SIMPLE FORMULAS FOR FOLDED ANTENNAS

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The folded antenna may be regarded as the ultimate electromagnetic structure because it integrates all three major families of components, that is linear radiators, transmission lines and lumped impedances, into a single device. It should come as no surprise, therefore, that it offers the engineer exceptional design freedom compared with other antennas. It provides six more independent design variables than a simple linear radiator of the same length. They are: 1- the resistance and 2- the reactance of the base load impedance; 3- the resistance and 4- the reactance of the top load impedance; 5- the differential mode characteristic impedance; and 6- the common mode current transformation ratio. Precise meanings for each of these will become clear as we develop and explore the model of this versatile antenna. Thanks to the additional variables, efficient antennas can be designed that are much shorter or longer than the traditional quarter-wave monopole or half-wave dipole.

In practice, extremely compact folded antennas have been fabricated. This article will conclude with a design example that is only 0.107 wavelength, or 38.4 electrical degrees, in height. The feasibility of such tiny antennas has obvious attractions at both long wavelengths, where available space and materials are primary limiting factors, and short wavelengths, where covertness may be highly desirable.

Folded antennas are acknowledged briefly in many handbooks and text books. Until a recent book by the author [1], however, only a few specialized configurations had been thoroughly analyzed with detailed mathematical formulas. Leonhard, Mattuck and Pote derived an equivalent circuit to model folded monopoles shorter than a quarter-wavelength [2]. In their model, the base load impedance is an inductor, and the top load impedance is a short circuit. Ronald W. P. King and Charles W. Harrison, Jr. used integral equations to model the folded antenna with nonzero top load impedance and a short circuit for a base load impedance [3].

In this article, we will develop an original mathematical model of the folded antenna, with arbitrary impedance loads at both top and base. The model applies to antennas both long and short compared with a wavelength. Although it is very general, the model is also mathematically simple, consisting chiefly of algebraic formulas. Extensive numerical analyses and an arsenal of computer programs are not necessary.
In particular, some of the questions our model will answer include:

1. What are the antenna input current and input impedance as functions of the impedance loads at the top and at the base?

2. What combinations of top and base load impedances result in a useable antenna input impedance?

3. What is the frequency response of the folded antenna?

Formulas for the Current and Voltage on the Antenna

Figure 1 shows the configuration of interest. The antenna consists of two parallel conductors, with equal lengths but not necessarily with equal diameters. The conductor connected to the feed line is called the "fold". The other conductor is called the "tower". The latter name probably originates from applications to vertical radiators in the VLF, LF, and MF bands. Our mathematical model will apply to any band, however, from ELF through EHF.

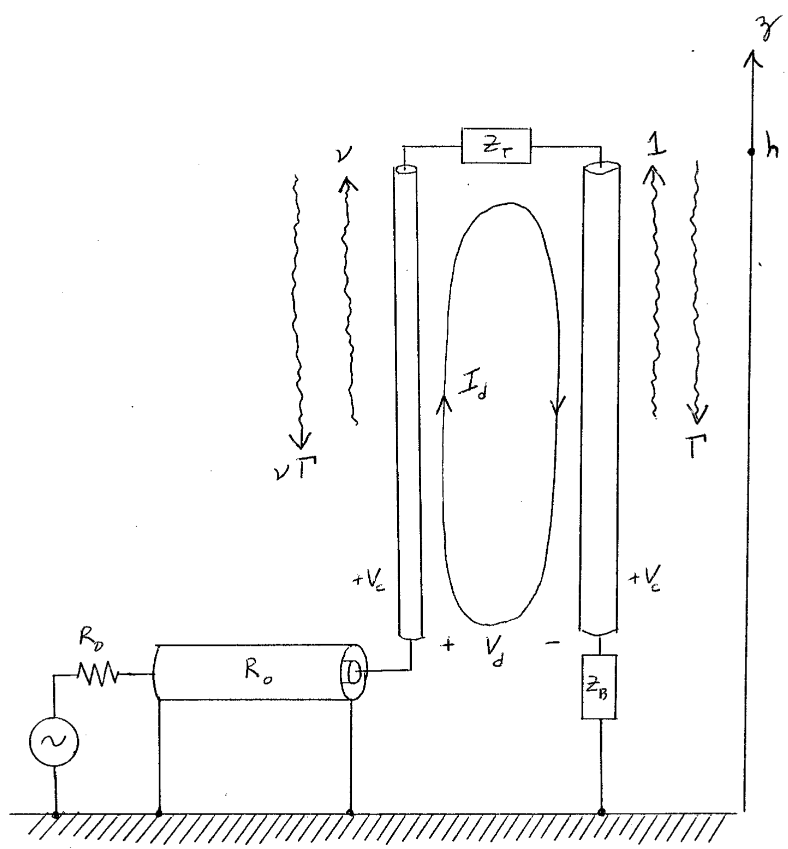

Figure 1. The folded antenna provides six more independent design variables than a monopole or dipole of the same height. Shown here are the essential geometry and design parameters. Also shown are two different modes of current, a common mode and a differential mode.

The fold and the tower are connected by the top load impedance Z_T. At its base, the fold is connected to a transceiver with output impedance R_0 via a transmission line with characteristic impedance equal to R_0. The tower is connected to ground through the base load impedance Z_B.

The figure also shows how the folded antenna fundamentally differs from other linear radiators. There are two different modes of current, a common (or unbalanced, or antenna) mode and a differential (or balanced, or transmission line) mode. Other linear antennas do not have a differential mode. The common mode currents on the fold and tower are in phase with each other; however, they do not necessarily have the same amplitude. The relative amplitudes are described by the *current transformation ratio* ν.

The superposition of common mode and differential mode currents provides great flexibility and control over the input impedance of the folded antenna. As a result, radiators ranging from very short to very long, compared with a wavelength, are feasible. The loading impedances Z_T and Z_B may be regarded as mixers for the two modes. As our mathematical model will show, the proper combination of these loads can provide useable antenna impedances for just about any wavelength.

Both current modes are the superposition of forward and reflected traveling waves. The common mode current on the tower is:

$$I_c(z) = e^{-jkz} - \Gamma e^{jkz} \tag{1}$$

The common mode current on the fold is $\nu I_c(z)$.

The differential mode current is:

$$I_d(z) = I_F e^{-jkz} + I_R e^{jkz} \tag{2}$$

In Equations 1 and 2, the wave number is:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} \tag{3}$$

Where λ is the wavelength, *f* is the frequency and *c* is the speed of light.

In Equation 1, the amplitude of the forward wave has been arbitrarily set to unity without loss of generality. The amplitude Γ of the reflected wave is determined by the *antenna end effect*, discussed in detail by the author in a recent article [4]. It is readily calculated using the formula,

$$\Gamma = e^{-j2kh} \frac{Z_{0c} - Z_s}{Z_{0c} + Z_s} \tag{4}$$

In Equation 4, Z_{0c} is the common mode characteristic impedance and Z_s is the parallel combination of the antenna end capacitance and the radiation resistance of a virtual slot antenna:

$$Z_s = \frac{jXR_s}{jX + R_s} \tag{5}$$

In Equation 5, the radiation resistance R_s of the virtual annual slot is approximately 112.5 Ω for any antenna radius. The reactance of the end capacitance is:

$$X = -\frac{1}{16\pi f \varepsilon a_e} \tag{6}$$

In Equation 6, a_e is the radius of a single cylinder with the same common mode characteristic impedance as the folded antenna.. This can be calculated by equating their common mode characteristic impedances. The result is:

$$a_e = \frac{1}{k'} \exp\left(-\frac{2\pi Z_{0c}}{\eta}\right) \approx 0.179\lambda \exp\left(-\frac{2\pi Z_{0c}}{\eta}\right) \tag{7}$$

Equation 7 follows from a formula for the common mode characteristic impedance of a single cylinder that was derived and discussed by the author in a recent book [5]. Z_{0c} is the common mode characteristic impedance of the folded antenna, λ is the wavelength, η is the impedance of free space and *k'* is the *quasi static wave number*.

Traveling waves of voltage also propagate along the folded antenna and these are related to the currents by the usual transmission line formulas. The common mode voltage is:

$$V_c(z) = Z_{0c}(1+\nu)\left(e^{-jkz} + \Gamma e^{jkz}\right) \tag{8}$$

The differential mode voltage is:

$$V_d(z) = Z_{0d}(I_F e^{-jkz} - I_R e^{jkz}) \tag{9}$$

In Equation 8, the factor $1+\nu$ is required because common mode current flows on both the tower and the fold.

So far, we have uniquely specified all of the variables in the formulas except for the complex amplitudes I_F and I_R in Equation 2. To determine those, we need two independent equations, and those are boundary conditions at the two loads. At the base of the antenna, Kirchhoff's voltage law requires that:

$$Z_B[I_d(0)+\Gamma-1]+V_d(0)=V_c(0) \tag{10}$$

At the top of the antenna, the differential mode voltage and current must satisfy the constitutive relation determined by the impedance Z_T:

$$V_d(h)=I_d(h)Z_T \tag{11}$$

If we apply Equations 1, 2, 8 and 9 to Equations 10 and 11, then after some algebra, we obtain formulas for the complex amplitudes of the differential mode current. The amplitude of the forward traveling wave is:

$$\boxed{I_F=\frac{[Z_{0c}(1+\nu)(1+\Gamma)+Z_B(1-\Gamma)](Z_{0d}+Z_T)\psi_R}{(Z_{0d}+Z_B)(Z_{0d}+Z_T)\psi_R-(Z_{0d}-Z_B)(Z_{0d}-Z_T)\psi_F}} \tag{12}$$

The amplitude of the reflected or reverse traveling wave is:

$$\boxed{I_R=\frac{[Z_{0c}(1+\nu)(1+\Gamma)+Z_b(1-\Gamma)](Z_{0d}-Z_T)\psi_F}{(Z_{0d}+Z_B)(Z_{0d}+Z_T)\psi_R-(Z_{0d}-Z_B)(Z_{0d}-Z_T)\psi_F}} \tag{13}$$

In Equations 12 and 13, the forward traveling wave function is:

$$\psi_F=e^{-jkh} \tag{14}$$

The reverse traveling wave function is:

$$\psi_R=e^{jkh} \tag{15}$$

Characteristic Impedance and Current Transformation Ratio

To evaluate the formulas in the previous section for current, we need values for the common mode characteristic impedance Z_{0c}, the differential mode characteristic impedance Z_{0d}, and the current transformation ratio ν. These are functions of the detailed cross section of the folded antenna. With careful design of that cross section, they can be prescribed over a wide range of values. Typically, Z_{0c} can range from 200 to 1,200 Ω; Z0d can range from 20 to 200 Ω; and ν can range from 0.3 to 3. These are only typical values. Extreme values outside those ranges are also possible. A thorough discussion and formulas for the calculation of values for arbitrary cross sections made from multiple conductors can be found in a recent book by the author [6]. Here, we will summarize the results for a simple cross section for completeness of our present discussion.

For a fold of radius *a* separated from a tower of radius *b* by a distance *s,* the common mode characteristic impedance is [7]:

$$Z_{0c} = \frac{\eta}{2\pi} \frac{\ln[k'(s-a)]\ln[k'(s-b)] - \ln(k'a)\ln(k'b)}{\ln\left[\frac{ab}{(s-a)(s-b)}\right]} \tag{16}$$

The differential mode characteristic impedance is [8]:

$$Z_{0d} = \frac{\eta}{2\pi} \ln\left[\frac{(s-a)(s-b)}{ab}\right] \tag{17}$$

The current transformation ratio is [9]:

$$\nu = \frac{\ln\left(\frac{b}{s-a}\right)}{\ln\left(\frac{a}{s-b}\right)} \tag{18}$$

In Equation 16, *k'* is the *quasi static wave number* [10]:

$$k' \approx 0.891k \tag{19}$$

Antenna Input Current and Input Impedance

At the input terminal to the folded antenna, the current is:

$$I_{in} = \nu I_c(0) + I_d(0) \tag{20}$$

Using Equations 1 and 2, Equation 20 becomes:

$$I_{in} = \nu(1-\Gamma) + I_F + I_R \tag{21}$$

In Equations 20 and 21, the factor ν is required because the antenna input terminal is at the base of the fold, not the tower.

The voltage at the input terminal is:

$$V_{in} = V_c(0) = Z_{0c}(1+\nu)(1+\Gamma) \tag{22}$$

From Equations 21 and 22, the input impedance is:

$$Z_{in} = \frac{V_{in}}{I_{in}} = Z_{0c}\,\frac{(1+\nu)(1+\Gamma)}{\nu(1-\Gamma) + I_F + I_R} \tag{23}$$

Equation 23 does not include any contribution from radiation resistance. We will derive a formula for that shortly. If the impedance loads Z_B or Z_T include any resistance, however, then that will show up in Z_{in}. In any case, the input reactance to the antenna is simply the imaginary part of Z_{in}:

$$\boxed{X_{in} = Z_{0c}\,\mathrm{Im}\left[\frac{(1+\nu)(1+\Gamma)}{\nu(1-\Gamma) + I_F + I_R}\right]} \tag{24}$$

We can obtain a formula for the input resistance R_{in} by first using the current distribution described by Equation 1 to determine the radiated electric field [11]:

$$E_\theta = \frac{j\eta\, e^{-jkr}\sin\theta}{4\pi r}\,F(\theta)$$

(25)

In Equation 25, the *vertical radiation characteristic* is:

$$F(\theta) = \int_{kl} I(z) e^{+jkz\cos\theta} d(kz) \tag{26}$$

Eqn. 26 can be evaluated exactly in terms of simple functions [12]; however, with modern desktop computers, it is quickly evaluated numerically as well. The integrand need only include the common mode currents on the fold and tower. The differential mode current does not contribute significantly to the radiated field because it is balanced.

From the radiated electric field, the radiated power density, or Poynting vector, is readily obtained:

$$S = \frac{|E_\theta|^2}{\eta} \tag{27}$$

and also the total radiated power:

$$P = \int_0^{2\pi} d\varphi \int_0^{\pi} S r^2 \sin\theta \, d\theta \tag{28}$$

For vertical or *z*-directed wires, using Equations 25 to 27, Equation 28 becomes:

$$P = \frac{\eta}{2} \int_0^{\pi} d\theta \sin^2\theta \, |F(\theta)|^2 \tag{29}$$

Finally, the input radiation resistance is:

$$\boxed{R_{in} = \frac{P}{|I_{in}|^2} = \frac{\eta}{2|I_{in}|^2} \int_0^{\pi} d\theta \sin^2\theta \, |F(\theta)|^2} \tag{30}$$

Equation 21 is used to compute I_{in} in Equation 30.

Our mathematical model is now ready to help us explore how the folded antenna performs as a function of its many independent design variables. ***Figures 2*** and ***3*** show the antenna input impedance as a function of frequency, for several different values of common mode characteristic impedance. For a simple monopole of the same length, the quarter-wave resonant frequency would be 82 MHz. For the folded antenna, however, we observe useable input resistances at much lower frequencies. Further, the input reactance is positive, or inductive, and this is easily tuned out with a practical capacitor. So, short folded antennas are inherently easier to match than short monopoles or dipoles.

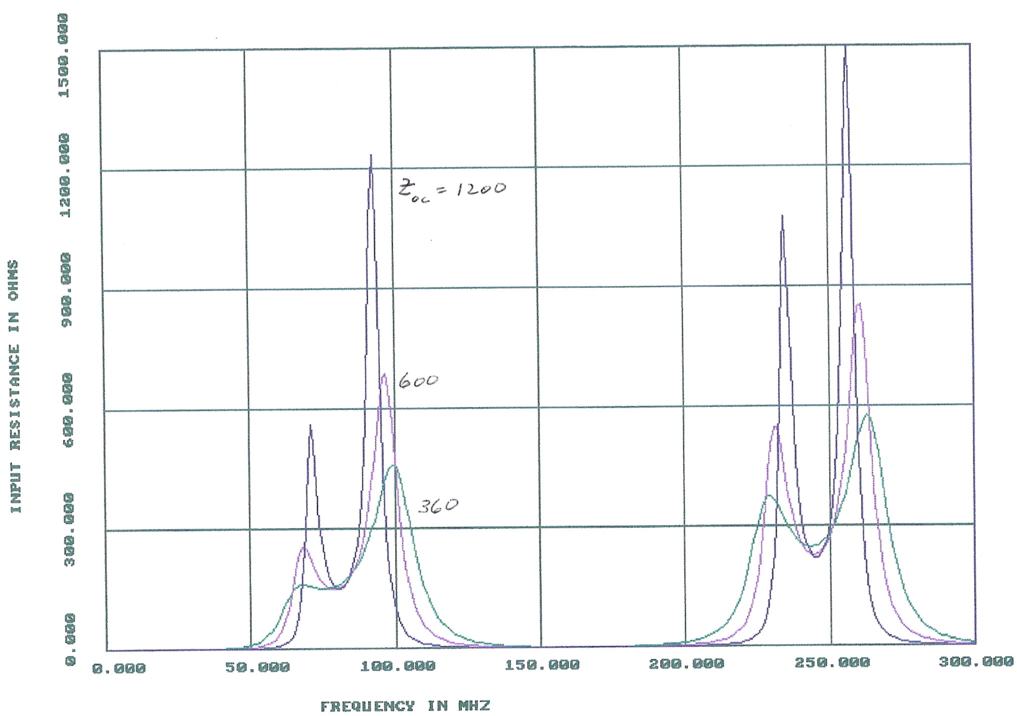

Figure 2. The antenna input resistance was computed for a 3-foot length, a differential characteristic impedance of 120 Ω, short circuits on both load impedances and a current transformation ratio of unity. Three distinct resonances are observed in the vicinity of a quarter-wavelength.

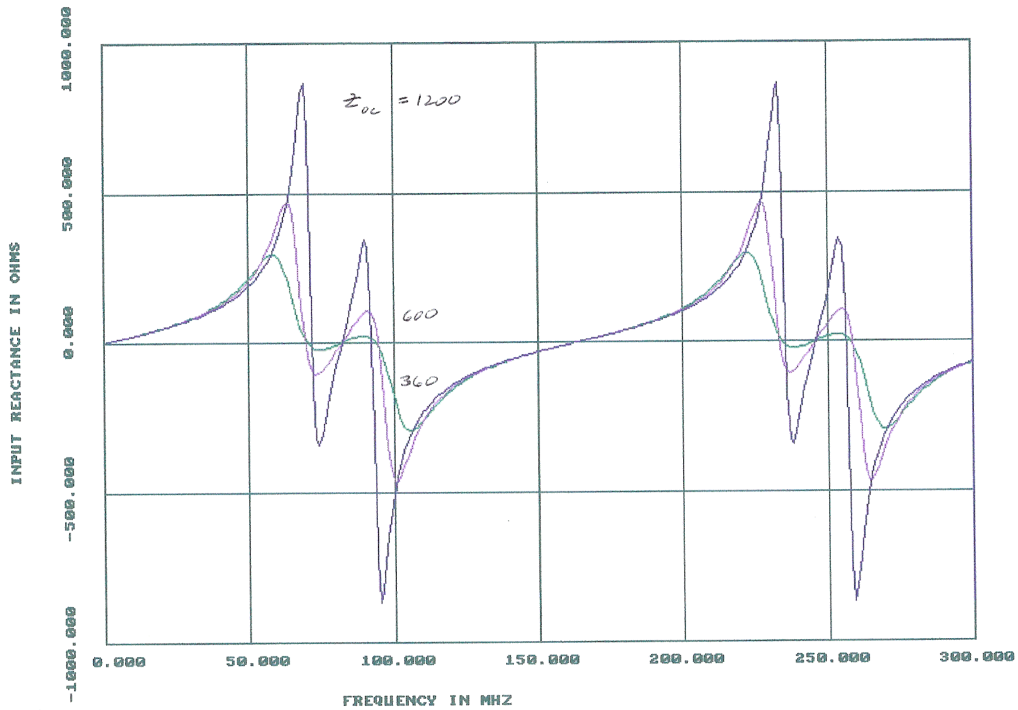

Figure 3. Antenna input reactance was computed for the same variables as figure 2. Three zero-crossings or resonances are observed in the vicinity of a quarter-wavelength.

Figure 2 also provides a good sanity check for our mathematical model. It is seen that, for all values of Z_{0c},, the resistance at 82 MHz converges to about four times the input resistance of a resonant monopole (36 Ω) This is in exact agreement with the theory of simple folded antennas, first published by van Roberts [13]. The figures also reveal some performance unique to folded antennas. While a simple monopole has a single resonance near a quarter-wavelength, the folded antenna has three. One of those extra resonances is below the quarter-wavelength, and the other is higher. The additional resonances appear more distinct as Z_{0c} increases.

Figures 4 and 5 show the antenna input impedance as a function of frequency, for several different values of current transformation ratio. Based upon our formulas, especially Equation 21 for input current, the results are not surprising. As the current transformation ratio decreases, so does the input current. As a direct result, input impedance increases.

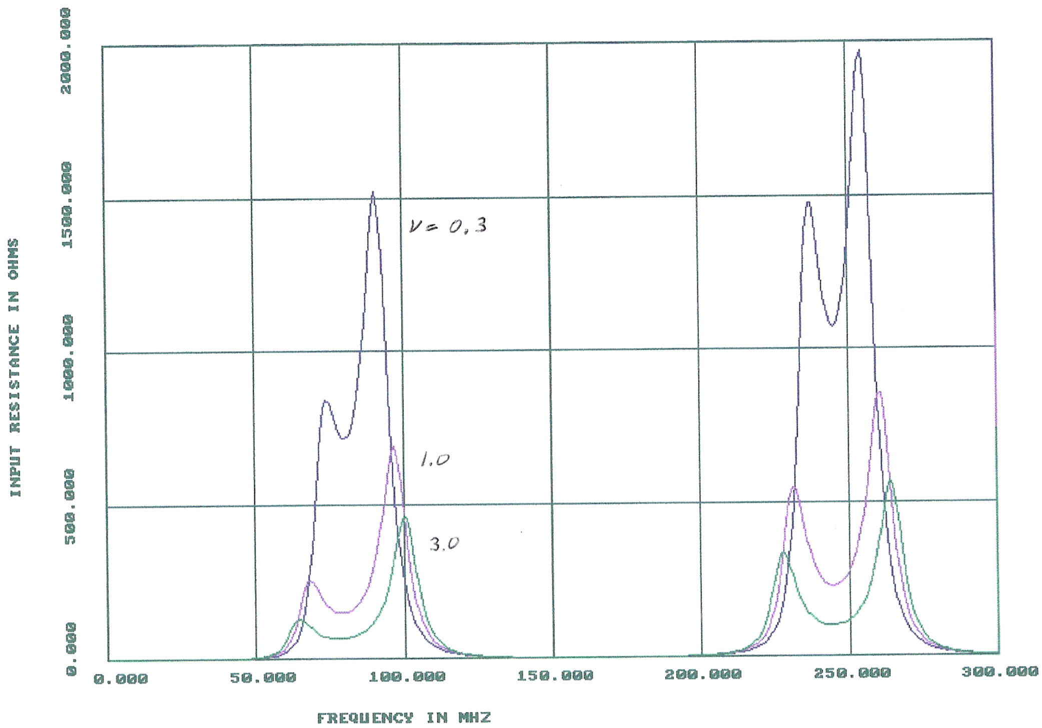

Figure 4. Antenna input resistance was computed for a 3-foot length, a common mode characteristic impedance of 600 Ω, a differential mode characteristic impedance of 120 Ω and short circuits for both load impedances. It is seen that resistance increases with decreasing current transformation ratio.

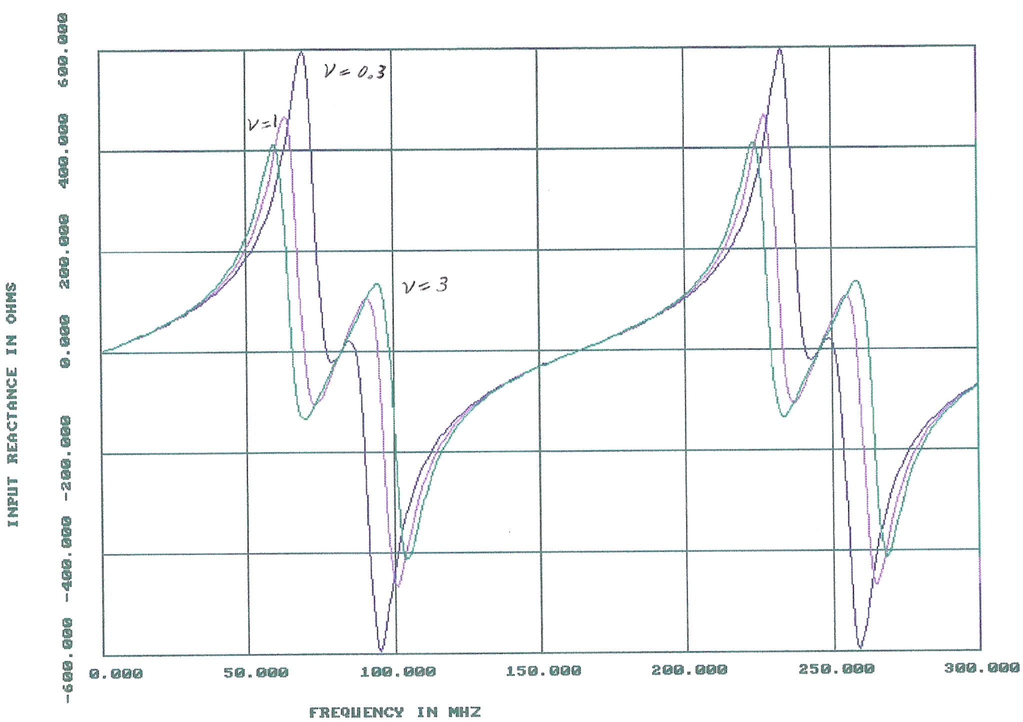

Figure 5. Antenna input reactance was computed for the same variables as in figure 4. It is seen that the resonances decrease in frequency with increasing current transformation ratio.

Examining especially Figures 2 and 4 for input resistance, it would seem that there is a "dead zone", or interval of near-zero input resistance, in the vicinity of a half-wavelength. This is an erroneous conclusion, however, because we have not yet begun to examine the benefits of varying the load impedances Z_B and Z_T.

Benefits of Base and Top Load Impedances

Each of the load impedances provides two independent design variables, a resistance and a reactance. Careful selection of these allows us to greatly extend the frequency performance of the folded antenna.

Figures 6 and ***7*** show the antenna impedance as a function of frequency, for several different values of base reactance X_B, that is, for $Z_B = jX_B$. It is seen that the spectrum for which there are useable input resistances has been extended well into the "dead zone" described in the above section, and below the quarter-wave resonance, into the realm of extremely short antennas.

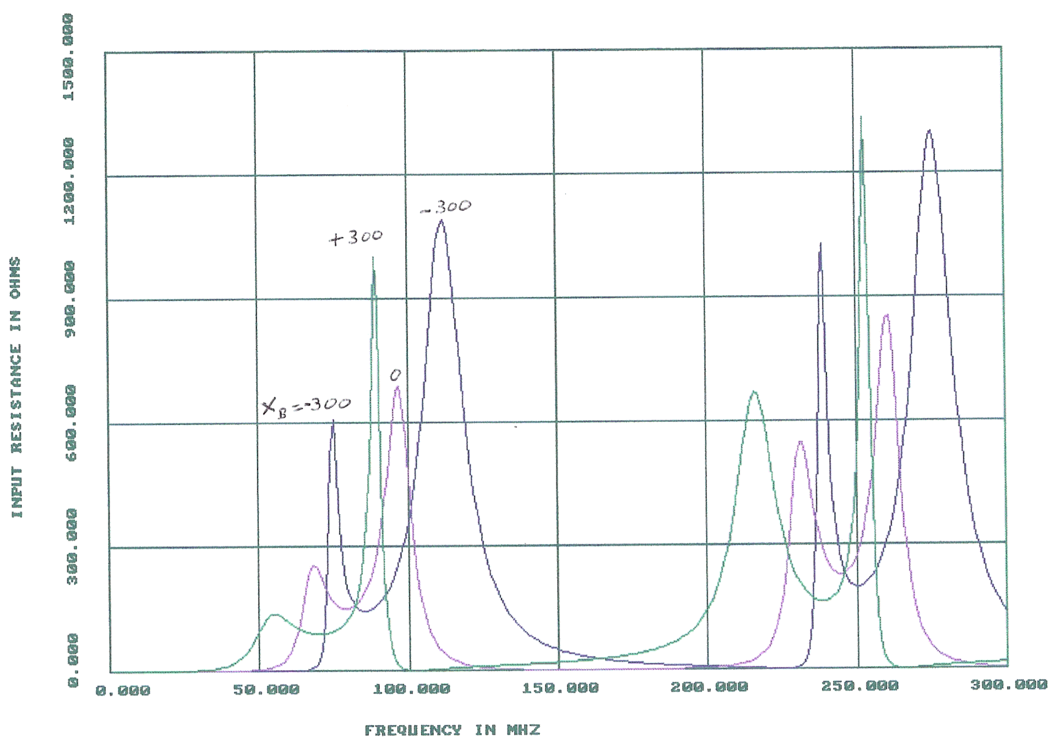

Figure 6. Antenna input resistance was computed for a 3-foot length, common mode characteristic impedance of 600 Ω, differential mode characteristic impedance of 120 Ω, a current transformation ratio of unity and a short circuit for the top load impedance. The base load impedance clearly influences frequency response.

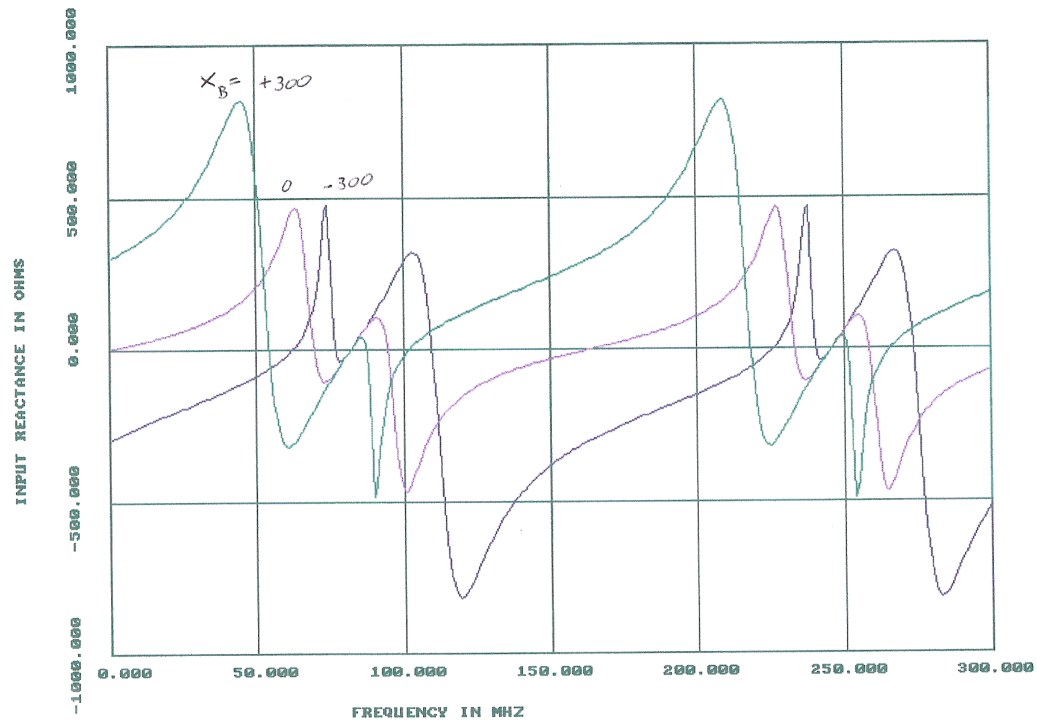

Figure 7. Antenna input reactance was computed for the same variables as in figure 6. The base load impedance clearly influences the zero crossings of the reactance or resonances of the antenna.

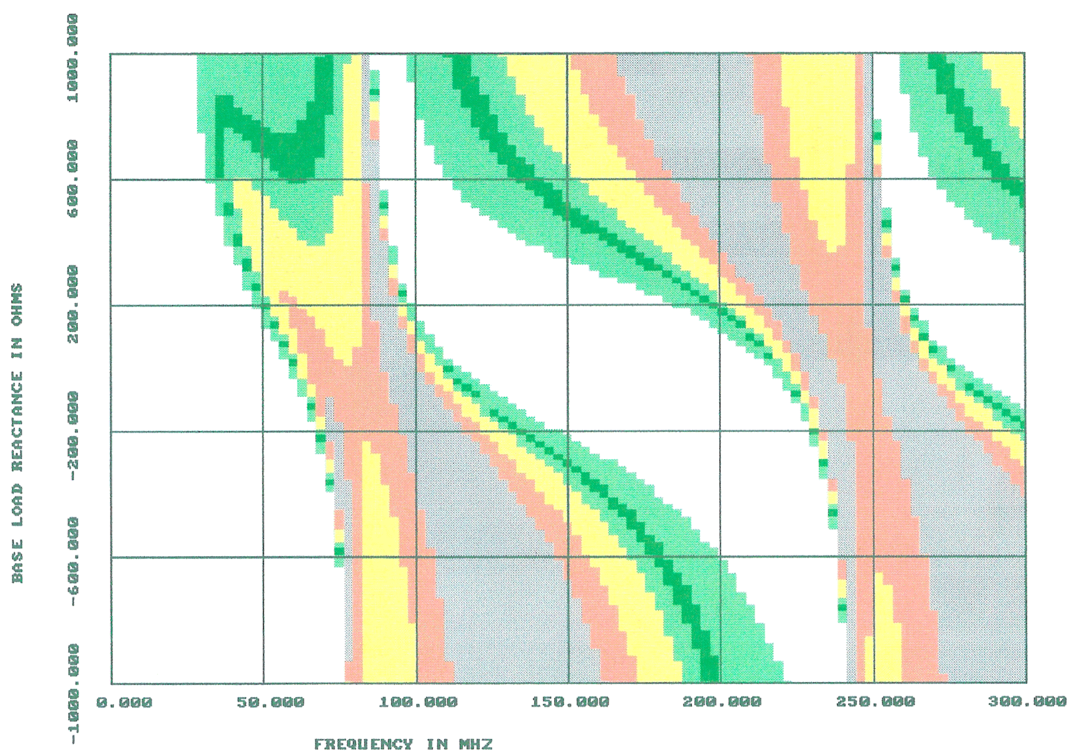

Figure 8. Color contour plots allow a more thorough investigation of the effect of base load reactance. Optimal (dark green) regions are observed at very long wavelengths compared with the length of the antenna.

A color contour plot, such as ***Figure 8***, is a more complete way to examine the effect of base reactance on input resistance. Compared with the conventional xy plots, such as the previous two figures, we are less likely to skip over useful design choices. Both base reactance X_B and frequency f are varied continuously and simultaneously. For this selection of colors, dark green denotes resistances from 45 to 55Ω. Light green denotes 25 to 75 Ω. Yellow denotes 75 to 150 Ω. Red denotes 150 to 300Ω. Gray denotes greater than 300 Ω. It is seen that there are highly desirable dark green regions for frequencies as low as about 35 MHz, or 0.107 wavelength.

Figure 9 is a similar color contour plot. The difference is that the top load reactance X_T is varied instead of X_B. Definite effects are seen, although they seem to be less extensive than for X_B. When both reactances are simultaneously varied, however, the synergistic design benefit is definitely apparent, as we will see in the following example.

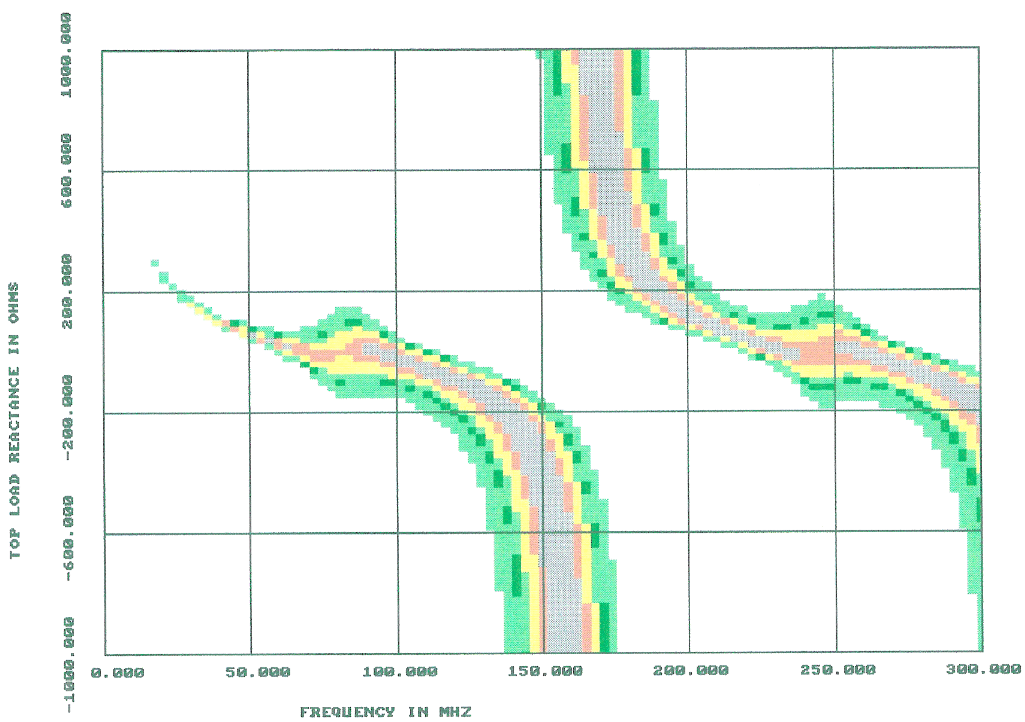

Figure 9. Color contour plots of input resistance similarly allow a more thorough investigation of the effect of top load reactance. The antenna length is 3 feet, only 0.107 wavelenth at 35 MHz. Common and differential mode characteristic impedances are 600 and 120 Ω, respectively. Current transformation ratio is unity.

Design Example: A Very Small Antenna

One of the unique advantages of folded antennas is the capability for extreme compactness compared with other linear radiators. In the previous section, we saw that useable resistances were possible for a 3-foot antenna at 35 MHz. At that frequency, the antenna is only 0.107 wavelength, or 38.4 electrical degrees.

Although our example is in the VHF band, the results scale readily to other bands both higher and lower. The linear dimensions decrease in direct proportion to the wavelength. This example also applies to 3.5 GHz and a 9.1 millimeter antenna, or to 350 kHz and a 300 foot antenna. ***Figure 10*** is a color contour plot for which the two continuous variables are the two load reactances X_B and X_T. The frequency is fixed at 35 MHz, and the antenna length is fixed at 3 feet. It is seen that there is a dark green region, corresponding to an input resistance of 45 to 55 Ω, in the vicinity of $X_B = 795$ Ω.

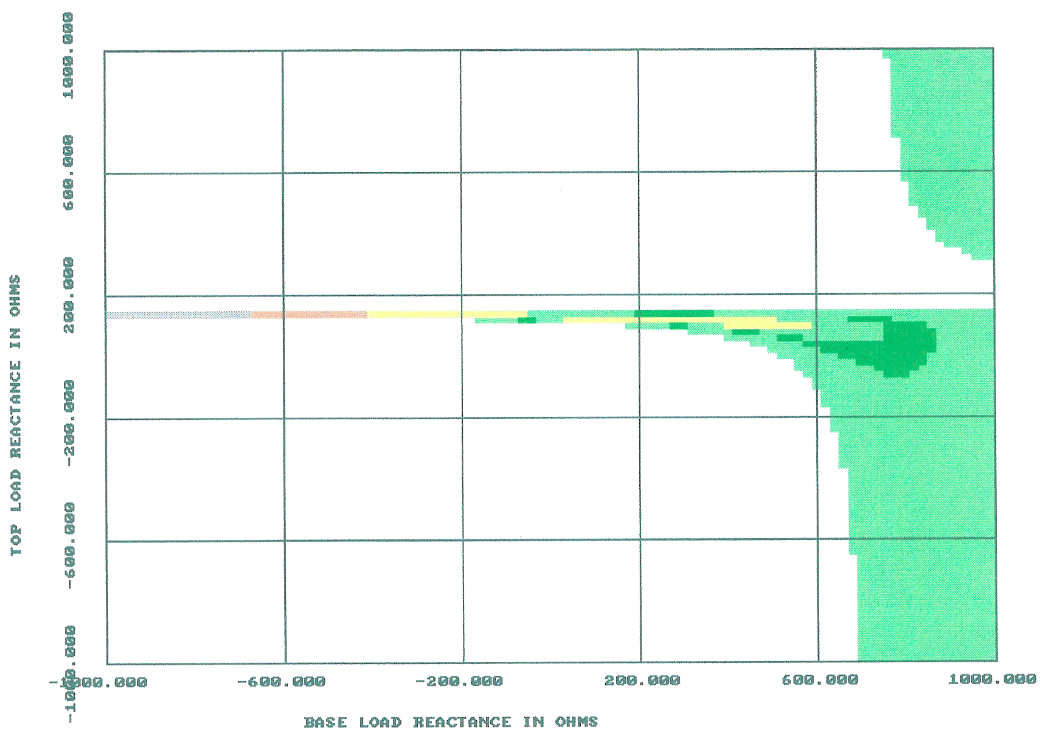

Figure 10. Color contour plots of input resistance shoe what happens when both base and top load reactances are varied simultaneously at 35 MHz. An optimal (dark green) region is observed in the vicinity of X_B = 795 Ω. Without this type of plot, what are the chances of discovering this region?

Let's see what happens when we hold X_B fixed and vary X_T. To that end, ***Figure 11*** shows the input resistance and reactance as a function of the top load reactance X_T, when X_B is fixed at 795 Ω. It is seen that the input reactance X_{in} changes sign twice. The input resistance R_{in} plunges at one of those zero crossings but remains flat and over 100 ohms at the other. The design choice obviously favors the latter.

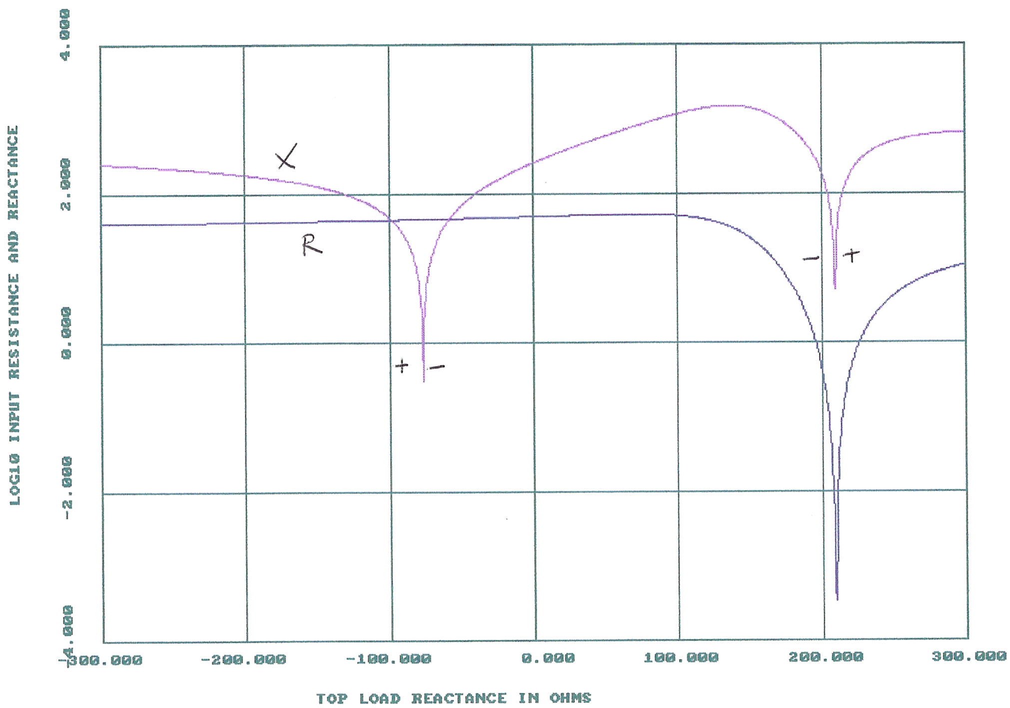

Figure 11. Antenna input resistance and reactance as a function of X_T when X_B is held fixed at 795 Ω. The input reactance changes sign twice. The zero-crossings for X_T = -77 Ω appears to correspond to an optimal resistance

As a confirmation of the above design choice, ***Figure 12*** shows the standing wave ratio or SWR as a function of top load reactance X_T. As expected, it dips to a highly acceptable level for X_T = *-77* ohms.

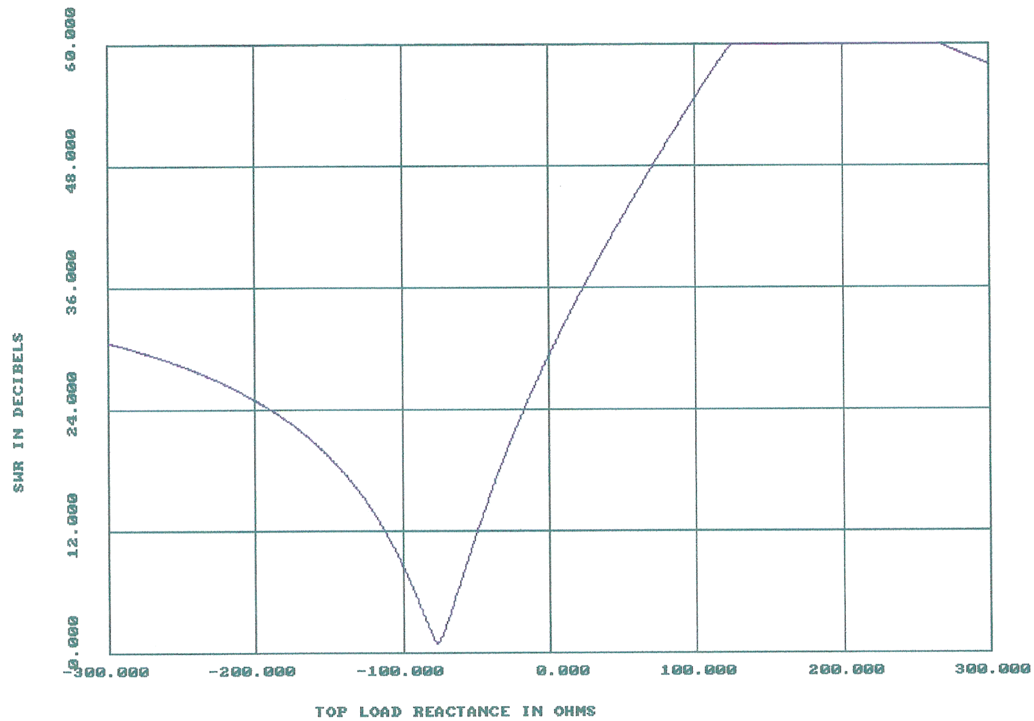

Figure 12. The standing wave ratio or SWR was computed for the same variables as figure 11. As expected, for $X_B = 795\ \Omega$ and $X_T = -77\ \Omega$, the SWR is just about minimal.

Comparison with Other Theories and Actual Installations

As discussed in the Introduction, no mathematical model as general as the one presented here is available in the literature; however, it is possible to compare our model with some special cases that have been published. One of these is a formula derived by the eminent antenna analysts King and Harrison [14]. For a monopole with short circuits for loads at both top and base, their formula is:

$$Z_{in} = \frac{1}{2}\left[\frac{1}{4Z_d} + \frac{1}{j2Z_{0d}\tan(kh)}\right]^{-1} \tag{31}$$

In Equation 31, Z_d is the input impedance of a dipole of length *2h*. The differential mode characteristic impedance Z_{0d} and the wave number *k* have the same meaning as we have used throughout this article. The factor of ½ outside the brackets is the transformation from the input impedance of a dipole to that of a monopole over an infinite ground plane.

Figure 13 compares input resistance computed using our mathematical model with that calculated using Equation 31, for a 3-foot antenna with short circuits at both top and base. Overall, the agreement between the two theories seems excellent. There are slight disagreements in the vicinity of the peak resonances; however, over most of the interval, the two curves lie exactly on top of one another. A similar plot was obtained for input reactance.

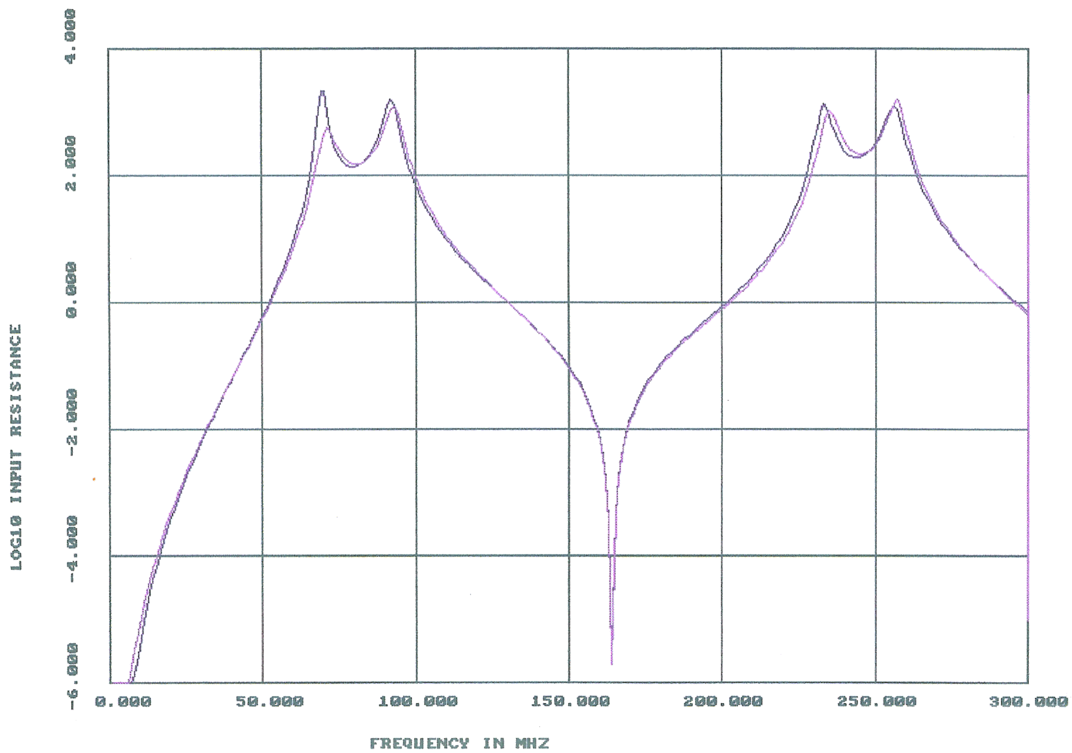

Figure 13. The input resistance calculated using our mathematical model (red) and the formula derived by King and Harrison (blue) agree very closely. In this example, the antenna is 3 feet long with short circuits loads at both top and base.

Further confirmation of our model comes from repeated experience in the field. The author's chief mentor concerning folded antennas was the late John H. Mullaney, (1920-1994), who installed hundreds of them around the world. He reported that there were multiple resonances around a quarter wavelength, not just one as in the case of the monopole. He further observed that the lowest resonance was "really steep". No one was ever able to explain these cumulative observations before; however, they are demonstrated quite readily by our model, and visible in Figures 2 and 3.

Conclusions

In this article, we developed a comprehensive yet mathematically simple model of one of the most versatile antennas available. The folded antenna combines the best features of linear radiators, transmission lines and lumped impedances into a single device. Compared with a simple monopole or dipole antenna of the same length, the folded antenna provides the engineer with six more independent design variables.

Thanks to these added variables, folded antennas are practical for lengths ranging from extremely short to extremely long, compared with a wavelength. Simple algebraic formulas were derived for antenna current, antenna voltage, input reactance, and input resistance. Color contour plots show that optimal (such as 50 Ω) resistances are obtainable for many combinations of design variables. With careful use of color contour plots, very compact folded antennas can be designed. We explored an example in which the antenna was 0.107 wavelength, 38.4 electrical degrees. The formulas agree almost exactly with the rigorous theory of King and Harrison. Further, they explain the multiple resonances in the vicinity of a quarter wavelength that have been observed during actual installation of these antennas around the world. It is hoped that these new, practical formulas will promote innovative designs for an antenna that was first described in the open literature in 1947, but that still has much unexplored potential.

Acknowledgments

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Biography

Jeremy Keith Raines received his BS degree in electrical science and engineering from MIT, his MS degree in applied physics from Harvard University and his Ph.D. degree in electromagnetics from MIT. He is a registered professional engineer in the state of Maryland. Since 1972, he has been a consulting engineer in electromagnetics. Antennas designed by him span the spectrum from ELF through SHF, and they may be found on satellites deep in space, on ships, on submarines, on aircraft, and at a variety of terrestrial sites. He may be contacted at www.rainesengineering.com.