Statistical notes on rifle group patterns

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2.1 General Comments

A shot pattern is well described by two independent normal distributions representing the vertical and horizontal dispersion of shots. The variances of the two distributions are not always equal, since gravity acts only vertically. In theory, the dimensions may be correlated, but in practice, the data seems to be well described by assuming independence.

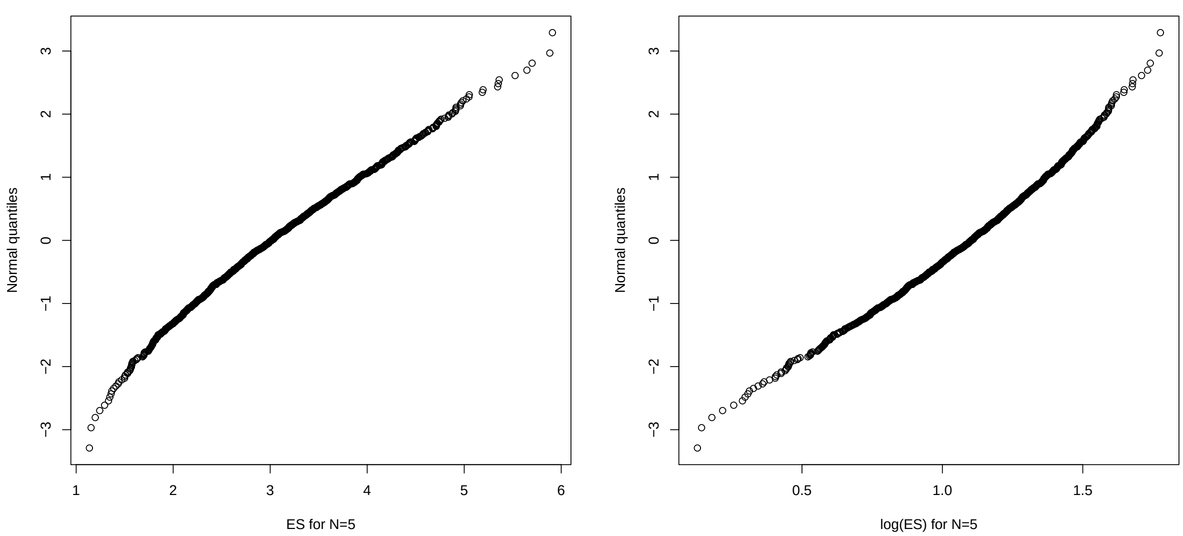
The accuracy of the shot pattern is not usually of much interest, since that may be adjusted by changing the aim point either by scope adjustment or by a visual offset.

The precision, is however of great interest, and several statistics appear in the literature. Six statistics are described in Grubbs (1964)[1]: their definitions are given in Section (4). The radial standard deviation, RSD, and the mean radius, MR, require the precise location of individual shots, which presents a problem on badly torn targets, making them impractical. The other four statistics, the radius of the covering circle, RC, the figure of merit, FOM, which is an average of the ranges in the two directions, the diagonal of the covering box, D, and the extreme spread, ES, are more practical. The efficiencies of the various statistics differ, but the differences are rather small, so that the less efficient and more practical statistics accrue no great loss in their use.

[1] Grubbs, Frank E.,Statistical measures of accuracy for rifelmen and missile engineers, 4109 Webster Rd.,Havre De Grace, MD 21078

In theory the expectations of all of the suggested statistics are scalar multiples of combinations of the standard deviations in the vertical and horizontal directions and all follow distributions with some similarity to the χ distribution. In practice however, even for small numbers of shots, their distributions are reasonably approximated by either the normal or lognormal distributions. Figure (1) shows normal and lognormal plots for 1,000 simulations of 5 shot groups for the ES statistic. It may be seen that either distribution may be chosen, and may be expected to represent the data reasonably well, except perhaps, in the extreme tails of the distributions. This may be improved by averaging a statistic for several groups, which by the central limit theorem will result in a more normal distribution. Plots for other statistics are similar.

Figure 1: Normal plots for ES and log(ES) for N=5



2.2 Tables

Tables (2) throw (13) show the means and standard deviations of the several statistics for varying N's, assuming the horizontal shot standard deviation, σ_h, to be unity. These tables were obtained from 10,000 simulated groups, and should be correct to the number of decimals that are shown. The simulations rounded the shot statistics to two decimal places before performing calculations, which should be more realistic since shooters usually measure to this accuracy. These may be compared with the tables in Graves (1964), many of which were obtained from theoretical evaluations.

2.3 Models and tests

Let y be an observation on any of the statistics, and let μ_N be the expectation from the appropriate table, then one may model y as

$$y = \sigma_h \mu_N + \text{error},$$

where σ_h is the actual horizontal shot standard deviation. Assuming normality the error is distributed as $N(0, \sigma_h \sigma_N)$, where σ_N is from the above cited tables. Unfortunately this error variance depends on σ_h, which requires a non-standard testing methodology. The situation is much better when the error is assumed to be lognormal for then the error variance does not depend on σ_h:

$$log(y) = log(\sigma_h) + \mu_{ln,N} + \text{error},$$

with $\mu_{ln,N}$ the expectation from the appropriate table. In this case the error is $N(0, \sigma_{ln,N})$ where $\sigma_{ln,N}$ is from the cited tables.

The maximum likelihood estimate of $log(\sigma_h)$ is thus $log(y) - \mu_{ln,N}$, and $\text{var}(log(y) - \mu_{ln,N}) = (\sigma_{ln,N})^2$. The lognormal assumption enables statistical testing. Let y_1 and y_2 be two observations involving N_1 and N_2 shots respectively, then

$$[log(y_1/y_2) - (\mu_{ln,N_1} - \mu_{ln,N_2})]/\sqrt{(\sigma_{ln,N_1})^2 + (\sigma_{ln,N_2})^2)} \quad (1)$$

is normally distributed, and the hypothesis of zero difference may be tested in the usual fashion.

EXAMPLE1:

Suppose $y_1 = 0.99$ in for $N = 5$, and $y_2 = 0.71$ in for $N = 10$, then from Table (13), one has $\mu_{ln,5} = 1.08$, $\mu_{ln,10} = 1.32$, hence the numerator of equation (1) is $log(0.99/0.71) - (1.08 - 1.32) = 0.57$. For the denominator, one has $\sigma_{ln,5} = 0.28$, $\sigma_{ln,10} = 0.19$, thus $\sqrt{(0.28)^2 + (0.19)^2} = 0.338$. Equation (1) then gives 1.69, which is significant at the 5% level.

EXAMPLE2:

The same as above, but assume $y_1 = 0.85$ is the geometric mean of 10 groups of size $N = 5$, then one has

$$(log(0.85/0.71) + 0.24)/\sqrt{(0.28)^2/10 + (0.19)^2)} = 0.42/0.21 = 2,$$

which is significant at about the 2% level.

2.4 Power

One may make power calculations using the logarithmic form, using

$$log(R) = K(\sigma_{ln,N}\sqrt{2/n}),$$

or

$$n = 2\left(\frac{K\sigma_{ln,N}}{log(R)}\right)^2,$$

where R is the ratio of two N shot group sizes, K is the sum of normal deviates corresponding to α and $1-\beta$, and n is the common sample size for each group.

EXAMPLE3

Let $N = 5$, and $R = 1.5$, $\alpha = 0.05$ and $\beta = 0.10$. Assume the test is one-sided, then the sum of the normal deviates for α and $1-\beta$ is 2.93, and one has $n = 2[2.93 \times 0.28)/log(1.5)]^2 = 8$. Thus 8 replicates of 5 shot groups when $ES = 0.6$ compared with 8 replicates of 5 shot groups with $ES = 0.9$ should be detected by a 5% test with a power of 90%. For a two-sided test, the sum of the normal deviates would be 3.6.

Figure (2) shows a plot of the ratio of group sizes that will be detectable for $\alpha = 0.05$, with a power of 0.90 for various numbers of 5 shot groups. For example, shooting 10 groups under each of two conditions will detect a ratio of about 1.4 in the group sizes between the two conditions: as, say, between 0.6 in and 0.84 in. It may be seen from this figure that detecting smaller ratios is extremely hard. To decrease it to 1.25 (0.6 in to 0.75 in) requires about 27 groups under each condition – that's 270 shots in all.

An additional consequence is that it is better to average several small groups than it is to shoot a single large one. For example, one can detect a ratio of 1.4 using the average of 10 groups of N=5, but the ratio is 1.6 for a single group of 50 shots. It is worth noting, however, that for group sizes between 5 and about 20 shots, it does not matter much how that sample size is divided into groups: for a total of 50 shots per condition, the ratio 1.4 holds for group sizes from N=5 up to about N=20. More or less shots per condition will effect this conclusion slightly.

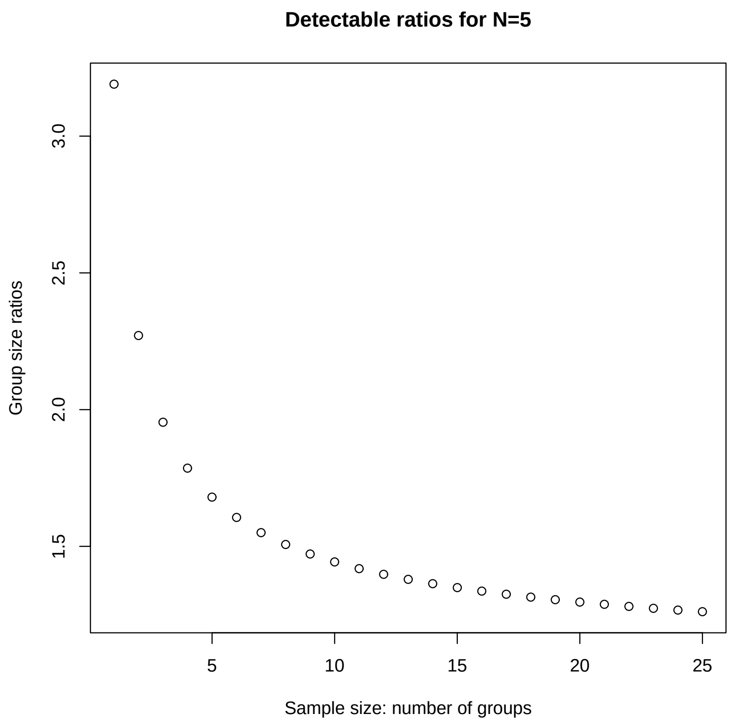
2.5 Practical rules

3 Powder throw variations

Let S_p be the standard deviation of the powder throw variations in grains, and V_p the increase in velocity per grain of powder, then $S_v = S_p V_p$ is the standard deviation in velocity due to powder throw variations. From ballistic tables, one can calculate the drop at $d_h = V + S_v$ and $d_l = V - S_v$, then take $S = (d_h - d_l)/2$ as an estimate of the vertical standard deviation due to powder throw variations. If ES_n is the extreme spread for an n round group assuming equal bivariate standard deviations of unity, and X is the expected value of the extreme spread for an n shot group for a given rifle, powder, distance configuration, then the standard deviation of the observed extreme spreads is

$$S_{ES} = ES_n\sqrt{(\frac{X}{ES_n})^2 + S^2}. \tag{2}$$

Figure 2: Detectable ratios for N=5



If the bivariate standard deviations are not equal, and R is the ratio of the vertical to the horizontal standard deviations, then ES_n should be replaced with $ES_n(R)/R$ in the above formula, where $ES_n(R)$ is the expected value of the extreme spread for an n round group with bivariate standard deviations in the ratio R.

EXAMPLE:

For an aim point of about 22gr, my Dillon 650 set up for 223 caliber produced a sample mean of 22.4 gr, and standard deviation of $S_p = 0.62$gr. A normal plot of the throw weights in gr is shown in Figure(3).

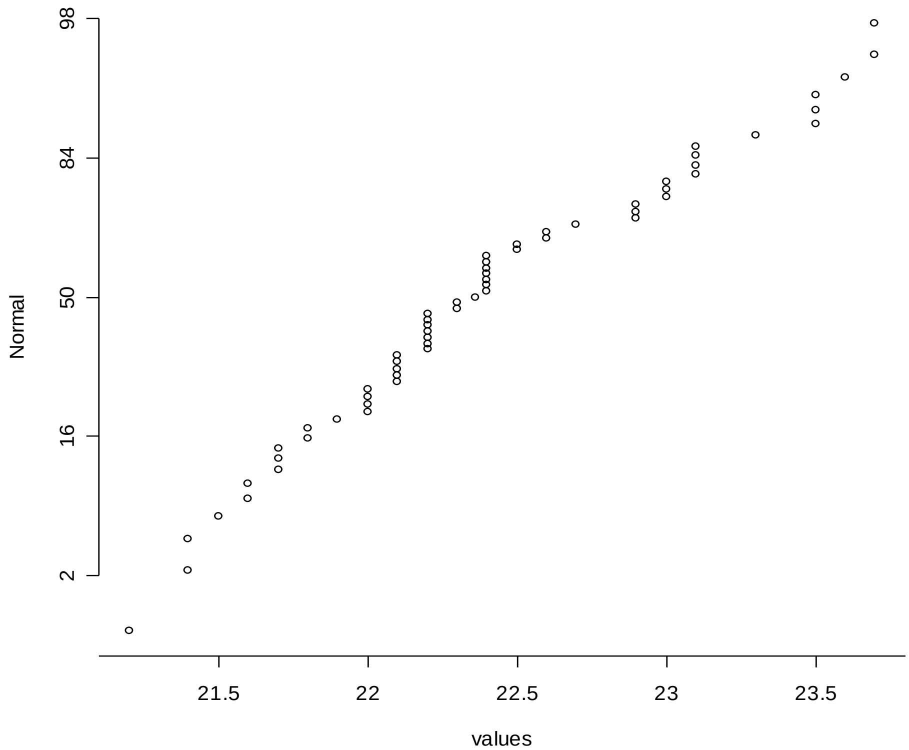
From the Sierra loading manual, 5th ed., the value of V_p for a 69gr HPBT bullet using 26.5gr of Viht N540 is 125 (f/s)/gr, thus $S_v = 0.62\text{gr} \times 125(f/s)/gr = 77.5f/s$. Using the Sierra ballistic tables for this load produces $d_h = -1.85''$, and $d_l = -2.04''$, and thus $S = 0.095''$.

From Table (12), $ES_5 = 3.04$, and assuming $X = 0.5''$ one has from equation (2), $S_{ES} = 0.58$, and thus the powder throw variations when using my Dillon 650 for this load increase the extreme spread from $0.5''$ to $0.58''$

OTHER LOADS:

Using RE15 with N=30, the sample mean was 22.2gr and the sample sd was

Figure 3: Powder throw, Dillon 650 for VihtaVuori 540 in 223



0.44gr. Using H335 with N=30, the sample mean was 25.3gr, and the sample sd was 0.47gr.

For a 308 with N=60 using RE15, the sample mean was 41.9gr, and the sample standard deviation was 0.18gr. For a Sierra 150gr HPBT, at 2700f/s, equation (2) gives 0.504″ when the expected ES is 0.50″, which is a negligible change.

CONCLUSION:

The effect of powder throw variations using my Dillon presses depends on the amount thrown, and the precision of the group. The smaller the amount thrown, the more powder variations effect the group size. The smaller the group size, the more powder variations effect the group size. Group sizes greater than 1 MOA are probably little effected by the Dillon throw. Group sizes less than 1/2 MOA are clearly effected.

4 Group size statistics

Let $(x_i, y_i), i = 1 \ldots N$ be the centers of the shots in an N shot group, then the following statistics appear in the literature. Historically, many of these statistics have only been considered for the case of equal horizontal and vertical dispersions. Many authors use N in these formulas in place of $N - 1$.

1. Horizontal, σ_h, and vertical, σ_v, standard deviation are defined as follows. Here $\bar{x}$ and $\bar{y}$ are the means in the two directions.

$$\sigma_h = \sqrt{\frac{1}{N-1} \Sigma_1^N (x_i - \bar{x})^2}$$

$$\sigma_v = \sqrt{\frac{1}{N-1} \Sigma_1^N (y_i - \bar{y})^2}$$

2. Radial standard deviation.

$$RSD = \sqrt{\sigma_h^2 + \sigma_v^2}$$

3. Mean radius.

$$MR = \Sigma_1^N (r_i) \quad | \quad r_i = \sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}$$

4. The diagonal. Let R_x and R_y be the ranges of the x_i and y_i values, then

$$D = \sqrt{R_x^2 + R_y^2}$$

5. The figure of merit

$$FOM = (R_x + R_y)/2$$

6. The radius of the covering circle.

$$RC = \text{Radius of the smallest circle containing all shot centers.}$$

   NOTE: This circle will either pass through two extreme points or through the points of a triangle. If a triangle, the center of the circle will be located at the intersection of the perpendicular bisectors of two sides.

7. The extreme spread

$$ES = \max_{i \neq j} \sqrt{(x_i - x_j)^2 - (y_i - y_j)^2} \quad i, j = 1, \ldots, N$$

Table 1: Relative variances for six measures of group size

Group size	RSD	MR	D	FOM	RC	ES
2	0.520	0.525	0.518	0.538	0.517	0.516
3	0.366	0.363	0.364	0.369	0.367	0.372
4	0.294	0.297	0.295	0.306	0.303	0.305
5	0.251	0.261	0.261	0.263	0.267	0.269
6	0.227	0.229	0.237	0.238	0.241	0.242
7	0.204	0.211	0.217	0.218	0.221	0.230
8	0.191	0.197	0.205	0.205	0.211	0.216
9	0.178	0.184	0.191	0.194	0.200	0.204
10	0.167	0.176	0.183	0.183	0.188	0.193
11	0.160	0.164	0.177	0.176	0.183	0.190
12	0.150	0.157	0.170	0.169	0.177	0.182
13	0.145	0.150	0.165	0.165	0.174	0.173
14	0.139	0.144	0.158	0.160	0.167	0.171
15	0.133	0.140	0.155	0.153	0.161	0.168
16	0.130	0.135	0.152	0.150	0.160	0.164
17	0.126	0.130	0.147	0.146	0.155	0.162
18	0.122	0.126	0.144	0.143	0.153	0.155
19	0.119	0.123	0.142	0.140	0.150	0.153
20	0.116	0.119	0.138	0.138	0.147	0.152
25	0.102	0.107	0.128	0.126	0.138	0.140
30	0.093	0.096	0.121	0.122	0.128	0.128
40	0.080	0.083	0.111	0.109	0.118	0.122
50	0.070	0.075	0.103	0.103	0.111	0.114
100	0.051	0.052	0.086	0.085	0.094	0.096

5 Tables

6 BIBLIOGRAPHY

1. Grubbs, Frank E. (1964). *Statistical measures of accuracy for riflemen and missle engineers*, 4109 Webster Rd. Havre De Grace, MD 21078

Table 2: RSD from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{RSD}	σ_{RSD}	μ_{RSD}	σ_{RSD}	μ_{RSD}	σ_{RSD}	μ_{RSD}	σ_{RSD}
2	1.25	0.65	1.59	0.87	1.93	1.13	2.66	1.71
3	1.34	0.49	1.67	0.64	2.06	0.84	2.91	1.31
4	1.36	0.40	1.73	0.54	2.11	0.71	2.96	1.09
5	1.38	0.35	1.73	0.46	2.14	0.62	3.01	0.97
6	1.38	0.31	1.76	0.43	2.17	0.56	3.03	0.87
7	1.38	0.28	1.76	0.39	2.18	0.52	3.06	0.81
8	1.39	0.27	1.77	0.36	2.18	0.48	3.07	0.74
9	1.39	0.25	1.77	0.34	2.20	0.46	3.09	0.71
10	1.40	0.23	1.77	0.32	2.19	0.42	3.10	0.67
11	1.40	0.22	1.78	0.30	2.21	0.41	3.09	0.63
12	1.40	0.21	1.78	0.29	2.20	0.38	3.11	0.60
13	1.40	0.20	1.78	0.27	2.21	0.38	3.11	0.57
14	1.40	0.19	1.78	0.27	2.21	0.36	3.11	0.56
15	1.40	0.19	1.79	0.25	2.21	0.34	3.12	0.53
16	1.40	0.18	1.78	0.25	2.21	0.33	3.12	0.52
17	1.41	0.18	1.79	0.24	2.22	0.32	3.12	0.50
18	1.40	0.17	1.78	0.23	2.22	0.31	3.13	0.49
19	1.41	0.17	1.79	0.23	2.21	0.30	3.14	0.47
20	1.40	0.16	1.80	0.22	2.22	0.30	3.13	0.46
25	1.41	0.14	1.79	0.19	2.22	0.27	3.14	0.41
30	1.41	0.13	1.79	0.18	2.23	0.24	3.14	0.37
40	1.41	0.11	1.80	0.16	2.23	0.21	3.14	0.32
50	1.41	0.10	1.80	0.14	2.23	0.19	3.15	0.29
100	1.41	0.07	1.80	0.10	2.23	0.13	3.16	0.20

Table 3: log(RSD) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnRSD}	σ_{lnRSD}	μ_{lnRSD}	σ_{lnRSD}	μ_{lnRSD}	σ_{lnRSD}	μ_{lnRSD}	σ_{lnRSD}
2	0.06	0.64	0.28	0.66	0.46	0.68	0.75	0.73
3	0.21	0.40	0.44	0.42	0.64	0.44	0.96	0.49
4	0.26	0.31	0.50	0.33	0.69	0.35	1.01	0.39
5	0.29	0.26	0.51	0.28	0.72	0.30	1.05	0.34
6	0.29	0.24	0.53	0.25	0.74	0.27	1.07	0.30
7	0.30	0.21	0.54	0.23	0.75	0.24	1.08	0.27
8	0.31	0.20	0.55	0.21	0.76	0.22	1.09	0.25
9	0.31	0.18	0.55	0.19	0.77	0.21	1.10	0.24
10	0.32	0.17	0.56	0.18	0.77	0.19	1.11	0.22
11	0.32	0.16	0.56	0.17	0.78	0.19	1.11	0.21
12	0.32	0.15	0.56	0.16	0.77	0.18	1.12	0.20
13	0.33	0.15	0.57	0.16	0.78	0.17	1.12	0.19
14	0.33	0.14	0.57	0.15	0.78	0.16	1.12	0.18
15	0.33	0.13	0.57	0.14	0.78	0.16	1.12	0.17
16	0.33	0.13	0.57	0.14	0.78	0.15	1.12	0.17
17	0.33	0.13	0.57	0.13	0.79	0.15	1.13	0.16
18	0.33	0.12	0.57	0.13	0.79	0.14	1.13	0.16
19	0.33	0.12	0.57	0.13	0.78	0.14	1.13	0.15
20	0.33	0.12	0.58	0.12	0.79	0.13	1.13	0.15
25	0.34	0.10	0.58	0.11	0.79	0.12	1.13	0.13
30	0.34	0.09	0.58	0.10	0.79	0.11	1.14	0.12
40	0.34	0.08	0.58	0.09	0.80	0.09	1.14	0.10
50	0.34	0.07	0.58	0.08	0.80	0.08	1.14	0.09
100	0.34	0.05	0.59	0.05	0.80	0.06	1.15	0.06

Table 4: MR from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{MR}	σ_{MR}	μ_{MR}	σ_{MR}	μ_{MR}	σ_{MR}	μ_{MR}	σ_{MR}
2	0.89	0.47	1.11	0.61	1.35	0.80	1.88	1.22
3	1.03	0.37	1.29	0.50	1.58	0.64	2.18	0.98
4	1.09	0.32	1.37	0.43	1.67	0.55	2.31	0.84
5	1.12	0.29	1.42	0.39	1.73	0.50	2.39	0.75
6	1.15	0.26	1.45	0.35	1.77	0.45	2.42	0.69
7	1.16	0.25	1.46	0.32	1.79	0.43	2.47	0.64
8	1.17	0.23	1.48	0.31	1.81	0.40	2.50	0.60
9	1.18	0.22	1.49	0.29	1.82	0.37	2.52	0.56
10	1.19	0.21	1.51	0.27	1.83	0.35	2.53	0.54
11	1.19	0.19	1.51	0.26	1.85	0.34	2.54	0.51
12	1.20	0.19	1.52	0.25	1.85	0.32	2.56	0.49
13	1.20	0.18	1.52	0.24	1.86	0.31	2.56	0.47
14	1.21	0.17	1.52	0.23	1.86	0.30	2.56	0.45
15	1.21	0.17	1.53	0.22	1.87	0.29	2.57	0.44
16	1.21	0.16	1.54	0.22	1.88	0.28	2.58	0.42
17	1.22	0.16	1.54	0.21	1.88	0.27	2.59	0.41
18	1.22	0.15	1.54	0.20	1.88	0.26	2.60	0.41
19	1.22	0.15	1.54	0.20	1.88	0.26	2.60	0.39
20	1.22	0.15	1.54	0.19	1.88	0.25	2.59	0.37
25	1.23	0.13	1.55	0.17	1.89	0.22	2.61	0.34
30	1.23	0.12	1.55	0.16	1.90	0.20	2.62	0.31
40	1.24	0.10	1.56	0.14	1.91	0.18	2.63	0.27
50	1.24	0.09	1.57	0.12	1.91	0.16	2.64	0.24
100	1.25	0.06	1.58	0.09	1.92	0.11	2.65	0.17

Table 5: log(MR) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnMR}	σ_{lnMR}	μ_{lnMR}	σ_{lnMR}	μ_{lnMR}	σ_{lnMR}	μ_{lnMR}	σ_{lnMR}
2	-0.29	0.64	-0.07	0.66	0.10	0.70	0.40	0.75
3	-0.05	0.40	0.18	0.42	0.37	0.44	0.67	0.49
4	0.04	0.32	0.26	0.33	0.46	0.35	0.77	0.39
5	0.07	0.27	0.31	0.29	0.50	0.30	0.82	0.33
6	0.11	0.24	0.34	0.25	0.54	0.27	0.84	0.29
7	0.13	0.22	0.35	0.22	0.55	0.24	0.87	0.26
8	0.14	0.20	0.37	0.21	0.57	0.23	0.89	0.25
9	0.15	0.19	0.38	0.20	0.58	0.21	0.90	0.23
10	0.16	0.18	0.39	0.18	0.59	0.19	0.91	0.22
11	0.16	0.17	0.40	0.18	0.60	0.19	0.91	0.20
12	0.17	0.16	0.40	0.16	0.60	0.18	0.92	0.19
13	0.17	0.15	0.41	0.16	0.61	0.17	0.92	0.19
14	0.18	0.15	0.41	0.15	0.61	0.16	0.93	0.18
15	0.18	0.14	0.41	0.15	0.61	0.16	0.93	0.17
16	0.18	0.14	0.42	0.14	0.62	0.15	0.93	0.17
17	0.19	0.13	0.42	0.14	0.62	0.15	0.94	0.16
18	0.19	0.13	0.42	0.13	0.62	0.14	0.94	0.16
19	0.19	0.12	0.42	0.13	0.62	0.14	0.94	0.15
20	0.19	0.12	0.43	0.13	0.62	0.13	0.94	0.15
25	0.20	0.11	0.43	0.11	0.63	0.12	0.95	0.13
30	0.20	0.10	0.44	0.10	0.64	0.11	0.96	0.12
40	0.21	0.08	0.44	0.09	0.64	0.09	0.96	0.10
50	0.21	0.08	0.45	0.08	0.64	0.08	0.97	0.09
100	0.22	0.05	0.45	0.05	0.65	0.06	0.97	0.06

Table 6: D from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_D	σ_D	μ_D	σ_D	μ_D	σ_D	μ_D	σ_D
2	1.79	0.92	2.24	1.23	2.74	1.59	3.77	2.41
3	2.53	0.92	3.24	1.25	3.97	1.64	5.52	2.51
4	3.04	0.90	3.85	1.22	4.76	1.62	6.62	2.51
5	3.41	0.89	4.31	1.19	5.33	1.58	7.44	2.47
6	3.68	0.87	4.67	1.17	5.76	1.57	8.11	2.38
7	3.92	0.85	4.98	1.14	6.13	1.54	8.61	2.38
8	4.12	0.84	5.22	1.13	6.45	1.50	9.00	2.31
9	4.27	0.82	5.43	1.11	6.73	1.49	9.51	2.33
10	4.42	0.81	5.62	1.10	6.96	1.47	9.81	2.28
11	4.55	0.80	5.79	1.09	7.16	1.46	10.12	2.25
12	4.68	0.80	5.95	1.07	7.37	1.42	10.37	2.22
13	4.80	0.79	6.08	1.05	7.56	1.43	10.59	2.21
14	4.89	0.77	6.23	1.05	7.65	1.40	10.82	2.17
15	4.97	0.77	6.33	1.06	7.82	1.40	11.02	2.15
16	5.05	0.77	6.44	1.03	7.96	1.38	11.23	2.14
17	5.13	0.75	6.53	1.02	8.09	1.38	11.39	2.13
18	5.20	0.75	6.61	1.02	8.19	1.35	11.55	2.08
19	5.26	0.75	6.71	1.01	8.29	1.34	11.70	2.10
20	5.34	0.74	6.78	0.99	8.40	1.34	11.85	2.05
25	5.60	0.72	7.13	0.97	8.81	1.30	12.47	2.05
30	5.82	0.71	7.40	0.95	9.18	1.29	12.94	2.00
40	6.13	0.68	7.85	0.93	9.70	1.24	13.72	1.94
50	6.39	0.66	8.15	0.90	10.09	1.21	14.25	1.84
100	7.11	0.61	9.08	0.83	11.24	1.11	15.84	1.71

Table 7: log(D) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnD}	σ_{lnD}	μ_{lnD}	σ_{lnD}	μ_{lnD}	σ_{lnD}	μ_{lnD}	σ_{lnD}
2	0.42	0.63	0.63	0.66	0.81	0.69	1.10	0.74
3	0.85	0.40	1.09	0.42	1.29	0.45	1.60	0.49
4	1.06	0.31	1.30	0.33	1.50	0.36	1.81	0.40
5	1.19	0.27	1.42	0.29	1.63	0.31	1.95	0.34
6	1.28	0.24	1.51	0.26	1.71	0.28	2.05	0.30
7	1.34	0.22	1.58	0.23	1.78	0.26	2.11	0.28
8	1.39	0.21	1.63	0.22	1.84	0.24	2.16	0.26
9	1.43	0.20	1.67	0.21	1.88	0.22	2.22	0.25
10	1.47	0.19	1.71	0.20	1.92	0.21	2.26	0.24
11	1.50	0.18	1.74	0.19	1.95	0.20	2.29	0.22
12	1.53	0.17	1.77	0.18	1.98	0.19	2.32	0.22
13	1.55	0.17	1.79	0.17	2.00	0.19	2.34	0.21
14	1.57	0.16	1.82	0.17	2.02	0.18	2.36	0.20
15	1.59	0.16	1.83	0.17	2.04	0.18	2.38	0.20
16	1.61	0.15	1.85	0.16	2.06	0.17	2.40	0.19
17	1.62	0.15	1.86	0.16	2.08	0.17	2.42	0.19
18	1.64	0.14	1.88	0.15	2.09	0.17	2.43	0.18
19	1.65	0.14	1.89	0.15	2.10	0.16	2.44	0.18
20	1.67	0.14	1.90	0.15	2.12	0.16	2.46	0.17
25	1.71	0.13	1.95	0.14	2.16	0.15	2.51	0.16
30	1.75	0.12	1.99	0.13	2.21	0.14	2.55	0.15
40	1.81	0.11	2.05	0.12	2.26	0.13	2.61	0.14
50	1.85	0.10	2.09	0.11	2.30	0.12	2.65	0.13
100	1.96	0.09	2.20	0.09	2.41	0.10	2.76	0.11

Table 8: FOM from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{FOM}	σ_{FOM}	μ_{FOM}	σ_{FOM}	μ_{FOM}	σ_{FOM}	μ_{FOM}	σ_{FOM}
2	1.12	0.60	1.42	0.78	1.68	0.95	2.25	1.33
3	1.68	0.62	2.12	0.81	2.51	0.98	3.36	1.39
4	2.07	0.63	2.58	0.79	3.08	0.98	4.12	1.39
5	2.32	0.61	2.91	0.78	3.50	0.97	4.65	1.36
6	2.53	0.60	3.17	0.76	3.78	0.94	5.06	1.35
7	2.71	0.59	3.39	0.76	4.05	0.95	5.43	1.30
8	2.85	0.58	3.56	0.74	4.28	0.92	5.67	1.27
9	2.97	0.58	3.71	0.73	4.44	0.90	5.92	1.26
10	3.08	0.56	3.85	0.72	4.62	0.90	6.17	1.27
11	3.17	0.56	3.97	0.71	4.77	0.88	6.35	1.24
12	3.26	0.55	4.08	0.69	4.89	0.87	6.53	1.22
13	3.33	0.55	4.17	0.70	5.01	0.87	6.66	1.21
14	3.40	0.54	4.26	0.69	5.11	0.85	6.82	1.20
15	3.47	0.53	4.34	0.68	5.20	0.85	6.94	1.21
16	3.54	0.53	4.42	0.68	5.29	0.84	7.05	1.19
17	3.59	0.53	4.47	0.67	5.39	0.85	7.18	1.19
18	3.64	0.52	4.54	0.66	5.47	0.81	7.28	1.16
19	3.70	0.52	4.62	0.66	5.54	0.82	7.36	1.15
20	3.73	0.52	4.66	0.65	5.60	0.81	7.46	1.15
25	3.93	0.50	4.90	0.64	5.90	0.79	7.90	1.13
30	4.09	0.50	5.10	0.62	6.12	0.77	8.17	1.08
40	4.31	0.47	5.40	0.60	6.49	0.74	8.66	1.07
50	4.49	0.46	5.63	0.59	6.76	0.72	8.99	1.03
100	5.01	0.43	6.27	0.54	7.51	0.67	10.03	0.96

Table 9: log(FOM) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnFOM}	σ_{lnFOM}	μ_{lnFOM}	σ_{lnFOM}	μ_{lnFOM}	σ_{lnFOM}	μ_{lnFOM}	σ_{lnFOM}
2	-0.06	0.64	0.17	0.66	0.33	0.67	0.61	0.70
3	0.45	0.41	0.67	0.42	0.84	0.43	1.12	0.45
4	0.68	0.33	0.90	0.33	1.07	0.34	1.36	0.36
5	0.80	0.27	1.03	0.28	1.21	0.29	1.49	0.31
6	0.90	0.25	1.12	0.25	1.30	0.26	1.59	0.27
7	0.97	0.22	1.19	0.23	1.37	0.24	1.66	0.25
8	1.03	0.21	1.25	0.21	1.43	0.22	1.71	0.23
9	1.07	0.20	1.29	0.20	1.47	0.21	1.76	0.22
10	1.11	0.19	1.33	0.19	1.51	0.20	1.80	0.21
11	1.14	0.18	1.36	0.18	1.55	0.19	1.83	0.20
12	1.17	0.17	1.39	0.17	1.57	0.18	1.86	0.19
13	1.19	0.17	1.41	0.17	1.60	0.18	1.88	0.18
14	1.21	0.16	1.44	0.16	1.62	0.17	1.90	0.18
15	1.23	0.15	1.45	0.16	1.63	0.17	1.92	0.17
16	1.25	0.15	1.47	0.15	1.65	0.16	1.94	0.17
17	1.27	0.15	1.49	0.15	1.67	0.16	1.96	0.17
18	1.28	0.14	1.50	0.14	1.69	0.15	1.97	0.16
19	1.30	0.14	1.52	0.14	1.70	0.15	1.98	0.16
20	1.31	0.14	1.53	0.14	1.71	0.15	2.00	0.16
25	1.36	0.13	1.58	0.13	1.77	0.13	2.06	0.14
30	1.40	0.12	1.62	0.12	1.80	0.13	2.09	0.13
40	1.46	0.11	1.68	0.11	1.86	0.11	2.15	0.12
50	1.50	0.10	1.72	0.10	1.90	0.11	2.19	0.11
100	1.61	0.08	1.83	0.09	2.01	0.09	2.30	0.09

Table 10: RC from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{RC}	σ_{RC}	μ_{RC}	σ_{RC}	μ_{RC}	σ_{RC}	μ_{RC}	σ_{RC}
2	0.89	0.46	1.12	0.61	1.35	0.78	1.87	1.18
3	1.22	0.45	1.55	0.61	1.93	0.82	2.69	1.25
4	1.41	0.43	1.80	0.58	2.24	0.80	3.23	1.25
5	1.55	0.41	2.00	0.57	2.50	0.80	3.59	1.26
6	1.65	0.40	2.13	0.57	2.69	0.78	3.89	1.24
7	1.74	0.38	2.24	0.55	2.85	0.77	4.12	1.21
8	1.81	0.38	2.36	0.55	3.00	0.77	4.36	1.21
9	1.87	0.37	2.44	0.55	3.10	0.77	4.52	1.19
10	1.94	0.36	2.50	0.53	3.20	0.75	4.68	1.15
11	1.98	0.36	2.56	0.53	3.30	0.75	4.83	1.17
12	2.01	0.36	2.64	0.52	3.38	0.75	4.95	1.15
13	2.06	0.36	2.69	0.52	3.45	0.73	5.07	1.12
14	2.10	0.35	2.74	0.51	3.53	0.73	5.17	1.13
15	2.13	0.34	2.78	0.51	3.58	0.71	5.28	1.12
16	2.15	0.35	2.83	0.51	3.63	0.71	5.34	1.10
17	2.19	0.34	2.87	0.51	3.70	0.72	5.43	1.10
18	2.21	0.34	2.90	0.51	3.74	0.71	5.50	1.09
19	2.23	0.33	2.93	0.50	3.79	0.70	5.59	1.09
20	2.26	0.33	2.97	0.49	3.82	0.69	5.66	1.07
25	2.35	0.32	3.11	0.49	4.02	0.68	5.95	1.05
30	2.43	0.31	3.22	0.48	4.18	0.67	6.17	1.03
40	2.54	0.30	3.39	0.46	4.41	0.65	6.54	1.00
50	2.64	0.29	3.52	0.45	4.58	0.63	6.78	0.97
100	2.88	0.27	3.88	0.42	5.08	0.60	7.55	0.90

Table 11: log(RC) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnRC}	σ_{lnRC}	μ_{lnRC}	σ_{lnRC}	μ_{lnRC}	σ_{lnRC}	μ_{lnRC}	σ_{lnRC}
2	-0.28	0.63	-0.06	0.65	0.11	0.68	0.41	0.73
3	0.12	0.40	0.36	0.43	0.56	0.46	0.87	0.50
4	0.29	0.32	0.54	0.34	0.74	0.37	1.09	0.41
5	0.40	0.28	0.65	0.29	0.86	0.33	1.21	0.37
6	0.47	0.25	0.72	0.27	0.95	0.30	1.30	0.33
7	0.53	0.23	0.78	0.25	1.01	0.28	1.37	0.30
8	0.57	0.21	0.83	0.23	1.06	0.26	1.43	0.29
9	0.61	0.20	0.87	0.23	1.10	0.25	1.47	0.27
10	0.64	0.19	0.90	0.21	1.14	0.23	1.51	0.25
11	0.66	0.18	0.92	0.21	1.17	0.23	1.55	0.25
12	0.68	0.18	0.95	0.20	1.19	0.22	1.57	0.24
13	0.71	0.17	0.97	0.19	1.22	0.21	1.60	0.23
14	0.73	0.17	0.99	0.19	1.24	0.21	1.62	0.22
15	0.74	0.16	1.01	0.18	1.25	0.20	1.64	0.22
16	0.75	0.16	1.02	0.18	1.27	0.20	1.65	0.21
17	0.77	0.15	1.04	0.18	1.29	0.19	1.67	0.20
18	0.78	0.15	1.05	0.17	1.30	0.19	1.69	0.20
19	0.79	0.15	1.06	0.17	1.32	0.18	1.70	0.20
20	0.80	0.15	1.08	0.17	1.33	0.18	1.72	0.19
25	0.85	0.14	1.12	0.16	1.38	0.17	1.77	0.18
30	0.88	0.13	1.16	0.15	1.42	0.16	1.81	0.17
40	0.92	0.12	1.21	0.13	1.47	0.15	1.87	0.15
50	0.96	0.11	1.25	0.13	1.51	0.14	1.90	0.14
100	1.05	0.09	1.35	0.11	1.62	0.12	2.01	0.12

Table 12: ES from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{ES}	σ_{ES}	μ_{ES}	σ_{ES}	μ_{ES}	σ_{ES}	μ_{ES}	σ_{ES}
2	1.78	0.92	2.22	1.24	2.74	1.61	3.77	2.37
3	2.42	0.90	3.05	1.20	3.81	1.63	5.40	2.53
4	2.79	0.85	3.57	1.18	4.47	1.61	6.43	2.52
5	3.07	0.83	3.94	1.16	4.98	1.60	7.18	2.47
6	3.28	0.79	4.23	1.13	5.35	1.58	7.76	2.45
7	3.44	0.79	4.45	1.12	5.69	1.58	8.30	2.45
8	3.59	0.78	4.66	1.12	5.94	1.55	8.70	2.40
9	3.71	0.76	4.85	1.09	6.19	1.53	9.02	2.38
10	3.81	0.74	4.99	1.09	6.39	1.54	9.38	2.37
11	3.91	0.74	5.13	1.08	6.55	1.51	9.65	2.33
12	3.99	0.73	5.24	1.06	6.73	1.50	9.90	2.31
13	4.06	0.70	5.33	1.04	6.89	1.48	10.13	2.27
14	4.13	0.71	5.46	1.05	7.02	1.47	10.36	2.28
15	4.20	0.70	5.53	1.04	7.14	1.47	10.51	2.24
16	4.26	0.70	5.61	1.02	7.24	1.44	10.70	2.22
17	4.32	0.70	5.69	1.02	7.37	1.43	10.87	2.21
18	4.35	0.68	5.77	1.02	7.49	1.45	11.01	2.18
19	4.40	0.68	5.85	1.03	7.55	1.41	11.14	2.13
20	4.44	0.67	5.89	1.01	7.65	1.41	11.29	2.16
25	4.63	0.65	6.17	1.00	8.03	1.37	11.89	2.14
30	4.79	0.62	6.41	0.97	8.31	1.35	12.37	2.06
40	5.02	0.61	6.74	0.97	8.79	1.30	13.08	1.99
50	5.19	0.59	6.98	0.92	9.17	1.30	13.58	1.93
100	5.68	0.55	7.72	0.87	10.16	1.20	15.14	1.80

Table 13: log(ES) from bivariate normal for several sd ratios

σ_v/σ_h	1		1.5		2		3	
Group size	μ_{lnES}	σ_{lnES}	μ_{lnES}	σ_{lnES}	μ_{lnES}	σ_{lnES}	μ_{lnES}	σ_{lnES}
2	0.41	0.62	0.62	0.67	0.81	0.68	1.10	0.72
3	0.81	0.41	1.03	0.42	1.24	0.46	1.57	0.51
4	0.98	0.32	1.21	0.35	1.43	0.38	1.78	0.41
5	1.08	0.28	1.33	0.30	1.55	0.33	1.91	0.36
6	1.16	0.25	1.41	0.27	1.63	0.30	2.00	0.33
7	1.21	0.23	1.46	0.26	1.70	0.28	2.07	0.30
8	1.25	0.22	1.51	0.24	1.75	0.26	2.12	0.28
9	1.29	0.21	1.55	0.23	1.79	0.25	2.16	0.27
10	1.32	0.19	1.58	0.22	1.83	0.24	2.21	0.26
11	1.35	0.19	1.61	0.21	1.85	0.23	2.24	0.25
12	1.37	0.18	1.64	0.20	1.88	0.22	2.27	0.24
13	1.39	0.17	1.65	0.19	1.91	0.22	2.29	0.23
14	1.40	0.17	1.68	0.19	1.93	0.21	2.31	0.22
15	1.42	0.17	1.69	0.19	1.94	0.21	2.33	0.22
16	1.44	0.16	1.71	0.18	1.96	0.20	2.35	0.21
17	1.45	0.16	1.72	0.18	1.98	0.19	2.37	0.20
18	1.46	0.16	1.74	0.18	1.99	0.19	2.38	0.20
19	1.47	0.15	1.75	0.17	2.00	0.19	2.39	0.19
20	1.48	0.15	1.76	0.17	2.02	0.18	2.41	0.19
25	1.52	0.14	1.81	0.16	2.07	0.17	2.46	0.18
30	1.56	0.13	1.85	0.15	2.10	0.16	2.50	0.17
40	1.61	0.12	1.90	0.14	2.16	0.15	2.56	0.15
50	1.64	0.11	1.93	0.13	2.21	0.14	2.60	0.14
100	1.73	0.10	2.04	0.11	2.31	0.12	2.71	0.12