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Polynomial Transformations of Tschirnhaus, Bring and Jerrard

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Abstract

Tschirnhaus gave transformations for the elimination of some of the intermediate terms in a polynomial. His transformations were developed further by Bring and Jerrard, and here we describe all these transformations in modern notation. We also discuss their possible utility for polynomial solving, particularly with respect to the Mathematica poster on the solution of the quintic.

1 Introduction

A recent issue of the BULLETIN contained a translation of the 1683 paper by Tschirnhaus [10], in which he proposed a method for solving a polynomial equation $P_n(x)$ of degree n by transforming it into a polynomial $Q_n(y)$ which has a simpler form (meaning that it has fewer terms). Specifically, he extended the idea (which he attributed to Decartes) in which a polynomial of degree n is *reduced* or *depressed* (lovely word!) by removing its term in degree $n-1$. Tschirnhaus's transformation is a polynomial substitution $y = T_k(x)$, in which the degree of the transformation $k < n$ can be selected. Tschirnhaus demonstrated the utility of his transformation by apparently solving the cubic equation in a way different from Cardano.

Although Tschirnhaus's work is described in modern books [9], later works by Bring [2, 3] and Jerrard [7, 6] have been largely forgotten. Here, we present the transformations in modern notation and make some comments on their utility. The investigations described here were stimulated by work on the Quintic poster [11] and therefore the discussion is directed towards the quintic. We shall deal with the following forms of a quintic:

$$
\begin{array}{lrcll}
x^5 + a_4\,x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0 & = & 0\,, & \text{General quintic form} & (1)\\
x^5 \qquad\qquad + b_3\,x^3 + b_2\,x^2 + b_1\,x + b_0 & = & 0\,, & \text{Reduced quintic form} & (2)\\
x^5 \qquad\qquad\qquad\qquad + c_2\,x^2 + c_1\,x + c_0 & = & 0\,, & \text{Principal quintic form} & (3)\\
x^5 \qquad\qquad\qquad\qquad\qquad\qquad + d_1\,x + d_0 & = & 0\,. & \text{Bring—Jerrard quintic form} & (4)
\end{array}
$$

2 Tschirnhaus's solution of the cubic

Before considering the quintic, we look at the cubic. Tschirnhaus transformed the depressed cubic $x^3 + px + q = 0$ into the binomial $y^3 + r = 0$. This can be done efficiently using the resultant, which is a tool from after Tschirnhaus's time, of course. Let $P_3(x)$ be the cubic and let $T_2(x,y)$ be Tschirnhaus's transform, then we have

$$\begin{aligned} P_3(x) &= x^3+px+q=0\ , && (5)\\ T_2(x,y) &= x^2+\alpha x+\tfrac{2}{3}p+y=0\ , && (6)\\ \mathrm{res}_x(P_3,T) &= y^3+\left(p\alpha^2-\tfrac{1}{3}p^2+3\alpha q\right)y+\tfrac{2}{3}\alpha^2p^2-\alpha^3q+q^2+\alpha pq+\tfrac{2}{27}p^3\ . && (7)\end{aligned}$$

Setting the coefficient of y in (7) to zero, we obtain a quadratic in α. Either root for α yields the equation

$$y^3=9q^3\alpha/p^2+\tfrac{4}{3}\alpha pq-\tfrac{8}{27}p^3-2q^2\ ,$$

and so 3 values for y are obtained. Tschirnhaus substituted the value(s) for y into (6) and solved for x. However, he failed to point out that this results in 6 values for x. The equation obeyed by these 6 values can be obtained by eliminating y between (7) and (6), and it is

$$(x^3+px+q)(x^3+3\,\alpha\,x^2+x[2p^2-9\alpha q]/p+\tfrac{4}{3}\,\alpha\,p+9\,\alpha\,q^2/p^2-2\,q)=0\ . \qquad (8)$$

After each root has been computed, it must be tested to see which factor it satisfies.

3 General quintic to principal quintic

To apply Tschirnhaus's idea to the quintic, we use the formulae for the power-sums of the quintic roots. This method is more interesting than simply using the resultant again, and in addition is actually easier to work with. Denote the roots of (1) by $x_i,\ i=1,...,5$, and let

$$S_n=S_n(x_k)=\sum_{k=1}^{5}x_k^n$$

be the sum of the nth powers. Following Newton [5], a general representation for S_n for $n\in\mathbb{N}$ is

$$S_n=-n\,a_{5-n}-\sum_{j=1}^{n-1}S_{n-j}\,a_{5-j}\ ,$$

with $a_j=0$ for $j<0$. Special cases are $S_1=-a_4$, $S_2=a_4^2-2a_3$, $S_3=-a_4^3+3a_3a_4-3a_2$,

$$\begin{aligned} S_4 &= a_4^4-4\,a_3\,a_4^2+4\,a_2\,a_4+2\,a_3^2-4\,a_1\ , && (9)\\ S_5 &= -a_4^5+5\,\left(a_3\,a_4^3-a_2\,a_4^2-a_3^2\,a_4+a_1\,a_4-a_0+a_2\,a_3\right)\ . && (10)\end{aligned}$$

Supposing that the roots x_k of (1) are related to roots y_k of (3) by a quadratic Tschirnhaus transformation

$$y_k={x_k}^2+\ \alpha\,x_k\ +\ \beta \qquad (11)$$

we wish to determine α and β subject to the requirement that they are expressed algebraically in terms of the coefficients. The power sums for (3) are $S_1(y_k)=S_2(y_k)=0$,

$$S_3(y_k)=-3b_2\ ,\qquad S_4(y_k)=-4b_1\ ,\qquad S_5(y_k)=-5b_0\ . \qquad (12)$$

Evaluating $S_1(y_k)$ and $S_2(y_k)$ using (11), we obtain equations for α and β. After some simplification,

$$\begin{aligned} \alpha\,a_4-5\,\beta+2\,a_3-a_4^2 &= 0\ , && (13)\\ \alpha^2\,a_3-10\,\beta^2+\alpha\,(3\,a_2-a_3\,a_4)+2\,a_1-2\,a_2\,a_4+a_3^2 &= 0\ . && (14)\end{aligned}$$

This system is quadratic with respect to α and β, and thus will produce two sets of coefficients. We are free to choose either of these. After finding α and β, we use the last three equations, those in (12), to obtain $b_0,\ b_1,\ b_2$.

As Tschirnhaus pointed out, transformation (11) can be applied to a general polynomial of degree n to remove terms in x^{n-1} and x^{n-2}.

4 Transformation to a Bring-Jerrard form

Removing, from a general quintic, the three terms in x^4, x^3 and x^2, brings it to Bring—Jerrard form. Tschirnhaus clearly thought that he would be able to do this by using the cubic transformation

$$z_k = {x_k}^3 + \alpha\, {x_k}^2 + \beta\, x_k + \gamma\,. \tag{15}$$

This section shows that it is not always possible to solve for the coefficients α, β, γ in terms of radicals, which are the quantities that Tschirnhaus would have been expecting to use.

To reduce the size of the equations, we shall start with a principal form cubic $x^5 + c_2\, x^2 + c_1\, x + c_0 = 0$ and eliminate the x^2 term. Extending the approach of the previous section in the obvious way, we use the power sums for the new quintic

$$\begin{aligned} S_1(z_k) &= S_2(z_k) \;=\; S_2(z_k) \;=\; 0, \\ S_4(z_k) &= -4d_1, \\ S_5(z_k) &= -5d_0. \end{aligned} \tag{16}$$

From the first three equations in (16) we determine α, β, γ. The remaining two equations will determine the new coefficients. Using (15) to evaluate S_1, S_2, S_3, we obtain three equations for the parameters:

$$5\,\gamma - 3\,c_2 = 0, \tag{17}$$

$$5\,\gamma^2 - 10\,\alpha\,c_0 - 4\,\alpha^2\,c_1 - 8\,\beta\,c_1 - 6\,\alpha\,\beta\,c_2 - 6\,\gamma\,c_2 + 3\,c_2^2 = 0, \tag{18}$$

$$\begin{aligned} 5\,\gamma^3 - 15\,\alpha^2\,\beta\,c_0 - 15\,\beta^2\,c_0 - 30\,\alpha\,\gamma\,c_0 - 12\,\alpha\,\beta^2\,c_1 - 12\,\alpha^2\,\gamma\,c_1 - 24\,\beta\,\gamma\,c_1 + 9\,c_0\,c_1 + \\ 12\,\alpha\,c_1^2 - 3\,\beta^3\,c_2 - 18\,\alpha\,\beta\,\gamma\,c_2 - 9\,\gamma^2\,c_2 + 24\,\alpha\,c_0\,c_2 + 21\,\alpha^2\,c_1\,c_2 + \\ 21\,\beta\,c_1\,c_2 + 3\,\alpha^3\,c_2^2 + 18\,\alpha\,\beta\,c_2^2 + 9\,\gamma\,c_2^2 - 3\,c_2^3 = 0\,. \end{aligned} \tag{19}$$

In general this system is not solvable by radicals. We can easily see this by eliminating β and γ. This will produce a sixth degree polynomial equation in α.

In 1796, Bring [2, 3] found a way around this problem. He considered a quartic transformation,

$$z_k = x_k^4 + \alpha\, x_k^3 + \beta\, x_k^2 + \gamma\, x_k + \delta, \tag{20}$$

that offers an extra parameter. Varying this parameter Bring was able to decrease the degrees of the equations for the rest of parameters. Substituting (20) into (16), we get a system of five equations with six unknown variables. From the equation

$$S_1(z_k) = 5\,\delta - 4\,c_1 - 3\,\alpha\,c_2 = 0\,,$$

we find $\delta = \frac{4}{5}c_1 + \frac{3}{5}c_2\alpha$. The second equation

$$\begin{aligned} S_2(z_k) \;=\; & -10\alpha\,\beta\,c_0 - 4\,\beta^2\,c_1 + \tfrac{4}{5}\,c_1^2 + 8\,c_0\,c_2 + \tfrac{46}{5}\,\alpha\,c_1\,c_2 \\ & + \tfrac{6}{5}\,\alpha^2\,c_2^2 + 6\,\beta\,c_2^2 - 2\,\gamma\,(5\,c_0 + 4\,\alpha\,c_1 + 3\,\beta\,c_2) = 0 \end{aligned} \tag{21}$$

relates β and γ. The trick is to choose a β that will make the coefficient of γ in (21) vanish. The value is

$$\beta = -\frac{5\,c_0}{3\,c_2} - \frac{4\,c_1}{3\,c_2}\,\alpha\,. \tag{22}$$

Therefore, equation (21) now depends only on α and is a quadratic.

$$\alpha^2(27c_2^4 - 160c_1^3 + 300c_0\,c_1\,c_2) + \alpha\,(27c_1\,c_2^3 - 400c_0\,c_1^2 + 375c_0^2\,c_2) + 18c_1^2\,c_2^2 - 45c_0\,c_2^3 - 250c_0^2\,c_1 = 0\,. \tag{23}$$

Finally, setting the sum of the cubes of (20) to zero, $S_3(z_k) = 0$, we obtain a cubic equation for γ. The equation will not be shown. Therefore all of the intermediate quantities can be found in terms of radicals.

This work was generalized by Jerrard [7, 6] (it is most likely that Jerrard was not aware of Bring's research) to show that the transformation (20) could be applied to a polynomial of degree n to remove the terms in x^{n-1}, x^{n-2} and x^{n-3}. In particular, Jerrard claimed that he developed a method for solving a general quintic. Hamilton was asked to report on this. In his detailed report, Hamilton showed that Jerrard had not completely solved the general quintic equation by radicals. The report is available at
`http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Jerrard/`.

5 Discussion

Starting with a general quintic (1), the transformation (20) produces the Bring–Jerrard form. In the case of a solvable quintic, the roots can be found using the 1771 formulae of Malfatti [8], who was the first to "solve" the quintic using a resolvent of sixth degree. In 1948, G. Watson [1] gave a public lecture in which he outlined step-by-step a procedure for solving a (solvable) quintic in radicals. In 1991, Dummit [4] developed exact formulas for the roots of solvable quintics.

The irreducible quintic can be solved in terms of Jacobi theta functions, as was first done by Hermite in 1858. Once we have the 5 solutions, we must invert the Tschirnhaus-Bring transformation to obtain the solutions to the original quintic. Thus we shall in general obtain 20 candidates for five solutions. There is no way of knowing which ones are correct without numerical testing. It is interesting to note that if one used Tschirnhaus's cubic transformation to solve a quintic (using something other than radicals), then one would obtain 15 solution candidates. By using a quartic transformation, Bring and Jerrard simplified the intermediate expressions at the price of now generating 20 solution candidates. Clearly, as symbolic solutions, any of these methods have their limitations, since we (or a computer system) can present only a list of solution candidates, rather than guaranteed solutions.

There is a Mathematica implementation of the Tschirnhaus-Bring transformation available from the Wolfram Research web cite at `http://library.wolfram.com/infocenter/TechNotes/158/`.
On this page you will find a set of notebooks that demonstrate various approaches to the solution of the quintic.

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