'$ is not absolutely dense since $\omega/2$, which

lies between the members of No_P and the members of ${No_P}'$, has character $\left(\omega, \omega^*\right)$ in

$No_P \cup \{\omega/2\} \cup {No_P}'$.[36]

As the above two examples intimate and PC 3 shows, the presence in a densely

ordered class of a set-gap or an element having character $\left(\alpha, \beta^*\right)$ where α or β is a non-

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[35] Wayne Myrvold [1995, pp. 535-537] introduced an ordered class that satisfies Peirce's continuity condition but likewise contains set-gaps. For example, there is a cut (X, Y) of Myrvold's line in which X contains $\left\{n\varepsilon^2 : n \in \mathbb{Z}^+\right\}$ as a cofinal subset and Y contains $\left\{\varepsilon/n : n \in \mathbb{Z}^+\right\}$ as a coinitial subset, where $\mathbb{Z}^+$ is the set of positive integers and ε and ε^2 are a pair of infinitesimals for which ε^2 is infinitesimal relative to ε. This cut, as is evident from the above description, is an $\left(\omega, \omega^*\right)$-gap.

[36] The two examples presented in the main body of the text have been selected for their simplicity. We hasten to add, however, that more robust examples can be found. To begin with, in accordance with a classical result from the theory of ordered fields, every open interval (a, b) of an ordered field is order isomorphic to the given ordered field (cf. [Sikorski, 1948, p. 74 (iv)]). Plainly then, if A is an ordered field containing a proper class of members, every nontrivial convex subclass of A likewise contains a proper class of members. Accordingly, by limiting such an ordered field A to its substructure A_P consisting of its finite and infinitesimal members, one will obtain a structure satisfying Peirce's continuity condition. Moreover, if A contains an isomorphic copy of the ordered field of real numbers, A_P will contain a copy of the reals that is both coinitial and cofinal with A_P. An example of such an A_P that contains a set-gap is obtained by letting A be the smallest subfield of *No* containing *No*'s reals and *No*'s ordinals, and an instance of such an A_P *all* of whose elements have characters that are sets is obtained by letting A be any of the ordered fields referred to in [Ehrlich 1992: Theorem 7, p. 175] other than $\mathbb{R}$ and *No*.

zero ordinal $< On$ indicates the presence of a *vacancy* that is definable in terms of sets. By such a vacancy we of course mean a pair of nonempty subsets L and R where $L < R$ for which there is no y in the class such that $L < \{y\} < R$. Absolutely dense ordered classes, by contrast, are characterized by the complete absence of such vacancies, and in this technical sense their elements are "welded together" or "cemented together", as Peirce picturesquely described the elements of his envisioned continuum. On the other hand, of course, if one allows for vacancies defined as above where either L or R are of necessity proper classes, then absolutely dense ordered classes have a great many vacancies indeed (see Postscript).

While Peirce did not describe the welded-togetherness of the points of his linear continuum in terms of the absence of vacancies that are definable in terms of sets, there is a strong family resemblance. Peirce thought the points of his linear continuum are "welded together" insofar as "there is no longer any room for…inserting any more...." [Peirce *circa* 1897, p. 95], and as the definition of an absolutely dense ordered class makes clear, it is indeed impossible to insert any more elements into such an ordered class, assuming such insertions amount to filling vacancies that are defined in terms of sets, the only vacancies Peirce's metaphysics would countenance.

It is perhaps worth adding that there is another important sense in which the elements of an absolutely dense ordered class, and a Peircean linear continuum in particular, may be said to be set-theoretically "welded together". Namely, insofar as no element of an absolutely dense ordered class has character $< On$, *it is impossible to whittle down a nontrivial interval of an absolutely dense ordered class to a point by means of a set number of cuts that divide nontrivial intervals into pairs of nontrivial*

subintervals. Indeed, to separate a point from a nontrivial interval by means of such divisions, *a proper class of such cuts is always required,* [37] something Peirce's metaphysics would not allow!

10. The Peircean Arithmetic Continuum

While Peirce staunchly advocated the use of infinitesimals in the calculus and pictured them manipulated algebraically [Peirce 1893, pp. 128-131; 1892, p. 98], he never attempted to impose an ordered algebraic structure on his envisioned linear continuum. Be that as it may, we believe that much as it is instructive to compare $\mathbb{R}$ and *No* from an ordered-algebraic point of view, it is likewise instructive to so compare $\mathbb{R}$ with No_P. When comparing $\mathbb{R}$ and No_P from this perspective, it is the conception of an *ordered (integral) domain*[38] that is fundamental rather than the conception of an ordered field, which was basic when $\mathbb{R}$ and *No* were the basis of comparison.

Whereas the properties of ordered fields generalize the familiar algebraic, order-theoretic and compatibility properties of the system of rational numbers, the properties of ordered domains generalize the analogous, though more general, properties of the system of integers. Indeed, while every ordered field is an ordered domain, there are ordered domains that are not ordered fields, insofar as, like the integers, they contain non-zero elements having no multiplicative inverse in the domain.

Every ordered domain A contains a canonical copy of the ordered domain $\mathbb{Z}$ of integers, namely, the subdomain of all elements of A of the form $n \cdot 1_A$ where n is an

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[37] This holds true whether we use the strict mathematical definition of a cut in which there is no overlap between the nontrivial subintervals or if we allow that the nontrivial subintervals overlap solely at the point of the cut.

[38] For the definition of an ordered domain, see the Appendix.

integer and 1_A is the unit element of *A*. Henceforth, we will refer to the elements of this subdomain of *A* as *the integers of A*. Moreover, an ordered domain *A* will be said to be *finitely bounded* if every element of *A* lies between two of *A*'s integers. While a finitely bounded ordered domain may contain elements that are infinitesimal, it cannot contain elements that are infinitely large.[39]

It is well known that an ordered field *K* is real-closed if and only if it satisfies the *intermediate value theorem for polynomials* (*in one variable*) *with coefficients in K*, i.e. the condition: if f is a polynomial with coefficients in *K*, $[a,b] \subset K$ and $f(a) \neq f(b)$, then for any $d \in K$ between $f(a)$ and $f(b)$, there is a c between a and b such that $f(c) = d$ (cf. [Gamboa, 1987]). From a geometrical point of view, this means that if the graph of a polynomial with coefficients in *K* has points on the opposite sides of a line, then the portion of the graph lying between the two points intersects the given line.[40]

Cherlin and Dickmann [1983] extended this idea to ordered domains more generally, likewise calling an ordered domain *K real-closed* if it satisfies the intermediate

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[39] An element a of an ordered domain is said to be *infinitesimal* if $n|a| < 1$ for all positive integers n, and it is said to be *infinite* if $|a| > n$ for all positive integers n.

[40] While Peirce does not appear to have investigated the intermediate value theorem, he likely would have appreciated its connection with the idea of continuity. In a marginal note to one of his essays on the continuum he observes, "it is impossible to get the idea of continuity without two dimensions. An oval line is continuous, because it is impossible to pass from the inside to the outside without passing a point of the curve" [Peirce April 1892, p. 115].

In connection with this, it is perhaps also worth mentioning that insofar as the intermediate value theorem fails for even polynomial functions in models of *smooth infinitesimal analysis* (SIA) (cf. [Bell 1998, pp. 105-106; 2005, p. 297]), it is by no means obvious that Peirce would favor (for the development of his theory) the theory of infinitesimals from SIA over the theory of infinitesimals employed in nonstandard analysis (and, of course, in *No*), as Herron [1997, p. 623] contends. There are other reasons for our doubts but we will not pursue them at this time.

value theorem for polynomials (in one variable) with coefficients in K. They also characterized this important class of ordered domains by means of the following result:

(Cherlin and Dickmann, 1983: Theorem 1). An ordered domain is real-closed if and only if it is a convex ordered subdomain of a real-closed ordered field.

Accordingly, if K is a real-closed ordered domain, then either K is a real-closed ordered field or K is an ordered subdomain of a real-closed ordered field K^* where K is a proper subclass of K^* having the property: every member of K^* that lies between two members of K is likewise a member of K.

As we suggested above in Section 2, the relation that exists between *No* and $\mathbb{R}$ considered as arithmetic continua foreshadows to a great extent the corresponding relation that exists between our proposed model of Peirce's continuum considered as an ordered algebraic system and its classical counterpart. Central to the latter relation is

PC 4. Whereas $\langle \mathbb{R},+,\cdot,<,0,1 \rangle$ is (up to isomorphism) the unique finitely bounded, real-closed ordered domain that is a Cantor-Dedekind linear continuum, $\langle No_P,+,\cdot,<,0,1 \rangle$ is (up to isomorphism) the unique finitely bounded, real-closed ordered domain that is a Peircean linear continuum.

Proof. As is well known, $\mathbb{R}$ is (up to isomorphism) the unique ordered domain that is a Cantor-Dedekind linear continuum. Thus, to complete the first part of the proof we need only note that $\mathbb{R}$ is both finitely bounded and real-closed. Now note that, by definition, No_P is finitely bounded. Moreover, No_P is a Peircean linear continuum. Furthermore, since No_P is a convex ordered subdomain of the real-closed ordered field *No*, by the just-

stated theorem of Cherlin and Dickmann, No_P is real-closed. Now suppose A is a finitely bounded, real-closed ordered domain that is a Peircean linear continuum. Again by Cherlin and Dickmann's theorem, A is a convex ordered subdomain of a real-closed field B. Suppose (a,b) is an open interval of A. By Lemma B (from Section 8), (a,b) is an absolute linear continuum. But then B is an absolute linear continuum insofar as every open interval of an ordered field is order isomorphic to the ordered field itself (see Note 36). Therefore, since No is (up to isomorphism) the unique real-closed ordered field that is an absolute linear continuum, B is isomorphic to No. Consequently, A is isomorphic to the unique finitely bounded, convex subdomain of No; that is, A is isomorphic to No_P.

The next of our results that casts light on the relation between $\mathbb{R}$ and No_P considered as arithmetic continua highlights their comparative structures as inclusive Archimedean and non-Archimedean finitely bounded ordered domains. In its formulation, the notions of *universally extending* and *homogeneous universal* applied to Archimedean finitely bounded ordered domains and to finitely bounded ordered domains more generally are defined in the analogous fashion as their field-theoretic counterparts employed in Theorem 4.

PC 5. (I) The following sets of axioms constitute (categorical) axiomatizations of $\langle \mathbb{R},+,\cdot,<,0,1\rangle$; (II) by deleting the Archimedean axiom from the following axiomatizations one obtains categorical axiomatizations of $\langle No_P,+,\cdot,<,0,1\rangle$.

Axioms for finitely bounded ordered domains

Archimedean axiom

Axiom of Homogeneous Universality

or, alternatively,   } (*Continuity Axioms*)

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where by the Axiom of Homogeneous Universality (Axiom of Universal Extensibility) we mean the assertion: The collection of numbers together with the corresponding relations defined on it constitutes a Homogeneous Universal (Universally Extending) model of the above stated axioms.[41]

To prepare the way for the proof of PC 5, we need a number of preliminary results beginning with the following simple consequence of the fact that (i) *No* is a homogeneous universal ordered field, and the fact that (ii) every ordered domain *D* admits (up to isomorphism) a unique extension to its *ordered field of fractions* and each embedding of *D* into an ordered field *K* has a unique extension to an embedding of the ordered field of fractions of *D* into *K* (cf. [Warner 1965/1990, p. 220]).

LEMMA 0. No_P is a homogeneous universal ordered domain.[42]

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[41] In his little known paper "Continuity and the Theory of Measurement," José Benardete [1968] attempts to develop a theory of "absolute continuity" that resembles, in some respects, Peirce's conception. For example, Benardete envisions the possibility that his continuum *may* contains a proper class of points (albeit only potentially), each of which is no more than an infinitesimal distance from a point on a classical linear continuum. In his attempt to characterize his absolute continuum, Benardete asks, "What are the true axioms of continuity?" and answers, "They are the standard axioms with the Archimedean axiom both deleted and denied" [1968, pp. 424-425]. What Benardete means by the "standard axioms" he does not say thought I am not aware of any standard axiomatization of the classical arithmetic continuum that yields his intended result. It is interesting to note, however, that the above axiomatizations, which make use of continuity axioms that are equivalent to the classical continuity axioms in the context of an axiomatization of $\langle \mathbb{R},+,\cdot,<,0,1 \rangle$, realizes Benardete's goal without having to explicitly deny the Archimedean axiom.

[42] In fact, using Theorem 4 in conjunction with the above stated classical result regarding ordered fields of fractions, one can readily prove the following stronger

THEOREM. $\langle No,+,\cdot,<,0,1 \rangle$ is (up to isomorphism) both the unique universally extending ordered domain and the unique homogeneous universal ordered domain.

Also employed in our proof of PC 5 are the following three lemmas regarding finitely bounded ordered domains, the proof of each of which uses the stated corresponding classical result regarding real-closed ordered fields and Cherlin and Dickmann's above stated theorem. The proof of Lemma 1 also uses the just-stated classical result regarding the ordered field of fractions of an ordered domain; and in the statement of Lemma 2, by $[X]_A$ we mean *the ordered subfield (ordered subdomain) of an ordered field (ordered domain) A generated by a subset X of A*, i.e., the least ordered subfield (ordered subdomain) of A containing the members of a subset X of A.

LEMMA 1. If A is an ordered field (a finitely bounded ordered domain), there is up to isomorphism a least real-closed ordered field (real-closed finitely bounded ordered domain) extending A--a *real-closure* of A. Moreover, if $f : A \rightarrow B$ is an isomorphism of ordered fields (finitely bounded ordered domains), there is an isomorphism from a real-closure of A onto a real closure of B that extends f.

LEMMA 2. If A and A' are real-closed ordered fields (real-closed finitely bounded ordered domains) where $A' \subset A$ and $a \in A - A'$, there is a least real-closed ordered subfield (real-closed finitely bounded ordered subdomain) of A containing the members of $A' \cup \{a\}$, henceforth written $(A' \cup \{a\})_A$. $(A' \cup \{a\})_A$, which has the same cardinality as $A' \cup \{a\}$, is the real-closure in A of $[A' \cup \{a\}]_A$.

See, [Ehrlich 1989, pp. 43-44 and Ehrlich 1992, p. 171: (vi)].

LEMMA 3. If A, A' and B are real-closed ordered fields (real-closed finitely bounded ordered domains) where $A' \subset A$ and $f : A' \to B$ is a monomorphism of ordered fields (ordered domains), then if $a \in A - A'$ and $h : A' \cup \{a\} \to B$ is an order injection that extends f, there is a unique monomorphism $g : (A' \cup \{a\})_A \to B$ of ordered fields (ordered domains) that extends h.

Finally, we make use of the following special case for ordered domains of a general classical result regarding the embedding one algebraic structure into another (cf. [Warner 1965/1990, pp. 54-56]).

ELEMENTARY OBSERVATION. If A, A' and B are ordered domains where $A \subset A'$, and $f : A \to B$ is a surjection of ordered domains, there is an ordered domain B' where $B \subset B'$ and a surjection $f' : A' \to B'$ of ordered domains extending f.

Proof of PC 5. To prove I it suffices to note that every Archimedean ordered domain is a finitely bounded ordered domain and the well-known fact that there is one and only one isomorphism of an Archimedean ordered domain into $\mathbb{R}$ [Warner 1965/1990, pp. 479-483].

The proof of II consists of three parts. First note that: (i) No_P is a homogeneous universal finitely bounded ordered domain. Indeed, since No_P a homogeneous universal ordered domain (Lemma 0) and since the image of every isomorphism of a finitely bounded ordered domain into No lies in No_P, No_P is universal. Moreover, since every isomorphism between finitely bounded ordered subdomains of No_P whose universes are

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sets can be extended to an automorphism of *No* one obtains the requisite automorphisms of No_P by restricting the former to No_P.

Next note that: (ii) If *A* is a homogeneous universal finitely bounded ordered domain, then *A* is likewise a universally extending finitely bounded ordered domain. Indeed, suppose *B* is a finitely bounded ordered subdomain of *A* whose universe is a set and *C* is a finitely bounded ordered domain that extends *B*. Then there is an isomorphism f from *C* to *A*. Moreover, since *C* extends *B*, the image of the restriction of f to *B* is an isomorphic copy of *B* in *A*. Furthermore, in virtue of the homogeneity of *A*, there is an automorphism *g* of *A* such that $g(x) = f(x)$ for all $x \in B$. But then, the function $g^{-1}f$ is an isomorphism from *C* to *A* that is an extension of the identity on *B*; thereby proving *A* is universally extending.

In virtue of (i) and (ii), to complete the proof of II it suffices to prove: (iii) If *A* is a universally extending finitely bounded ordered domain, then *A* is isomorphic to No_P. We first show: if *A* is a universally extending finitely bounded ordered domain, then *A* is real-closed. Let $A_{\mathbb{Z}}$ be *A*'s ordered domain of integers and $\bar{a}$ be a well ordering of $A - A_{\mathbb{Z}}$. Since the union of any chain of real-closed finitely bounded ordered domains is itself a real-closed finitely bounded ordered domain, to complete the first part of the proof we need only note that $A = \bigcup_{\alpha < On} A_{\alpha}$, where A_0 is a real-closure of $A_{\mathbb{Z}}$ in *A*, $A_{\alpha} = \bigcup_{\beta < \alpha} A_{\beta}$ if α is an infinite limit ordinal, and for each ordinal α, $A_{\alpha+1}$ is a real-closure in *A* of $\left[A_{\alpha} \cup \{a_{\alpha}\}\right]_A$, where a_{α} is the first member of $\bar{a}$ not contained in A_{α}. The existence of A_0 and the various $A_{\alpha+1}$s follows from the existence of the real-closures of

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$A_{\mathbb{Z}}$ and the various $\left[A_\alpha \cup \{a_\alpha\}\right]_A$'s (see Lemmas 1 and 2) and the fact that A is universally extending.

Now let $No_P = B$, A_0 and B_0 be the real-closures in A and B of A's and B's integers, and $\bar{a}$ and $\bar{b}$ be well orderings of $A - A_0$ and $B - B_0$, respectively. We obtain the desired surjection $F : A \to B$ by defining a chain $f_\alpha(\alpha < On)$ of isomorphisms where $F = \bigcup_{\alpha < On} f_\alpha$ as follows. If $\alpha = 0$, we let f_0 be the unique isomorphism from A_0 onto B_0; if $\alpha = 2\beta + 1$, we take the first unused element of $\bar{a}$, call it $a_{2\beta+1}$, and let

$f_{2\beta+1} : A_{2\beta+1} = \left(A_{2\beta} \cup \{a_{2\beta+1}\}\right)_A \to B$ be the unique embedding extending an order

injection $h_{2\beta+1} : A_{2\beta} \cup \{a_{2\beta+1}\} \to B$ extending $f_{2\beta}$ that sends $a_{2\beta+1}$ to the first unused

element of $\bar{b}$, call it $b_{2\beta+1}$, that fills the corresponding cut in $B_{2\beta} = f_{2\beta}\left(A_{2\beta}\right)$ that $a_{2\beta+1}$

fills in $A_{2\beta}$ (see Note 30); if $\alpha = 2\beta + 2$, we take the first unused element of $\bar{b}$, call it

$b_{2\beta+2}$, and let $f_{2\beta+2} = g_{2\beta+2}^{-1}$ where $g_{2\beta+2} : B_{2\beta+2} = \left(f_{2\beta+1}\left(A_{2\beta+1}\right) \cup \{b_{2\beta+2}\}\right)_B \to A$ is the

unique embedding extending an order injection $h_{2\beta+2} : f_{2\beta+1}\left(A_{2\beta+1}\right) \cup \{b_{2\beta+2}\} \to A$

extending $f_{2\beta+1}^{-1}$ that sends $b_{2\beta+2}$ to an unused element of $\bar{a}$, call it $a_{2\beta+2}$, that fills the

corresponding cut in $A_{2\beta+1}$ that $b_{2\beta+2}$ fills in $B_{2\beta+1} = f_{2\beta+1}\left(A_{2\beta+1}\right)$; and if α is an infinite

limit ordinal, we let $f_\alpha = \bigcup_{\tau < \alpha} f_\tau$. The existence of f_0 is evident; the existence of the

$f_{2\beta+1}$s and $f_{2\beta+2}$s follow from Lemma 3, the fact that A and B are real-closed, and the

existence of the $b_{2\beta+1}$s and $a_{2\beta+2}$s, respectively; the existence of the $b_{2\beta+1}$s is a simple

consequence of the absolute density of Peircean linear continua; and the existence of the

$a_{2\beta+2}$s is a consequence of the aforementioned Elementary Observation and fact that B is universally extending.

Our final result that sheds light on the relation between $\mathbb{R}$ and No_P considered as arithmetic continua is the analog for Peircean arithmetic continua of Theorem 5.

PC 6 (I) The following set of axioms constitutes a categorical axiomatization of $\langle\mathbb{R},+,\cdot,<,0,1\rangle$; (II) by deleting the Archimedean axiom from the following axiomatization one obtains a categorical axiomatization of $\langle No_P,+,\cdot,<,0,1\rangle$.

Axioms for real-closed finitely bounded ordered domains (*Elementary Continuity Axioms*)
Archimedean axiom
Axiom of maximal s-density (*Continuity Axiom*)

where, as the reader will recall from the statement of Theorem 5, the axiom of maximal s-density asserts that: The collection of numbers together with the corresponding relations defined on it constitutes a maximally s-dense model of the above stated axioms.

Proof. Plainly, (I) follows from the classical result that $\mathbb{R}$, which is a real-closed finitely bounded Archimedean ordered domain, is (up to isomorphism) the unique Archimedean ordered domain that admits no proper extension to an Archimedean ordered domain. And (II) follows from the fact that No_P is an absolutely dense real-closed finitely bounded ordered domain, and the fact that every real-closed finitely bounded ordered domain A that is not isomorphic to No_P admits an extension to an isomorphic copy of No_P that fills

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a vacancy in *A* that is definable in terms of sets (see Section 9). The existence of such an extension is a consequence of the second part of PC 4 and the absolute density of No_P.

Conclusion

In his Presidential Address on "Peirce's Continuum" delivered to the *Charles S. Peirce Sesquicentennial International Congress* held at Harvard University in 1989, Hilary Putnam proclaimed:

> *What answers to our conception of a continuum is a possibility of repeated division which can never be exhausted in any possible world, not even in a possible world in which one can complete abnumerably* [i.e., non-denumerably] *infinite processes. That is what I take Peirce's daring metaphysical hypothesis to be.* [Putnam 1995, p. 17; Putnam 1992, p. 51].

We suspect that what Putnam calls "Peirce's daring metaphysical hypothesis" is indeed something very much like what Peirce was after. On the other hand, as we hope we have made clear, if Peirce thought that "the possibility of determining more than any given multitude of points…at every part of the line" is sufficient for characterizing his "daring metaphysical hypothesis," then his "daring metaphysical hypothesis" is not nearly daring enough to underwrite his idea that a "continuum is all that is possible, in whatever dimension it be continuous," even if we ignore the Peircean presupposition that the points are no more than a finite distance from one another. In fact, given that there are finitely bounded ordered domains that are not real-closed but which are nevertheless Peircean linear continua, it is doubtful that any attempt to unpack the latter solely in terms of divisions that arise by repeatedly taking cuts of sets of points would be adequate, even if "one can complete abnumerably infinite processes." This observation is not unlike the late nineteenth-century realization that simple density--the set-theoretic counterpart of

infinite divisibility--is not sufficient to characterize continuity, even the modest sort of

continuity required by the geometry of Euclid.[43]

What then is sufficient for characterizing the sort of continuum so daringly

envisioned by Peirce? The central contention of this work is that, if one brings Peirce's

theory into harmony with the standard geometrical practices of our day (and thereby

downplays Peirce's commitment to the potential nature of points on a line), the most

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[43] Euclid's continuity needs reduce to the following two principles that Euclid tacitly employed.

The Circular Continuity Principle: If a circle C has one point inside and one point outside another circle C', then the two circles intersect in two points.

The Line-Circle Continuity Principle: If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle at one point.

During the twentieth century it became clear that a model of Hilbert's axioms for classical Euclidean geometry less the continuity axioms satisfies the circular continuity principle if and only if it satisfies the line-circle continuity principle if and only if it is isomorphic to a Cartesian space over a *Euclidean ordered field* (i.e. an ordered field in which every positive element is the square of some element of the field).
While there is little doubt Peirce would have embraced both the Circular Continuity Principle and the Line-Circle Continuity Principle (see Note 40), it is worth noting that being a Peircean linear continuum ensures the satisfaction of neither. Indeed, as is well known, M is a model of Hilbert's axioms of Euclidean geometry less the continuity axioms if and only if M is isomorphic to a Cartesian space over a *Pythagorean ordered field* (i.e. an ordered field in which $\sqrt{a^2 + b^2}$ is a member of the field whenever a and b are members of the field). But there are Pythagorean ordered fields that are absolute linear continua but are not Euclidean. To obtain such an ordered field, one simply has to employ the construction described in Note 16 for the case where A_0 is any (of the many) Pythagorean ordered fields that is not Euclidean. Within the finitely bounded portion of a Cartesian space over such an ordered field there are segments of Peircean linear continua containing two points, one inside and one outside a given circle, that do not intersect the circle. Of course, since every real-closed field is a Euclidean ordered field, spatial anomalies of this sort cannot arise in the Cartesian space over *No* and, hence, in absolutely continuous elementary Euclidean geometry (see Note 10).
For further discussions of these geometrical matters including references to the relevant literature, see [Ehrlich 1997] and the section of [Ehrlich 2005] entitled *Modern Euclidean Geometry and the Continuum*.

natural, most compelling and theoretically most significant candidate is No_P--the Peircean counterpart of the absolute arithmetic continuum (modulo NBG).

Postscript

As we mentioned in the Introduction, whereas the absolute arithmetic continuum and its Peircean counterpart may be said to exhibit all possible types of algebraic and *set-theoretically* defined order-theoretic gradations consistent with their structures as an ordered field and a finitely bounded ordered domain, respectively, if one is willing to allow for algebraico-order-theoretic gradations that are definable in terms of proper classes as well as sets, there are ordered fields and corresponding finitely bounded ordered domains whose algebraico-order-theoretic gradations are even more refined still. Moreover, as the following examples nicely illustrate, some of these structures are highly distinguished to boot.

Let $\mathbb{R}(No)$ be the class of all formal series of the form

$$\sum_{\alpha < \beta} \omega^{y_\alpha} . r_\alpha$$

where $\{y_\alpha : \alpha < \beta \leq On\}$ is a descending sequence of elements of No and $r_\alpha \in \mathbb{R} - \{0\}$ for each $\alpha < \beta$, $\mathbb{R}$ being the ordered field of real numbers. In addition to all the Conway names of surreal numbers, i.e., the members of $\mathbb{R}(No)$ for which $\beta < On$, $\mathbb{R}(No)$ contains a vast array of "Conway names" of "numbers" where $\beta = On$, including

$$\sum_{\alpha < On} \omega^{-\alpha} = 1 + \frac{1}{\omega^1} + ... + \frac{1}{\omega^\alpha} + ... \quad (\alpha < On).$$

Moreover, if one defines sums, products and order for members of $\mathbb{R}(No)$ in same manner they are defined for Conway names of surreal numbers (see Section 7, Theorem 10), then by applying straightforward adaptations of classical arguments whose roots lie

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in the work of Hans Hahn [1907], Wolfgang Krull [1932] and Norman Alling [1962], one may show that $\mathbb{R}(No)$ *is a real-closed ordered field that is an absolute linear continuum.* We hasten to add, however, that $\mathbb{R}(No)$ has power $2^{\aleph_{On}}$, which is greater than $\aleph_{On}$, the power of *No*; moreover, with the exception of the Conway names of surreal numbers, the members of $\mathbb{R}(No)$ are not sets but proper classes! For this reason alone, in NBG one cannot even construct $\mathbb{R}(No)$, let alone talk about the cardinal $2^{\aleph_{On}}$ or prove the above result. On the other hand, $\mathbb{R}(No)$ can be constructed and the just-stated theorem can be proved in the equiconsistent set theory of Ackermann (see Note 11), where, unlike in NBG, besides the class V of all sets, there exists the power class PV of all subclasses of V, the power class PPV of all subclasses of PV and so on [Lévy and Vaught 1961, p. 1061; also see Fraenkel, Bar-Hillel and Lévy 1973, p. 153 and Lévy 1976, p. 212]. Moreover, these classes are "well-behaved" in the sense that they behave in the manner they would be expected to behave assuming they were sets [Lévy and Vaught 1961, p. 1061; Lévi 1959]; in particular, if $\aleph_{On}$ is the "cardinal" of V, then $2^{\aleph_{On}}$ is indeed the "cardinal" of PV.

In Ackermann's set theory, $\mathbb{R}(No)$ is a subclass of PV, and by limiting $\mathbb{R}(No)$ to its substructure consisting of its finite and infinitesimal members, one obtains a *Peircean arithmetic continuum of power* $2^{\aleph_{On}}$ that is likewise a subclass of PV. This Peircean arithmetic continuum arises by filling $2^{\aleph_{On}}$ vacancies (see Section 9) in No_P, each of which corresponds to a gap in No_P having character $\left(On, {}^*On\right)$. Thus, while it must be admitted that No_P has a great many "holes" indeed, none of them is definable in terms of sets of NBG!

Appendix

The concepts of partially ordered class, ordered class, ordered integral domain and ordered field play central roles in the text. For discussions of partially ordered classes and ordered classes, the reader may consult [Harzheim 2005] and [Rosenstein, 1982], and for the basic properties of ordered integral domains and ordered fields, the reader may consult [Mendelson 1973] or [Birkhoff and MacLane 1977/1997]. For the reader's convenience, however, the definitions of these basic concepts are given below where "ordered" without the modifier "partial" always means, "totally ordered." This convention is employed throughout the paper.

A *partially ordered class* is a structure $\langle A, < \rangle$, where A is a class and $<$ is a binary relation on A that satisfies the following conditions: (i) $\forall xy (x < y \rightarrow x \neq y)$; (ii) $\forall xyz ((x < y \wedge y < z) \rightarrow x < z)$.

An *ordered class* is a partially ordered class that also satisfies: (iii) $\forall xy (x \neq y \rightarrow (x < y \vee y < x))$.

A *cut* of an ordered class $\langle A, < \rangle$ is an ordered pair (X, Y) of subclasses of A such that every member of X precedes every member of Y, and $X \cup Y = A$. If X and Y are nonempty, the cut is said to be a *Dedekind cut*. If (X, Y) is a Dedekind cut, then the cut is said to be a *gap*, if X has no greatest member and Y has no least member, and it is said to be a *continuous cut*, if either X has no greatest member and Y has a least member or X has a greatest member and Y has no least member. A Dedekind cut that is neither a gap nor a continuous cut is often said to be a *jump*.

The reader will notice that, in accordance with these definitions, partially ordered classes and ordered classes may be empty or contain a single element. Throughout the paper, such structures will be said to be *trivial* and partially ordered classes and ordered classes containing at least two members will be said to be *nontrivial*.

An *ordered integral domain* (or, more simply, an *ordered domain*) is a structure $\langle A,+,\cdot,<,0,1 \rangle$, where $\langle A,< \rangle$ is an ordered class, $0,1 \in A$ where $0 \neq 1$, and $+$ and $.$ are commutative, associative binary operations on A for which the following conditions hold: (i) $\forall x(x + 0 = x)$; (ii) $\forall x(x \cdot 1 = x)$; (iii) $\forall x \exists y(x + y = 0)$; (iv) $\forall x \forall y \forall z[x \cdot (y + z) = (x \cdot y) + (x \cdot z)]$; (v) $\forall x \forall y((x \neq 0 \wedge y \neq 0) \rightarrow x \cdot y \neq 0)$; (vi) $\forall x \forall y \forall z(x < y \rightarrow x + z < y + z)$; (vii) $\forall x \forall y \forall z((x < y \wedge 0 < z) \rightarrow x \cdot y < y \cdot z)$.

The elements 0 and 1 of A, which more appropriately should be written 0_A and 1_A, are the additive and multiplicative identities of the domain, respectively, and need not be the familiar numbers thus denoted.

An *ordered field* is an ordered domain that also satisfies the property:
$$\forall x[(x \neq 0 \rightarrow \exists y(x \cdot y = 1))].$$

Thus, ordered fields are ordered domains each of whose non-zero elements has a multiplicative inverse. In axiomatizations of ordered fields one frequently finds condition (v) for ordered domains omitted since it is a consequence of the remaining axioms.

Whereas the properties of ordered domains generalize the familiar algebraic, order-theoretic and compatibility properties of the system of integers, the properties of ordered fields generalize the analogous properties of the system of rational numbers.

An ordered domain $\langle A,+,\cdot,<,0,1 \rangle$ is said to be *Archimedean* if it satisfies the condition: for all $a,b \in A$, if $0 < a < b$, there is a positive integer n such that $na > b$. An

element a of A is said to be *infinitesimal* if $n|a| < 1$ for all positive integers n, and it is said to be *infinite* if $|a| > n$ for all positive integers n. Since every ordered field A contains a unique isomorphic copy $\mathbb{Q}_A$ of the ordered field of rational numbers, it follows from the above definition of an infinitesimal that an element a of an ordered field A is *infinitesimal* if $-1/n < a < 1/n$ for all positive integers n. An ordered domain is Archimedean if and only if it contains neither infinite nor infinitesimal elements. Non-Archimedean ordered fields, by contrast, contain infinite as well as non-zero infinitesimal elements, the latter being the multiplicative inverses of the former. Non-Archimedean ordered domains always contain infinitesimal elements but need not contain infinite elements.

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